Abelian torsion groups with artinian primary components and their automorphisms

by

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I. Introduction. Throughout this paper T denotes an abelian torsion group and $A(T)$ its automorphism group.

As in [9], we are interested in theorems of the type:

$$T \text{ is a } \boldsymbol{P}\text{-group if and only if } A(T) \text{ is a } \boldsymbol{Q}\text{-group},$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are group theoretical properties.

In [3], R. Baer has proven that T is artinian if and only if every torsion group of automorphisms of T is finite. This nice result holds true even if the finiteness of the torsion subgroups of $A(T)$ is replaced by the weaker postulate that elementary abelian 2-groups of automorphisms of T are finite (see [4]). But it is clear that the torsion subgroups of $A(T)$ become arbitrarily complicated if infinitely many primary components of T are nontrivial. So R. Baer raised the question, what group theoretical property $\boldsymbol{Q}$ imposed on $A(T)$ is necessary and sufficient in order that every primary component of T is artinian.

An answer to this question shall be given in the present paper. We will prove the following result.

MAIN THEOREM. *Every primary component of T is artinian if and only if $A(T)$ is residually finite and, for every primary normal subgroup Γ of $A(T)$, the centralizer $\mathrm{c}\Gamma$ of Γ in $A(T)$ has finite index in $A(T)$.*

The investigations in section III of the automorphism group of abelian p-groups might be of independent interest. We shall show that for every abelian p-group G, $p > 3$, every normal p'-subgroup of its automorphism group $A(G)$ is contained in the center of $A(G)$.

II. Preliminaries. Our notation and terminology concerning abelian group follows [5].

A group is called *artinian* if its subgroups satisfy the minimum condition.

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If G is an abelian p-group for some prime p, the maximal divisible subgroup of G shall be denoted by $\mathrm{d}G$. The rank $\mathrm{rk}(G)$ of G is the dimension of $G[p]$ as a vectorspace over the primefield of characteristic p. By a wellknown theorem, G is artinian if and only if $\mathrm{rk}(G)$ is finite (cf. [5], p. 65, Theorem 19.2).

If Y is a normal subgroup of a group X, we write $Y \lhd X$. The group of all automorphisms of X shall be denoted by $A(X)$.

If X is abelian and S some subgroup of X, the set of all automorphisms of X which fix S elementwise and induce the 1-automorphism in X/S is a subgroup of $A(X)$, called the *stabilizer* of S in X, and shall be denoted here by $\Sigma(X/S, S)$. By a wellknown theorem of Kaloujnine, $\Sigma(X/S, S)$ is abelian (cf. [13], p. 88, Satz 19), and furthermore $\Sigma(X/S, S) \cong \mathrm{Hom}(X/S, S)$ (cf. [8], p. 153, Hilfssatz 1.4).

T_p denotes the p-component of the abelian torsion group T. A torsion group without elements of order p is called a *p'-group*. $A \oplus B$ denotes the direct sum of the abelian groups A and B, Σ°, Π° and Π^* are our symbols for a direct sum, a direct product, and a cartesian or unrestricted direct product. If C is a Γ-admissible factor group of X, for $\Gamma \subseteq A(X)$, then $\Gamma|_C$ denotes the group of automorphisms of C induced by Γ.

If $\Delta \subseteq A(X)$, we write $\mathrm{c}_\Delta \Gamma$ for the centralizer of Γ in Δ. The centralizer of Γ in $A(X)$ simply is denoted by $\mathrm{c}\Gamma$ and the center of Γ by $\mathrm{z}\Gamma$. $\Gamma \circ \Delta$ is the subgroup generated by the set of all commutators $\gamma^{-1}\delta^{-1}\gamma\delta$ where $\gamma \in \Gamma$ and $\delta \in \Delta$.

A group is called *residually finite* if the intersection of all subgroups of finite index is trivial (see [7], p. 16).

III. Central automorphisms. In this section we are concerned with groups of automorphisms of an abelian p-group G.

The proof of our main theorem will vitally depend on the fact that, if $p > 3$, every normal p'-subgroup of $A(G)$ is contained in the center of $A(G)$.

In order to establish this result we need several lemmas.

LEMMA 1. *Let G be an elementary abelian p-group of finite rank n. If either $n \neq 2$ or $n = 2$ and $p > 3$, then every normal p'-subgroup of $A(G)$ is contained in the center of $A(G)$.*

Proof. If G is cyclic, then $A(G)$ is abelian and our proposition holds. Therefore, we may assume that $\mathrm{rk}(G) = n \geqslant 2$. Since every elementary abelian p-group is a vectorspace over the primefield K_p of characteristic p, $A(G)$ is isomorphic to the group $GL(n, p)$ of all invertible matrices of size $n \times n$ over K_p. It is wellknown that, for $n \geqslant 3$ or $n = 2$ and $p > 3$, every normal subgroup of $GL(n, p)$ either contains the subgroup $SL(n, p)$ of all matrices with determinant 1 or it is contained in the center of $GL(n, p)$ (cf. [1], p. 165, Theorem 4.9). But, for $n \geqslant 2$, $SL(n, p)$ contains elements of order p (see [1], p. 170, Theorem 4.11). Therefore, if $n \geqslant 3$ or $n = 2$ and $p > 3$, every normal p'-subgroup of $GL(n, p)$ is contained in the center, and the analogue statement holds for $A(G) \cong GL(n, p)$. This proves the lemma.

LEMMA 2. *If G is an infinite elementary abelian p-group, then every normal p'-subgroup of $A(G)$ is contained in the center of $A(G)$.*

Proof. Let Γ be a normal p'-subgroup of $A(G)$ and $\gamma \in \Gamma$. Since the order of γ is finite, any two elements x_1 and x_2 of G can be imbedded into a finite γ-admissible subgroup F of G, and we can choose F to have a rank $\neq 2$. Let Φ be the set of all automorphisms of G mapping F onto itself. Then $\gamma \in \Phi$ and, since F is a direct summand, Φ induces in F its full group $A(F)$ of automorphisms. Therefore, the group of automorphisms of F induced by $\Gamma \cap \Phi \lhd \Phi$ is a normal p'-subgroup of $A(F)$ which, accordingly to Lemma 1, is contained in $\mathrm{z}A(F)$. Hence, $\gamma \in \Gamma \cap \Phi$ induces in F a central automorphism. But the center of $A(F)$ consists just of the multiplications with p-adic units (see [2], p. 111, Theorem), so $x_1\gamma = kx_1$ and $x_2\gamma = kx_2$ for a suitable integer k prime to p. This being true for every pair of elements implies $x\gamma = kx$ for all $x \in G$. Hence $\gamma \in \mathrm{z}A(G)$ and $\Gamma \subseteq \mathrm{z}A(G)$ as stated above.

A consequence of these two results is the following

COROLLARY 3. *Let G be an elementary abelian p-group, $p > 3$. Then every normal p'-subgroup of $A(G)$ is contained in the center of $A(G)$.*

LEMMA 4. *For $n \geqslant 0$ an integer, every automorphism of*

$$F^n = (p^nG)[p]/(p^{n+1}G)[p]$$

is induced by some automorphism of G.

Proof. Since every automorphism of p^nG is induced by an automorphism of G (see [6], p. 123, Lemma 1), it suffices to prove the proposition for $n = 0$.

Let B be a maximal elementary abelian direct summand of G (cf. [5], p. 99). Then, for some subgroup H of G, we have

$$G = B \oplus H ,$$

where $(pG)[p] = H[p]$, and the mapping $\sigma\colon B \to F^\circ$ defined by

$$b\sigma = b + H[p] \quad \text{for} \quad b \in B ,$$

is an isomorphism from B onto $F^\circ = G[p]/(pG)[p]$.

Let Φ be an automorphism of F°. Then G possesses an automorphism α satisfying

$$b\alpha = b\sigma\Phi\sigma^{-1} \quad \text{for} \quad b \in B$$
$$h\alpha = h \quad \text{for} \quad h \in H .$$

If $\bar{a}$ denotes the automorphism of F° induced by a, then

$$(b+H[p])\bar{a} = ba+H[p] = ba\sigma = b\sigma\Phi = (b+H[p])\Phi$$

for every $b \in B$. Hence, $\bar{a} = \Phi$ and the lemma is proven.

Lemma 5. *Let G be a reduced abelian p-group for $p > 3$ and Γ a normal p'-subgroup of $A(G)$. Then $\Gamma \subseteq zA(G)$.*

Proof. Let Δ_n denote the set of all automorphisms of G inducing the 1-automorphism in $F^n = (p^nG)[p]/(p^{n+1}G)[p]$. Then Δ_n is a normal subgroup of $A(G)$ and, by Lemma 4, we have

$$(1) \qquad A(F^n) \cong A(G)/\Delta_n .$$

For every n, Γ induces a p'-group $\Gamma_n \cong \Gamma\Delta_n/\Delta_n$ of automorphisms in F^n, which, according to (1), is a normal subgroup of $A(F^n)$. Hence, by Corollary 3, Γ_n is contained in $zA(F^n)$; it follows that $\Gamma_n \circ A(F^n) = 1$ and

$$(2) \qquad \Gamma \circ A(G) \subseteq \Gamma \cap \Delta_n \quad \text{for} \quad n = 0, 1, \ldots ,$$

using (1) and the normality of Γ. Consequently,

$$(3) \qquad \Gamma \circ A(G) \subseteq \Gamma \cap \Big(\bigcap_{n\geqslant 0} \Delta_n\Big) .$$

But according to [11], p. 101, the order of every torsion automorphism in $\bigcap_{n\geqslant 0} \Delta_n$ is a power of p, which implies $\Gamma \cap (\bigcap_{n\geqslant 0} \Delta_n) = 1$. Using (3) we obtain $\Gamma \circ A(G) = 1$, i.e. $\Gamma \subseteq zA(G)$.

Theorem 6. *Let G be an abelian p-group for $p > 3$ and Γ a normal p'-subgroup of $A(G)$. Then Γ is contained in the center of $A(G)$.*

Proof. Let $G = D \oplus R$ where D is divisible and R is reduced. Using the injectivity of divisible groups (cf. [5], p. 59, Theorem 16.1) one verifies easily that every automorphism of $D[p]$ is induced by an automorphism of D and therefore by an automorphism of G. Hence, Γ induces in $D[p]$ a p'-group θ of automorphisms which is normal in $A(D[p])$. By Corollary 3 we have

$$\Gamma|_{D[p]} = \theta \subseteq zA(D[p]) ,$$

and consequently,

$$(1) \qquad [\Gamma \circ A(G)]|_{D[p]} = 1 .$$

In particular, $\Gamma \circ A(G) \subseteq \Gamma$ is a p'-group. A closer examination of the proof of a lemma by R. Baer ([3], p. 525) shows, that every p'-group of automorphisms of a divisible p-group P operates faithfully on $P[p]$. Hence, (1) implies

$$(2) \qquad [\Gamma \circ A(G)]|_D = 1 .$$

Likewise, every automorphism of $G/D \cong R$ is induced by an automorphism of G, and $\Gamma|_{G/D}$ is essentially a normal p'-subgroup of $A(R)$. Lemma 5 then implies that $\Gamma|_{G/D} \subseteq zA(G/D)$ and hence,

$$(3) \qquad [\Gamma \circ A(G)]|_{G/D} = 1 .$$

Comparing (2) and (3) we obtain

$$(4) \qquad \Gamma \circ A(G) \subset \Sigma(G/D, D) ,$$

where $\Sigma(G/D, D)$ denotes the stabilizer of D in G. But $\Sigma(G/D, D) \cong \mathrm{Hom}(G/D, D)$ (cf. [8], p. 153, Hilfssatz 1.4), and the order of every torsion element of $\Sigma(G/D, D)$ consequently is a power of p. Since $\Gamma \supseteq \Gamma \circ A(G)$ is a p'-group, (4) implies $\Gamma \circ A(G) = 1$ or $\Gamma \subseteq zA(G)$ as stated in the theorem.

The stabilizer $\Sigma(G/G[p], G[p])$ of $G[p]$ in G is an elementary abelian normal p-subgroup of $A(G)$. We will need the following

Lemma 7. *Let θ denote the group of automorphisms of $G[p]$ that is induced by the centralizer of $\Sigma(G/G[p], G[p])$ in $A(G)$. If $pG \neq p^2G$, then $\theta = zA(G[p])$; in particular, θ is cyclic of order $p-1$.*

Proof. For the sake of shortness, let $\Sigma = \Sigma(G/G[p], G[p])$. We claim that

$$(*) \qquad x\xi \in \{x\} \quad \text{for every} \quad x \in G[p] \text{ and } \xi \in c\Sigma .$$

In order to prove this statement, let $x \in G[p]$, $x \notin dG$. Then G has a decomposition $G = dG \oplus R$, where $x \in R$ (cf. [5], p. 63) and $pR \neq 0$. By our hypothesis $G/G[p] \cong dG \oplus R/R[p]$ is not divisible, and consequently G contains a subgroup $U \supseteq G[p]$ such that G/U is cyclic of order p (cf. [5], p. 67, Exercise 2). Hence $G = \{g\}+U$ for $g \in G$, $g \notin U$, and $pG \subseteq U$. It is easy to see that there exists an automorphism σ of G satisfying

$$g\sigma = g+x$$
$$u\sigma = u \quad \text{for all} \quad u \in U .$$

Since $x \in G[p] \subseteq U$ it follows that $\sigma \in \Sigma$. Let $\xi \in c\Sigma$ and $g\xi = kg+u$ for some integer k and $u \in U$. Since ξ commutes with σ we obtain

$$0 = g(\xi\sigma - \sigma\xi) = g\xi + kx - g\xi - x\xi = kx - x\xi$$

and consequently

$$(1) \qquad x\xi \in \{x\} \quad \text{for every} \quad x \in G[p],\ x \notin dG .$$

If $d \in (dG)[p]$, then, using our hypothesis $pG \neq p^2G$, there exists $x \in G[p]$, $x \notin dG$, and we have $d = d-x+x$ where $(d-x)\xi \in \{d-x\}$ and $x\xi \in \{x\}$ according to (1). A simple computation shows that $d\xi \in \{d\}$ and hence $x\xi \in \{x\}$ for all $x \in G[p]$ and every $\xi \in c\Sigma$. This proves $(*)$.

As an immediate consequence of (⁎) we state

(⁎⁎) If $\xi \in c\Sigma$, then $S\xi = S$ for every $S \subseteq G\{p\}$.

By a theorem of R. Baer an automorphism ξ of an abelian p-group A induces the 1-automorphism in the lattice of all subgroups of A if and only if ξ is the multiplication with an invertible p-adic integer ([2], p. 110, Theorem 5.2). If A is elementary abelian (or $p \neq 2$), then the central automorphisms of A are precisely the multiplications with p-adic units ([2], p. 111, Theorem). Hence, (⁎⁎) implies

$$c\Sigma|_{G[p]} = \theta = zA(G[p])$$

as stated in the proposition. It is wellknown that the multiplications with p-adic integers operate as a cyclic group of order $p-1$ of automorphisms on an elementary abelian p-group.

Lemma 7 is proven.

IV. The main results. We are now ready to give a new characterization of artinian p-groups by means of their automorphism groups.

A group is called *bounded* if there exists a finite upper bound for the orders of its elements.

THEOREM 8. *The following properties of the abelian p-group G are equivalent.*

(1) *G is artinian.*

(2) *Every torsion subgroup $A(G)$ is finite.*

(3) *$A(G)$ is residually finite and every normal torsion subgroup of $A(G)$ is finite.*

(4) *$A(G)$ is residually finite and $A(G)/c\Gamma$ is finite for every normal torsion subgroup Γ of $A(G)$.*

(5) *$A(G)$ is residually finite and $A(G)/c\Gamma$ is finite for every primary normal subgroup Γ of $A(G)$.*

(6) *$A(G)$ is residually finite and $A(G)/c\Sigma(G/G[p], G[p])$ is finite.*

(7) *$A(G)$ induces in $G[p]$ a bounded group of automorphisms.*

Remark. It is easy to see that the finiteness postulated in (2) and (3) can be replaced by countability (see [12], Main Theorem). In (4), (5), and (6) the finiteness of the occuring factorgroups can be weakened to the condition that they are bounded.

Remark. The residual finiteness of $A(G)$ postulated in (3)–(6) is indispensable. This is due to the fact that for a divisible p-group D of arbitrary rank every normal torsion subgroup of $A(D)$ is contained in the center of $A(D)$ (see [10]). Even if we restrict G to be reduced, the postulate of residual finiteness in Theorem 8 cannot be omitted completely (cf. Lemma 9 and Corollary 10).

Proof of Theorem 8. The equivalence of (1) and (2) was proven by R. Baer ([3], p. 521, Theorem). It is easy to verify that the automorphism group of an artinian abelian group is residually finite (see [9], Lemma 9). Hence, (2) implies (3).

Let us assume the validity of (3). Since $A(G)/c\Gamma$ is isomorphic to a group of automorphisms of Γ, it follows that $A(G)/c\Gamma$ is finite, and we have derived (4) from (3).

It is obvious that (4) implies (5) and, recalling $\Sigma(G/G[p], G[p]) \cong \mathrm{Hom}(G/G[p], G[p])$ (cf. [8], p. 153, Hilfssatz 1.4), that (5) in turn implies (6).

So, let us assume the validity of (6). Then there exists a natural number n such that

(a) $$\alpha^n \in c\Sigma(G/G[p], G[p]) \quad \text{for all} \quad \alpha \in A(G).$$

Let us distinguish two cases.

Case 1. $pG = p^2G$. Then $G = dG \oplus R$, where $pR = 0$ and both, dG and R, are direct sums of pairwise isomorphic groups of rank 1. If $\mathrm{rk}(dG)$ or $\mathrm{rk}(R)$ were infinite, the group of all permutations on a countably infinite set would be isomorphic to a subgroup of $A(G)$, contradicting the residual finiteness of $A(G)$ (see [9], Lemma 1 and Main Theorem). Hence, both $(dG)[p]$ and $R[p]$ are finite and so is $G[p] = (dG)[p] \oplus R[p]$. In this case the validity of (7) is obvious.

Case 2. $pG \neq p^2G$. Then, by Lemma 7, the centralizer of $\Sigma(G/G[p], G[p])$ induces in $G[p]$ a cyclic group θ of automorphisms of order $p-1$. Hence, by (a), $(\alpha^n)^{p-1} = \alpha^{n(p-1)}$ induces the identical automorphism in $G[p]$ for every $\alpha \in A(G)$; i.e. $A(G)$ induces in $G[p]$ a bounded group of automorphisms. We have derived (7) from (6).

It remains to show that (7) implies (1). So, let us finally assume the existence of an integer $n \geqslant 1$ such that

(b) $$\alpha^n|_{G[p]} = 1 \quad \text{for every} \quad \alpha \in A(G).$$

Let $G = D \oplus R$ where D is divisible and R is reduced. Clearly, every automorphism of $D[p]$ is induced by some automorphism of G, and the automorphism group of an infinite elementary abelian group is not bounded. Hence, $D[p]$ is finite and

(c) $$D \text{ is artinian}.$$

It remains to show that R is finite.

Let us assume, by way of contradiction, that R is infinite. Then there exist elements $z_i \in R$ of order $p^{e_i} \neq 1$ such that

(d) $$R = \sum_{i=1}^{n+1}{}^{\circ} \{z_i\} \oplus H, \quad e_1 \leqslant e_2 \leqslant \ldots \leqslant e_{n+1}.$$

It is easy to check that G possesses an automorphism α such that

$$\begin{cases} z_i\alpha = z_i + p^{e_{i+1}-e_i} z_{i+1} & \text{for} \quad 1 \leqslant i \leqslant n \\ \alpha|_{[D\oplus\{z_{n+1}\}\oplus H]} = 1 \end{cases}$$

and by complete induction one verifies that

$$z_1\alpha^n = z_1 + \sum_{i=1}^{n} \binom{n}{i} p^{e_{i+1}-e_1} z_{i+1}. \tag{e}$$

By (b), we have $(p^{e_1-1}z_1)\alpha^n = p^{e_1-1}z_1$, and hence, using (e) and the direct decomposition (d),

$$p^{e_1-1}p^{e_{n+1}-e_1}z_{n+1} = p^{e_{n+1}-1}z_{n+1} = 0,$$

which contradicts $p^{e_{n+1}}$ to be the order of z_{n+1}. This shows the finiteness of R, and because of (c), $G = D\oplus R$ is artinian.

The proof of Theorem 8 is completed.

LEMMA 9. *Let G be a reduced abelian p-group. Then G is finite or elementary abelian if and only if $A(G)/\mathrm{c}\Gamma$ is finite for every primary normal subgroup Γ of $A(G)$.*

Proof. Clearly, if G is finite its automorphism group has the stated property. So, let us assume that G is infinite and elementary abelian. Let Γ be a normal q-subgroup of $A(G)$, for q a prime. If $q \neq p$, then $\Gamma \subseteq \mathrm{z}A(G)$ according to Lemma 2. If $q = p$, then $\Gamma = 1$, as it is easy to verify (cf. [14]). Hence, $1 = A(G)/\mathrm{c}\Gamma$ is finite for every primary normal subgroup Γ of $A(G)$.

If conversely, $A(G)$ has the property stated above, then in particular $A(G)/\mathrm{c}\Sigma(G/G[p], G[p])$ is finite. Since G is reduced, either $pG = 0$ or $pG \neq p^2G$. In the latter case we apply Lemma 7 and obtain, that $A(G)$ induces a bounded group of automorphisms in $G[p]$. By Theorem 8 then G is artinian and consequently finite.

This completes the proof of the lemma.

Lemma 9 shows that the residual finiteness of $A(G)$ postulated in (5) of Theorem 8 is indispensable, even if we restrict G to be reduced. However, for a reduced group G, residual finiteness in (4) of Theorem 8 can be omitted.

COROLLARY 10. *A reduced abelian p-group G is finite if and only if $A(G)/\mathrm{c}\Gamma$ is finite for every normal torsion subgroup Γ of $A(G)$.*

Proof. In view of Lemma 9 the only thing that remains showing is that an infinite elementary abelian p-group P possesses a torsion group Δ of automorphisms such that Δ is normal in $A(P)$ and $A(P)/\mathrm{c}\Delta$ is infinite. It is easily seen that the set of all $\alpha \in A(P)$ which induce the identity automorphism in a subgroup of P of finite index forms such a group. Lemma 9 therefore implies the corollary.

In order to generalize our results to arbitrary abelian torsion groups we need the following

LEMMA 11. *If all primary components of the abelian torsion group T are artinian, then $A(T)$ is residually finite.*

Proof. According to Lemma 9 in [9] the automorphism group of an artinian p-group is residually finite and the property of residual finiteness is inherited by cartesian products ([9], Lemma 6). Therefore, $A(T) \cong \prod_p^* A(T_p)$ is residually finite.

We are now ready to prove our

MAIN THEOREM. *Every primary component of the abelian torsion group T is artinian if and only if $A(T)$ is residually finite and $A(T)/\mathrm{c}\Gamma$ is finite for every primary normal subgroup Γ of $A(T)$.*

Proof. First, let us assume that all primary components of T are artinian. Lemma 11 then implies that $A(T)$ is residually finite. Let Γ be a normal q-subgroup of $A(T)$ for some prime q. Clearly, for every prime p, Γ induces a group Γ_p of automorphisms in the p-component T_p of T. Since every automorphism of T_p is induced by an automorphism of T it follows, that Γ_p is a normal q-subgroup of $A(T_p)$. Hence, by Theorem 8,

$$A(T_p)/\mathrm{c}_{A(T_p)}\Gamma_p \text{ is finite for all } p. \tag{1}$$

But according to Theorem 6 we have $\Gamma_p \subseteq \mathrm{z}A(T_p)$ for every $p \neq q$ and $p > 3$, and consequently

$$A(T_p)|\mathrm{c}_{A(T_p)}\Gamma_p = 1 \quad \text{for} \quad 3 < p \neq q. \tag{2}$$

Let us, as usual, identify $A(T_p)$ with the group of automorphisms of T fixing $\sum_{r\neq p}^{\circ} T_r$ elementwise. Then $\Gamma_p \subseteq A(T) = \prod_p^* A(T_p)$ and $\Gamma \subseteq \prod^* \Gamma_p$. Clearly, we have

$$\mathrm{c}\Gamma = \mathrm{c}\Big(\prod_p^* \Gamma_p\Big) = \prod_p^* (\mathrm{c}_{A(T_p)}\Gamma_p), \tag{3}$$

and therefore

$$\begin{aligned} A(T)/\mathrm{c}\Big(\prod_p^* \Gamma_p\Big) &= \Big[\prod_p^* A(T_p)\Big]\Big/\Big[\prod_p^* (\mathrm{c}_{A(T_p)}\Gamma_p)\Big] \\ &\cong \prod_p^* [A(T_p)/\mathrm{c}_{A(T_p)}\Gamma_p]. \end{aligned} \tag{4}$$

This, together with (2), implies

$$A(T)/\mathrm{c}\Big(\prod_p^* \Gamma_p\Big) \cong \prod_{p\in\{2,3,q\}}^{\circ} [A(T_p)/\mathrm{c}_{A(T_p)}\Gamma_p], \tag{5}$$

and hence, we obtain from (1) that

$$A(T)/\mathrm{c}\Big(\prod_p{}^{*}\Gamma_p\Big) \text{ is finite}. \tag{6}$$

Using (3), the finiteness of $A(T)/\mathrm{c}\Gamma$ follows.

Let, on the other hand, T be an abelian torsion group whose automorphism group has the properties stated above. Since subgroups of residually finite groups are residually finite (cf. [9], Lemma 1), $A(T_p)$ is residually finite for every prime p. And the finiteness of $A(T)/\mathrm{c}\Gamma$ for every primary normal subgroup Γ of $A(T)$ implies in particular the finiteness of $A(T_p)/\mathrm{c}_{A(T_p)}\Gamma_p$ for every primary normal subgroup Γ_p of $A(T_p)$. Hence, for every prime p, $A(T_p)$ satisfies condition (5) of Theorem 8, and it follows, that all primary components of T are artinian.

The proof of the Main Theorem is completed.

Closely related to this result is the following

THEOREM. *Let T be a reduced abelian torsion group. Then every primary component of T is either finite or elementary abelian if and only if $A(T)/\mathrm{c}\Gamma$ is finite for every primary normal subgroup Γ of $A(G)$.*

Proof. Let first T have the stated property and let Γ be a normal q-subgroup of $A(T)$ for q a prime. Then, as shown in the proof of the Main Theorem,

$$A(T)/\mathrm{c}\Gamma \cong \prod_{p\in\{2,3,q\}}^{\circ} A(T_p)/\mathrm{c}_{A(T_p)}\Gamma_p\,,$$

where Γ_p is the group of automorphisms of T_p induced by Γ. Since T_p is either finite or elementary abelian, Lemma 9 implies the finiteness of $A(T_p)/\mathrm{c}_{A(T_p)}\Gamma_p$ for every p. Hence $A(T)/\mathrm{c}\Gamma$ is finite.

If, conversely, $A(T)$ satisfies the stated condition, then, for every prime p, $A(T_p)/\mathrm{c}_{A(T_p)}\Gamma_p$ is finite for every primary normal subgroup Γ_p of $A(T_p)$. Since T_p is reduced, Lemma 9 implies that T_p is either finite or elementary abelian.

This completes the proof of the theorem.

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