[5] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups*, II, Ann. of Math. 48 (1947), pp. 326–341.
[6] C. J. Everett Jr., *An extension theory for rings*, Amer. J. Math. 64 (1942), pp. 363–370.
[7] G. Hochshild, *On the cohomology groups of an associative algebra*, Ann. of Math. 46 (1945), pp. 58–67.
[8] — *On the cohomology theory for associative algebras*, Ann. of Math. 47 (1946), pp. 568–579.
[9] — *Lie algebra kernels and cohomology*, Amer. J. Math. 76 (1954), pp. 698–716.
[10] G. Hochshild and J. P. Serre, *Cohomology of groups extensions*, Trans. Amer. Math. Soc. 74 (1953), pp. 110–134.
[11] Nathan Jacobson, *Structure of rings*, A.M.S., Providence 1956.
[12] — *Lectures in abstract algebra*, Vol. 1, 1951.
[13] S. Mac Lane, *Homologie des anneaux et des modules*, Colloque de Topologie Alégbrique, Louvain (1956), pp. 55–80.
[14] — *Extensions and obstructions for rings*, Ill. J. of Math. 2 (1958), pp. 316–345.
[15] — *Homology*, Academic Press, New York 1963.
[16] Hakote Misan, *Simple Vector Spaces*, Sci. Rep Kanazawa Univ., 2 (1953), pp. 5–6.
[17] Umeshachandra Shukla, *Cohomologie des algébres associatives*, Ann. Sci. École Normale Supérieure, Paris, 78 (1961), pp. 163–209.
[18] Ferenc Szasz, *Ringe, deren echte Unterringe strong zyklische Rechts Ideale sind*, Magyar Tud. Akad. Kutató Int. Közl., 5 (1960), pp. 287–292.
[19] J. Szendrei, *Über eine allgemeine Ringkonstruktion durch Schiefes Produkt*, Acta Szeged, XIX (1958), pp. 63–76.
[20] J. Szep, *Über eine neue Erweiterung von Ringen*, I–II, Acta Szeged 19 (1958), pp. 51–62 and same Acta, 20 (1959), pp. 202–214.

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On the lattice of left annihilators of certain rings

by

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§ 1. Introduction. In this note we shall explore the connection between algebraic equivalence in a Rickart ring and certain lattice theoretic properties of its lattice of left annihilators—our goal being to place the whole theory in a lattice theoretic rather than a ring theoretic setting. In the case of the projection lattice of a von Neumann algebra, it is shown that the usual dimension relation of $*$-equivalence may be realized as perspectivity in a certain associated lattice. The parallel between von Neumann's dimension theory for a continuous geometry and the one for von Neumann algebras thus becomes apparent in that both are seen to be intrinsic—based on perspectivity.

§ 2. Rickart rings. Following terminology introduced by S. Maeda [8], we agree to call a ring $\mathfrak{A}$ a *Rickart ring* in case it satisfies the following two conditions:

(R_r) *The right annihilator of every element is the principal right ideal generated by an idempotent.*

(R_l) *The left annihilator of every element is the principal left ideal generated by an idempotent.*

For examples we refer the reader to Kaplansky [5] as well as S. Maeda [8]. Given the Rickart ring $\mathfrak{A}$, let $L(x)$ denote the left annihilator of x, $R(x)$ its right annihilator, $\mathfrak{L}(\mathfrak{A}) = \{L(x)\colon x \in \mathfrak{A}\}$ and $\mathfrak{R}(\mathfrak{A}) = \{R(x)\colon x \in \mathfrak{A}\}$. If $\mathfrak{L}(\mathfrak{A})$ and $\mathfrak{R}(\mathfrak{A})$ are each partially ordered by set inclusion, by [8], Theorem 1.1, p. 512, they form dual isomorphic relatively complemented lattices with 0 and 1. Our goal in this section is to extend [8], Lemma 4.3, p. 517.

First we need some additional terminology. Two elements e, f of a lattice L are said to form a *modular pair*, denoted $M(e,f)$, in case $a \leqslant f \Rightarrow a\vee(e\wedge f) = (a\vee e)\wedge f$; they form a *dual modular pair*, in symbols $DM(e,f)$, if $a \geqslant f \Rightarrow a\wedge(e\vee f) = (a\wedge e)\vee f$. In a lattice L with 0, two elements e and f are called *perspective* and written $e\sim f$ in case there is an element x such that $e\vee x = f\vee x$ with $e\wedge x = f\wedge x = 0$; they are called *strongly perspective* and denoted $e\sim^s f$ when they are perspective in

$[0, e\vee f]$; the symbol $e\approx f$ will be used to indicate that e and f are *projective* in the sense that there exist finitely many elements $e_1, e_2, \ldots, e_n$ such that $e\sim e_1\sim \ldots \sim e_n\sim f$. We agree to write $e\lesssim f$ to indicate the existence of an element g such that $e\vee g = f\vee g = e\vee f$, $e\wedge g = f\wedge g = 0$ and (f, g) as well as (g, f) both form modular and dual modular pairs; if in addition (e, g), (g, e) each form modular and dual modular pairs, we say that e and f are *modularly perspective*.

Now let $\mathfrak{A}$ be a Rickart ring with $L = \mathfrak{L}(\mathfrak{A})$. We adopt the convention that e, f, g, h (with or without subscripts) will always denote idempotents. If $\mathfrak{A}e, \mathfrak{A}f \in L$ we follow S. Maeda [7] and call $\mathfrak{A}e$, $\mathfrak{A}f$ *semi-orthogonal* if we can find e_0, f_0 such that $\mathfrak{A}e = \mathfrak{A}e_0$, $\mathfrak{A}f = \mathfrak{A}f_0$ and $e_0f_0 = f_0e_0 = 0$. A mapping φ: $L\to L$ is called *residuated* if φ is isotone and there exists an isotone mapping φ^+: $L\to L$ such that $(\mathfrak{A}g)\varphi^+\varphi \leqslant \mathfrak{A}g \leqslant (\mathfrak{A}g)\varphi\varphi^+$ for all $\mathfrak{A}g \in L$. As in [3], p. 94, each $x\in L$ induces a residuated map φ_x: $L\to L$ by the rule $(\mathfrak{A}g)\varphi_x = LR(gx)$ with $(\mathfrak{A}g)\varphi_x^+ = L\big(x(1-g)\big)$. As a final item, we agree to call x *range-closed* if $\mathfrak{A}g \leqslant LR(x)$ implies the existence of an element $\mathfrak{A}h$ such that $(\mathfrak{A}h)\varphi_x = \mathfrak{A}g$; we call x *dual range-closed* if $\mathfrak{A}g \geqslant 0\varphi_x^+ = L(x)$ implies $\mathfrak{A}g = (\mathfrak{A}h)\varphi_x^+$ for suitable $\mathfrak{A}h \in L$. It is easy to show that x dual range-closed is equivalent to the assertion that $e\mathfrak{A} \leqslant RL(x)$ implies $e\mathfrak{A} = RL(xf)$ for some $f\mathfrak{A}\in \mathfrak{R}(\mathfrak{A})$.

THEOREM 1. *If $RL(x) = e\mathfrak{A}$, $LR(x) = \mathfrak{A}f$ with $ef = fe = 0$, then $\mathfrak{A}e \lesssim \mathfrak{A}f$.*

Proof. Set $d = e+f$. By [8], Lemma 1.4, p. 512, $d\mathfrak{A}d$ is a Rickart ring with $\mathfrak{L}(d\mathfrak{A}d)$ isomorphic to $L[0, \mathfrak{A}d]$. Dropping down to $d\mathfrak{A}d$, we may assume $f = 1-e$. It is important to notice that $x = exf$, $x^2 = 0$, $R(x) = R(f) = e\mathfrak{A}$ and $L(x) = L(e) = \mathfrak{A}f$. Set $g = e-x$ and note that g is idempotent. Also, $ag = 0 \Rightarrow ae = ax$, so $ae = aef = 0$, while $ae = 0 \Rightarrow a \in L(e) = L(x) \Rightarrow ag = 0$. This shows that $L(g) = L(e)$, so $\mathfrak{A}(1-g) = \mathfrak{A}(1-e) = \mathfrak{A}f$. As in [6], Lemma 5.6, p. 165, or by [3], Theorem 27, p. 95, $\mathfrak{A}g$ and $\mathfrak{A}f$ are complements with $(\mathfrak{A}g, \mathfrak{A}f)$ as well as $(\mathfrak{A}f, \mathfrak{A}g)$ forming both a modular and a dual modular pair.

It remains to show that $\mathfrak{A}e$ and $\mathfrak{A}g$ are complements. If $\mathfrak{A}e\vee\mathfrak{A}g \leqslant \mathfrak{A}h$, then $e = eh$ and $g = gh$, so

$$g = e-x = (e-x)h = eh-xh = e-xh$$

and $x = xh$. Now $R(x) = R(f) = (1-f)\mathfrak{A} = e\mathfrak{A}$, so $x(1-h) = xh(1-h) = 0$ implies $1-h = e(1-h) = e-eh = 0$ and $h = 1$. To see that $\mathfrak{A}e \cap \mathfrak{A}g = 0$, let $b\in \mathfrak{A}e \cap \mathfrak{A}g$. Then $b = be = bg$ so $b = b(e-x) = be-bx = b-bx$ shows $bx = 0$, so

$$b \in L(x) = L(e) \quad \text{and} \quad b = be = 0\,.$$

COROLLARY 2. *With notation as in the theorem, if $\mathfrak{A}e_0 \leqslant \mathfrak{A}e$, then $\mathfrak{A}e_0 \lesssim (\mathfrak{A}e_0)\varphi_x$; furthermore, $\mathfrak{A}e$ and $\mathfrak{A}f$ have a common complement $\mathfrak{A}g$ in their join such that $(\mathfrak{A}e_0)\varphi_x = (\mathfrak{A}e_0\vee\mathfrak{A}g)\cap\mathfrak{A}f$ for all $\mathfrak{A}e_0 \leqslant \mathfrak{A}e$.*

Proof. If necessary (see [7], Lemma 4, p. 159) we may replace e_0 by ee_0, so we may assume with no loss of generality that $e_0 = ee_0 = e_0e$. Notice that $y(e_0x) = 0 \Rightarrow ye_0 \in L(x) = L(e) \Rightarrow ye_0 = ye_0e = 0$, while $ye_0 = 0$ clearly implies $ye_0x = 0$. Thus $L(e_0x) = L(e_0)$ and $RL(e_0x) = RL(e_0) = e_0\mathfrak{A}$.

We now observe that $LR(e_0x) \leqslant LR(x) = \mathfrak{A}f$, so let $\mathfrak{A}f_0 = LR(e_0x)$. We may assume as above, that $f_0 = ff_0 = f_0f$, so we have $LR(e_0x) = \mathfrak{A}f_0$, $RL(e_0x) = e_0\mathfrak{A}$ with $e_0f_0 = f_0e_0 = 0$. By the theorem, if $g_0 = e_0 - e_0x$, then $\mathfrak{A}g_0$ is a common complement of $\mathfrak{A}e_0$, $\mathfrak{A}f_0$ in $[0, \mathfrak{A}e_0\vee\mathfrak{A}f_0]$ such that $(\mathfrak{A}g_0, \mathfrak{A}f_0)$, $(\mathfrak{A}f_0, \mathfrak{A}g_0)$ form modular and dual modular pairs. Routine computation shows that $\mathfrak{A}g_0 \leqslant \mathfrak{A}g$, so

$$\mathfrak{A}e_0\vee\mathfrak{A}g = \mathfrak{A}e_0\vee\mathfrak{A}g_0\vee\mathfrak{A}g = \mathfrak{A}f_0\vee\mathfrak{A}g_0\vee\mathfrak{A}g = \mathfrak{A}f_0\vee\mathfrak{A}g\,.$$

Using $M(\mathfrak{A}g, \mathfrak{A}f)$, we have

$$\mathfrak{A}f_0 = (\mathfrak{A}f_0\vee\mathfrak{A}g)\cap\mathfrak{A}f = (\mathfrak{A}e_0\vee\mathfrak{A}g)\cap\mathfrak{A}f$$

as desired.

COROLLARY 3. *If $e\mathfrak{A} = RL(x)$, $\mathfrak{A}f = LR(x)$ with $\mathfrak{A}e$ semi-orthogonal to $\mathfrak{A}f$, then $\mathfrak{A}e \lesssim \mathfrak{A}f$.*

Proof. By hypothesis there exist $e_0, f_0 \in \mathfrak{A}$ such that $\mathfrak{A}e = \mathfrak{A}e_0$, $\mathfrak{A}f = \mathfrak{A}f_0$ and $e_0f_0 = f_0e_0 = 0$. Let $x_0 = e_0x$. Then

$$x_0y = 0 \Rightarrow e_0xy = 0 \Rightarrow xy = exy = ee_0xy = 0\,,$$
$$xy = 0 \Rightarrow x_0y = e_0xy = 0\,,$$
$$yx_0 = 0 \Rightarrow ye_0x = 0 \Rightarrow ye_0 \in L(x) = L(e) \Rightarrow ye_0 = ye_0e = 0\,,$$
$$ye_0 = 0 \Rightarrow yx_0 = ye_0x = 0\,.$$

This shows that $R(x_0) = R(x)$ and $L(x_0) = L(e_0)$, so $RL(x_0) = e_0\mathfrak{A}$, $LR(x_0) = \mathfrak{A}f_0$ and $e_0f_0 = f_0e_0 = 0$. Now invoke Theorem 1.

It seems worth mentioning that the symmetry of e and f in the above Corollary will also yield $\mathfrak{A}f \lesssim \mathfrak{A}e$. We return now to the notation of Theorem 1, and assume $RL(x) = e\mathfrak{A}$, $LR(x) = \mathfrak{A}f$ with $ef = fe = 0$, and $g = e-x$. By [4], Lemma 3.6, p. 1216, the assertion $DM(\mathfrak{A}e, \mathfrak{A}g)$ is equivalent to $x = e(1-g) = e(1-e+x)$ being range-closed, while $M(\mathfrak{A}g, \mathfrak{A}e)$ is equivalent to x being dual range-closed. Thus if x is both range-closed and dual range-closed, we have $\mathfrak{A}e$ and $\mathfrak{A}f$ modularly perspective, and $\mathfrak{A}e_0 \to (\mathfrak{A}e_0\vee\mathfrak{A}g)\cap\mathfrak{A}f$ an isomorphism of $[0, \mathfrak{A}e]$ onto $[0, \mathfrak{A}f]$ whose inverse is given by $\mathfrak{A}f_0 \to (\mathfrak{A}f_0\vee\mathfrak{A}g)\cap\mathfrak{A}e$. In particular, if e and f are *algebraically equivalent* (see [8], p. 517) in the sense that there is an element y such that $xy = e$ and $yx = f$, then x is both range-closed and dual range-closed, so the above remark applies. It should be noted that the isomorphism induced lattice theoretically by $\mathfrak{A}e_0 \to (\mathfrak{A}e_0 \vee \mathfrak{A}g) \cap \mathfrak{A}f$ coincides with that provided by the algebraic equivalence of e and f in the ring $\mathfrak{A}$ (see [4], Lemma 5.2, p. 1220). This extends [8], Lemma 4.3, p. 517 and is summarized in the next theorem.

THEOREM 4. *If $xy = e$, $yx = f$ and $ef = fe = 0$, then $\mathfrak{A}e$ and $\mathfrak{A}f$ are modularly perspective with the perspectivity implemented by a common complement $\mathfrak{A}g$ such that $\mathfrak{A}e_0 \to LR(e_0xf) = (\mathfrak{A}e_0 \vee \mathfrak{A}g) \cap Af$ is an isomorphism of $[0, \mathfrak{A}e]$ onto $[0, \mathfrak{A}f]$; furthermore, $\mathfrak{A}e_0$ and $(\mathfrak{A}e_0 \vee \mathfrak{A}g) \cap \mathfrak{A}f$ are modularly perspective for each $Ae_0 \leqslant Ae$.*

One can, of course, proceed as in Corollary 3 and weaken this to the case where $\mathfrak{A}e$ and $\mathfrak{A}f$ are semi-orthogonal.

§ 3. Rickart $*$-rings. A Rickart $*$-ring is an involution ring $\mathfrak{A}$ in which the right annihilator of each element x is the principal right ideal generated by the projection x' (see [8], pp. 522–525). It is immediate (see [2], Theorem 3, p. 651) that the lattice $L = P(A)$ formed by the projections in $\mathfrak{A}$ is *orthomodular*. We agree to call two projections *orthogonal* and write $e \perp f$ in case $ef = 0$. As in § 2, we call them *semi-orthogonal* if there exist idempotents e_0, f_0 such that $\mathfrak{A}e = \mathfrak{A}e_0$, $\mathfrak{A}f = \mathfrak{A}f_0$ and $e_0f_0 = f_0e_0 = 0$. We use the notation $e \sim^* f$ to denote the fact that e and f are $*$-*equivalent*, in the sense that there is an element x such that $xx^* = e$ and $x^*x = f$. Also, for each $x \in \mathfrak{A}$ we agree to let $x'' = (x')' = 1 - x'$. The results of § 2 when applied to a Rickart $*$-ring now yield the following:

THEOREM 5. *Let $\mathfrak{A}$ be a Rickart $*$-ring.*

(i) *If x'' is semi-orthogonal to $x^{*\prime\prime}$, then $x'' \precsim x^{*\prime\prime}$ and $a \precsim (ax^*)''$ for each $a \leqslant x''$; furthermore, x'' and $x^{*\prime\prime}$ have a common complement g in their join such that $(ax^*)'' = (a \vee g) \wedge x^{*\prime\prime}$ for all $a \leqslant x''$.*

(ii) *Let $xx^* = e$ and $x^*x = f$ with $e \perp f$ in $P(\mathfrak{A})$. Then e and f are modularly perspective with the perspectivity induced by an element g such that $x^*ax = (a \vee g) \wedge f$ for all $a \leqslant e$.*

It follows from part (ii) of the above theorem that if $e \perp f$ and $e \sim^* f$, the ortho-isomorphism of $[0, e]$ onto $[0, f]$ induced by their $*$-equivalence is also induced lattice theoretically.

COROLLARY 6. *If two projections of a Rickart $*$-ring $\mathfrak{A}$ are both semi-orthogonal and projective, then they are strongly perspective.*

Proof. If g is a common complement for e and f it is easy to see that $(eg'f)'' = f$ and $(fg'e)'' = e$. Making repeated use of this fact we see that if $e \approx f$ there exists an x in $\mathfrak{A}$ such that $x'' = e$ and $x^{*\prime\prime} = f$. Now apply Theorem 5.

§ 4. Dimension lattices. Our goal in this section is to show that the usual dimension relation of $*$-equivalence in a Baer $*$-ring satisfying (EP) and (SR) may be regarded as a purely lattice theoretic concept. First we must establish our basic terminology. A *Baer $*$-ring* $\mathfrak{A}$ is an involution ring in which the right annihilator of each subset is a principal right ideal generated by a projection. An element u of such a ring is called *unitary* if $uu^* = u^*u = 1$; two projections e and f are called *unitarily equivalent* if there exists a unitary element u such that $u^*eu = f$. Writing "CC" for "commutes with everything that commutes with", we now introduce the following axioms (due to Kaplansky [5], pp. 89–90) for $\mathfrak{A}$:

(EP) *For any non-zero element x there exists a self-adjoint element y with $y\,\mathrm{CC}x^*x$ and x^*xy^2 a non-zero projection (existence of projections).*

(SR) *For any element x we can write $x^*x = y^2$ with y self-adjoint and $y\,\mathrm{CC}x^*x$ (square root).*

THEOREM 7. *Let $\mathfrak{A}$ be a Baer $*$-ring satisfying (EP) and (SR). There exists a Baer $*$-ring $\mathfrak{B}$ such that $\mathfrak{A}$ is a $*$-subring of $\mathfrak{B}$, $P(\mathfrak{A})$ is ortho-isomorphic to an interval sublattice of $P(\mathfrak{B})$ and for $e, f \in P(\mathfrak{A})$ the following conditions are equivalent:*

(i) *There exists an element x of $\mathfrak{A}$ such that $x'' = e$, $x^{*\prime\prime} = f$.*

(ii) *$e \sim^* f$ in $\mathfrak{A}$.*

(iii) *e is unitarily equivalent to f in $\mathfrak{B}$.*

(iv) *There exists a projection g in $\mathfrak{B}$ such that e is modularly perspective to g and g is modularly perspective to f.*

(v) *$e \approx f$ in $P(\mathfrak{B})$.*

Proof. By [5], Theorem 10, p. 12 we may assume $\mathfrak{A}$ finite or purely infinite (see [5], pp. 10–11 for a definition of these terms). In the finite case we take $\mathfrak{B} = \mathfrak{A}$ and apply [5], Theorem 63, p. 99 and Theorem 71, p. 120 to conclude that (i) $\Longleftrightarrow$ (ii) which in turn is equivalent to e being perspective to f in $P(\mathfrak{A})$. The remaining equivalences are now obvious.

Thus we may as well assume $\mathfrak{A}$ purely infinite. In view of [5], Exercise 2(a), p. 66 we may take for $\mathfrak{B}$ the 2 by 2 matrix ring over $\mathfrak{A}$ and identify $\mathfrak{A}$ with the set of all matrices of the form $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ with $x \in \mathfrak{A}$.

(i) $\Rightarrow$ (ii). [5], Theorem 63, p. 99.

(ii) $\Rightarrow$ (iii). Let $xx^* = e$, $x^*x = f$. Notice that

$$\begin{bmatrix} x & 1-e \\ f-1 & x^* \end{bmatrix}\begin{bmatrix} x^* & f-1 \\ 1-e & x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} x^* & f-1 \\ 1-e & x \end{bmatrix}\begin{bmatrix} x & 1-e \\ f-1 & x^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\begin{bmatrix} x & 1-e \\ f-1 & x^* \end{bmatrix}$ is unitary. We now need only observe that

$$\begin{bmatrix} x^* & f-1 \\ 1-e & x \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 1-e \\ f-1 & x^* \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} x & 1-e \\ f-1 & x^* \end{bmatrix}\begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x^* & f-1 \\ 1-e & x \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}.$$

(iii) ⇒ (iv). If e is unitarily equivalent to f in $\mathfrak{B}$ there exists an element X of $\mathfrak{B}$ such that

$$X'' = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X^{*''} = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}.$$

Now

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so two applications of Theorem 5 (ii) will now produce the fact that $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ is modularly perspective to $\begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$ which in turn is modularly perspective to $\begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$.

(iv) ⇒ (v). Clear.

(v) ⇒ (i). If $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ in $P(\mathfrak{B})$, then there exists an element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{B}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}'' = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{*''} = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}.$$

But then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ae & 0 \\ ce & 0 \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} fa & fb \\ 0 & 0 \end{bmatrix}$$

shows $b = c = d = 0$ so

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{A}.$$

Remark 8. If L happens to be the projection lattice of a von Neumann algebra, we may invoke [1], Theorem 1, p. 383 to replace condition (v) by

(v′) $e \sim f$ in $P(\mathfrak{B})$.

Thus $*$-equivalence in the projection lattice of a von Neumann algebra $\mathfrak{A}$ coincides with the restriction to $P(\mathfrak{A})$ of perspectivity in the projection lattice of the 2 by 2 matrix ring over $\mathfrak{A}$.

All of this suggests that a suitable vehicle for lattice dimension theory ought to be a complete orthomodular lattice such that $e \perp f$, $e \approx f \Rightarrow e$ is modularly perspective to f.

References

[1] P. A. Fillmore, *Perspectivity in projection lattices*, Proc. Amer. Math. Soc. 16 (1965), pp. 383–387.
[2] D. J. Foulis, *Baer $*$-semigroups*, Proc. Amer. Math. Soc. 11 (1960), pp. 648–654.
[3] M.F. Janowitz, *Baer semigroups*, Duke Math. J. 32 (1965), pp. 85–96.
[4] — *A semigroup approach to lattices*, Canad. J. Math. 18 (1966), pp. 1212–1223.
[5] I. Kaplansky, *Rings of operators*, New York 1968.
[6] F. Maeda, *Decompositions of general lattices into direct summands of types* I, II *and* III, J. Sci. Hiroshima Univ., Ser. A, 23 (1959), pp. 151–170.
[7] S. Maeda, *On relatively semi-orthocomplemented lattices*, J. Sci. Hiroshima Univ., Ser. A, 24 (1960), pp. 155–161.
[8] — *On a ring whose principal right ideals generated by idempotents form a lattice*, J. Sci. Hiroshima Univ., Ser. A, 24 (1960), pp. 509–525.

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