Abelian torsion groups with
artinian primary components and their automorphisms

by
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I. Introduction. Throughout this paper \( T \) denotes an abelian torsion


group and \( A(T) \) its automorphism group.

As in [9], we are interested in theorems of the type:

\[
T \text{ is a } P\text{-group if and only if } A(T) \text{ is a } Q\text{-group ,}
\]

where \( P \) and \( Q \) are group theoretical properties.

In [3], R. Baer has proven that \( T \) is artinian if and only if every torsion group of automorphisms of \( T \) is finite. This nice result holds true even if the finiteness of the torsion subgroups of \( A(T) \) is replaced by the weaker postulate that elementary abelian 2-groups of automorphisms of \( T \) are finite (see [4]). But it is clear that the torsion subgroups of \( A(T) \) become arbitrarily complicated if infinitely many primary components of \( T \) are nontrivial. So R. Baer raised the question, what group theoretical property \( Q \) imposed on \( A(T) \) is necessary and sufficient in order that every primary component of \( T \) is artinian.

An answer to this question shall be given in the present paper. We will prove the following result.

**Main Theorem.** Every primary component of \( T \) is artinian if and only if \( A(T) \) is residually finite and, for every primary normal subgroup \( \Gamma \) of \( A(T) \), the centraliser \( c\Gamma \) of \( \Gamma \) in \( A(T) \) has finite index in \( A(T) \).

The investigations in section III of the automorphism group of abelian \( p \)-groups might be of independent interest. We shall show that for every abelian \( p \)-group \( G \), \( p > 3 \), every normal \( p' \)-subgroup of its automorphism group \( A(G) \) is contained in the center of \( A(G) \).

II. Preliminaries. Our notation and terminology concerning abelian group follows [5].

A group is called **artinian** if its subgroups satisfy the minimum condition.

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If \( G \) is an abelian \( p \)-group for some prime \( p \), the maximal divisible subgroup of \( G \) shall be denoted by \( dG \). The rank \( \text{rk}(G) \) of \( G \) is the dimension of \( dG/pdG \) as a vectorspace over the primefield of characteristic \( p \). By a well-known theorem, \( G \) is artinian if and only if \( \text{rk}(G) \) is finite (cf. [5], p. 65, Theorem 19.2).

If \( X \) is a normal subgroup of a group \( X \), we write \( X \triangleleft X \). The group of all automorphisms of \( X \) shall be denoted by \( \text{Aut}(X) \).

If \( X \) is abelian and \( S \) some subgroup of \( X \), the set of all automorphisms of \( X \) which fix \( S \) elementwise and induce the \( 1 \)-automorphism in \( X/S \) is a subgroup of \( \text{Aut}(X) \), called the stabilizer of \( S \) in \( X \), and shall be denoted by \( \Sigma(X/S, S) \). By a well-known theorem of Kalojbin, \( \Sigma(X/S, S) \) is abelian (cf. [13], p. 88, Satz 19), and furthermore \( \Sigma(X/S, S) \cong \text{Hom}(X/S, S) \) (cf. [5], p. 153, Hilfsätze 1.4).

\( T_\gamma \) denotes the \( p \)-component of the abelian torsion group \( T \). A torsion group without elements of order \( p \) is called a \( p \)-group. \( A \oplus B \) denotes the direct sum of the abelian groups \( A \) and \( B \), \( \Sigma, \Pi \) and \( L \) are our symbols for a direct sum, a direct product, and a cartesian or unrestricted direct product. If \( G \) is a \( \Gamma \)-admissible factor group of \( X \), for \( \Gamma \subset \text{Aut}(X) \), then \( \Gamma \subset \text{Aut}(X) \) denotes the group of automorphisms of \( G \) induced by \( \Gamma \).

If \( \Delta \subset \text{Aut}(X) \), we write \( \Delta = \text{Aut}(X) \) for the centralizer of \( \Delta \subset \text{Aut}(X) \). The centralizer of \( \Delta \subset \text{Aut}(X) \) is simply denoted by \( \Delta \) and the center of \( G \) by \( Z_G \). \( \Delta \) is the subgroup generated by the set of all commutators \( \gamma \cdot \delta \cdot \gamma \cdot \delta^{-1} \cdot \delta \cdot \gamma^{-1} \) where \( \gamma, \delta \in \Delta \).

A group is called residually finite if the intersection of all subgroups of finite index is trivial (see [7], p. 16).

III. Central automorphisms. In this section we are concerned with groups of automorphisms of an abelian \( p \)-group \( G \).

The proof of our main theorem will vitally depend on the fact that, if \( p > 3 \), every normal \( p \)-subgroup of \( A(G) \) is contained in the center of \( A(G) \).

In order to establish this result we need several lemmas.

LEMMA 1. Let \( G \) be an elementary abelian \( p \)-group of finite rank \( n \). If either \( n \neq 2 \) or \( n = 2 \) and \( p > 3 \), then every normal \( p \)-subgroup of \( A(G) \) is contained in the center of \( A(G) \).

Proof. If \( G \) is cyclic, then \( A(G) \) is abelian and our proposition holds. Therefore, we may assume that \( \text{rk}(G) = n > 2 \). Since every elementary abelian \( p \)-group is a vectorspace over the primefield \( K \) of characteristic \( p \), \( A(G) \) is isomorphic to the group \( GL(n, p) \) of all invertible matrices of size \( n \times n \) over \( K \). It is wellknown that, for \( n > 2 \) or \( n = 2 \) and \( p > 3 \), every normal subgroup of \( GL(n, p) \) either contains the subgroup \( SL(n, p) \) of all matrices with determinant \( 1 \) or it is contained in the center of \( GL(n, p) \) (cf. [1], p. 165, Theorem 4.9). But, for \( n > 2 \), \( SL(n, p) \) contains elements of order \( p \) (see [1], p. 170, Theorem 4.11). Therefore, if \( n > 3 \) or \( n = 2 \) and \( p > 3 \), every normal \( p \)-subgroup of \( GL(n, p) \) is contained in the center, and the analogue statement holds for \( A(G) \). This proves the lemma.

LEMMA 2. If \( G \) is an infinite elementary abelian \( p \)-group, then every normal \( p \)-subgroup of \( A(G) \) is contained in the center of \( A(G) \).

Proof. Let \( \Gamma \) be a normal \( p \)-subgroup of \( A(G) \) and \( \gamma \in \Gamma \). Since the order of \( \gamma \) is finite, any two elements \( x_1 \) and \( x_2 \) of \( G \) can be imbedded into a finite \( \gamma \)-admissible subgroup \( F \subset G \) and we can choose \( F \) to have a rank \( \neq 2 \). Let \( \Theta \) be the set of all automorphisms of \( G \) mapping \( F \) onto itself. Then \( \gamma \in \Theta \) and, since \( F \) is a direct summand, \( \Theta \) induces in \( F \) its full group \( A(F) \) of automorphisms. Therefore, the group of automorphisms of \( F \) induced by \( \Gamma \cap \Theta \subset \Theta \) is a normal \( p \)-subgroup of \( A(F) \) which, accordingly to Lemma 1, is contained in \( xA(F) \). Hence, \( \gamma \in \Theta \) induces in \( F \) a central automorphism. But the center of \( A(F) \) consists just of the multiplications with \( p \)-adic units (see [2], p. 111, Theorem 1), so \( x \gamma \cdot x^{-1} = x \gamma \cdot x^{-1} = x \gamma \cdot x^{-1} = x \gamma \cdot x^{-1} \) for a suitable integer \( k \) prime to \( p \). This being true for each pair of elements implies \( x \gamma \cdot x^{-1} = x \gamma \cdot x^{-1} \) for all \( x \in \Gamma \). Hence \( \gamma \in xA(F) \) and \( \Gamma \subset xA(G) \) as stated above.

A consequence of these two results is the following

COROLLARY 3. Let \( G \) be an elementary abelian \( p \)-group, \( p > 3 \). Then every normal \( p \)-subgroup of \( A(G) \) is contained in the center of \( A(G) \).

LEMMA 4. For \( n > 0 \) an integer, every automorphism of

\[
F^n = (p^n G)[p] \cong (p^n G)[p]
\]

is induced by some automorphism of \( G \).

Proof. Since every automorphism of \( p^n G \) is induced by an automorphism of \( G \) (see [6], p. 123, Lemma 1), it suffices to prove the proposition for \( n = 0 \).

Let \( H \) be a maximal elementary abelian direct summand of \( G \) (cf. [3], p. 99). Then, for some subgroup \( H \) of \( G \), we have

\[
G = B \oplus H,
\]

where \( (p^n G)[p] = H[p] \), and the mapping \( a : B \to F^n \) defined by

\[
nb = nb + H[p] \quad \text{for} \quad nb + H[p] \;
\]

is an isomorphism from \( B \) onto \( F^n = G[p]((p^n G)[p]) \).

Let \( \phi \) be an automorphism of \( F^n \). Then \( G \) possesses an automorphism \( \phi \) satisfying

\[
ba = b \phi(b) \quad \text{for} \quad b \in B,
\]

\[
h = h + H \quad \text{for} \quad h \in H.
\]
If $\bar{a}$ denotes the automorphism of $F^p$ induced by $a$, then

$$(b[H(p)]\bar{a} = ba + H(p)] = bab = b[a + H(p)] \Phi$$

for every $b \in B$. Hence, $\bar{a} = \Phi$ and the lemma is proven.

**Lemma 5.** Let $G$ be a reduced abelian $p$-group for $p > 3$ and $A$ a normal $p'$-subgroup of $A(G)$. Then $\Gamma \subset \pm A(G)$.

**Proof.** Let $A_n$ denote the set of all automorphisms of $G$ inducing the automorphism of $F^n = (p^n G)(p^n H)(G)(p^n H)$ in $F^n = (p^n G)(p^n H)(G)(p^n H)$, then $A_n$ is a normal subgroup of $A(G)$ and, by Lemma 4, we have

$$(1) \quad A(F^p) \cong A(G)/A_n.$$ 

For every $n$, $A_n$ induces a $p'$-group $A_n \cong A_n/A_n$ of automorphisms in $F^n$, which, according to (1), is a normal subgroup of $A(F^n)$. Hence, by Corollary 3, $A_n$ is contained in $A(F^n)$; it follows that $A_n \cdot A(F^n) = 1$ and

$$(2) \quad \Gamma \cdot A(G) \subset \Gamma \cap A_n \quad \text{for} \quad n = 0, 1, \ldots,$$

using (1) and the normality of $\Gamma$. Consequently,

$$(3) \quad \Gamma \cdot A(G) \subset \Gamma \cap \bigcap_{n > 0} A_n.$$ 

But according to (11), p. 101, the order of every torsion automorphism in $\bigcap_{n > 0} A_n$ is a power of $p$, which implies $\Gamma \cap \bigcap_{n > 0} A_n = 1$. Using (3) we obtain

$$\Gamma \cdot A(G) = 1, \quad \text{i.e.} \quad \Gamma \subset A(G).$$

**Theorem 6.** Let $G$ be an abelian $p$-group for $p > 3$ and $A$ a normal $p'$-subgroup of $A(G)$. Then $\Gamma$ is contained in the center of $A(G)$.

**Proof.** Let $G = D \oplus R$ where $D$ is divisible and $R$ is reduced. Using the injectivity of divisible groups (cf. [5], p. 59, Theorem 16.1) one verifies easily that every automorphism of $D$ is induced by an automorphism of $D$ and therefore by an automorphism of $G$. Hence, $\Gamma$ induces in $D$ a $p'$-group $\theta$ of automorphisms which is normal in $A(D)$. By Corollary 3 we have

$$(1) \quad \Gamma \cdot A(D) \subset \Gamma$$

and consequently,

$$(2) \quad [\Gamma \cdot A(D)]_{\Gamma_D = 1}.$$ 

In particular, $\Gamma \cdot A(G) \subset \Gamma$ is a $p'$-group. A closer examination of the proof of a lemma by R. Baer ([3], p. 526) shows, that every $p'$-group of automorphisms of a divisible $p$-group $P$ operates faithfully on $P$. Hence, (1) implies

$$(3) \quad [\Gamma \cdot A(G)]_{\Gamma_D = 1}.$$ 

Likewise, every automorphism of $G/D \cong R$ is induced by an automorphism of $G$, and $\Gamma|_{\Gamma_D = 1}$ is essentially a normal $p'$-subgroup of $A(E)$. Lemma 5 then implies that $\Gamma|_{\Gamma_D = 1} \subset \pm A(G(D))$ and hence,

$$(4) \quad [\Gamma \cdot A(G)]_{\Gamma_D = 1} = 1.$$ 

Comparing (2) and (3) we obtain

$$(5) \quad \Gamma \cdot A(G) \subset \Sigma(G(D), D),$$

where $\Sigma(G(D), D)$ denotes the stabilizer of $D$ in $G$. But $\Sigma(G(D), D) \cong \text{Hom}(G(D), D)$ (cf. [8], p. 153, Hilfsätze 1.4), and the order of every torsion element of $\Sigma(G(D), D)$ consequently is a power of $p$. Since $\Gamma \subset \Gamma \cdot A(G)$ is a $p'$-group, (4) implies $\Gamma \cdot A(G) = 1$ or $\Gamma \subset \pm A(G)$ as stated in the theorem.

The stabilizer $\Sigma(G(D), G(D))$ of $G(D)$ in $G$ is an elementary abelian normal $p'$-subgroup of $A(G)$. We need the following

**Lemma 7.** Let $\theta$ denote the group of automorphisms of $G(p)$ that is induced by the centralizer of $\Gamma(G(p), G(p))$ in $A(G)$. If $\theta \neq \pm G$, then $\theta = \pm A(G(p))$; in particular, $\theta$ is cyclic of order $p - 1$.

**Proof.** For the sake of shortness, let $\Sigma = \Sigma(G(D), G(p))$. We claim that

$$(x) \quad x \not\in \theta \quad \text{for every} \quad x \in G(D) \quad \text{and} \quad x \not\in \Sigma.$$ 

In order to prove this statement, let $x \in G(D)$, $x \not\in G(D)$. Then $G$ has a decomposition $G = D \oplus R$, where $x \in R$ (cf. [5], p. 63) and $R \neq 0$. By our hypothesis $G(p)[p] \cong \Sigma(G(D), G(p))$ is not divisible, and consequently $G$ contains a subgroup $U \supset G(p)$ such that $G/U$ is cyclic of order $p$ (cf. [5], p. 67, Exercise 2). Hence $G - \{g\} + U$ for $g \in G$, $g \not\in U$, and $pU \subset U$. It is easy to see that there exists an automorphism $\sigma$ of $G$ satisfying

$$g = g + x$$

for all $u \in U$.

Since $x \in G(p)[p] \subset U$ it follows that $\sigma \in \Sigma$. Let $\xi \in \Sigma$ and $g \xi = k\xi + u$ for some integer $k$ and $u \in U$. Since $\xi$ commutes with $e$ we obtain

$$0 = g(\xi - u) = k\xi + u - k\xi - u\xi$$

and consequently

$$(y) \quad x \not\in \theta \quad \text{for every} \quad x \in G(D).$$ 

If $d \in (dG)[p]$, then, using our hypothesis $dG \not\subset G$, there exists $x \in G(p)$, $x \not\in G(D)$, and we have $d = d - x + x$ where $(d - x) \xi \in \theta$ and $x \xi \not\in \theta$ according to (1). A simple computation shows that $d\xi \not\in \theta$ and hence $x \xi \not\in \theta$ for all $x \in G(p)$ and every $\xi \in \Sigma$. This proves (y).
As an immediate consequence of (4) we state

\tag{**} \quad \text{If } \xi \in \Sigma, \text{ then } S_\xi = S \text{ for every } S \subseteq \sigma(G) \text{.}

By a theorem of R. Baer an automorphism $\xi$ of an abelian $p$-group $A$ induces the 1-automorphism in the lattice of all subgroups of $A$ if and only if $\xi$ is the multiplication with an invertible $p$-adic integer ([12], p. 110, Theorem 5.2). If $A$ is elementary abelian (or $p \neq 2$), then the central automorphisms of $A$ are precisely the multiplications with $p$-adic units ([12], p. 111, Theorem). Hence, (**) implies

$$c \Sigma_{\text{cyclic}} = \theta = \text{Aut}(G[p])$$

as stated in the proposition. It is well-known that the multiplications with $p$-adic integers operate as a cyclic group of order $p - 1$ of automorphisms on an elementary abelian $p$-group.

Lemma 7 is proven.

IV. The main results. We are now ready to give a new characterization of artinian $p$-groups by means of their automorphism groups.

A group is called bounded if there exists a finite upper bound for the orders of its elements.

**Theorem 8.** The following properties of the abelian $p$-group $G$ are equivalent:

1. $G$ is artinian.
2. Every torsion subgroup $A(G)$ is finite.
3. $A(G)$ is residually finite and every normal torsion subgroup of $A(G)$ is finite.
4. $A(G)$ is residually finite and $A(G)/\alpha \Gamma$ is finite for every normal torsion subgroup $\Gamma$ of $A(G)$.
5. $A(G)$ is residually finite and $A(G)/\alpha \Gamma$ is finite for every primary normal subgroup $\Gamma$ of $A(G)$.
6. $A(G)$ is residually finite and $A(G)/\alpha \Sigma(G)/G[p], G[p]$ is finite.
7. $A(G)$ induces in $G[p]$ a bounded group of automorphisms.

Remark. It is easy to see that the finiteness postulated in (2) and (3) can be replaced by countability (see [12], Main Theorem). In (4), (5), and (6) the finiteness of the occurring factorgroups can be weakened to the condition that they are bounded.

Remark. The residual finiteness of $A(G)$ postulated in (3)–(6) is indispensable. This is due to the fact that for a divisible $p$-group $D$ of arbitrary rank every normal torsion subgroup of $A(D)$ is contained in the center of $A(D)$ (see [10]). Even if we restrict $G$ to be reduced, the postulate of residual finiteness in Theorem 8 cannot be omitted completely (cf. Lemma 9 and Corollary 10).

**Proof of Theorem 8.** The equivalence of (1) and (2) was proven by R. Baer ([3], p. 521, Theorem). It is easy to verify that the automorphism group of an artinian abelian group is residually finite (see [9], Lemma 9). Hence, (2) implies (3).

Let us assume the validity of (3). Since $A(G)/\alpha \Gamma$ is isomorphic to a group of automorphisms of $\Gamma$, it follows that $A(G)/\alpha \Gamma$ is finite, and we have derived (4) from (3).

It is obvious that (4) implies (5) and, recalling $\Sigma(G)/G[p], G[p] \cong \text{Hom}(G, G[p])$ (cf. [8], p. 153, Hilfssatz 1.4), that (5) in turn implies (6).

So, let us assume the validity of (6). Then there exists a natural number $n$ such that

\[ a^* \in \Sigma(G)/G[p], G[p] \quad \text{for all} \quad a \in A(G) \, . \]

Let us distinguish two cases.

Case 1. $pG = pG$. Then $G = dG$, where $dG = 0$ and both, $dG$ and $R$, are direct sums of pairwise isomorphic groups of rank 1. If $\text{rk}(dG)$ or $\text{rk}(R)$ were infinite, the group of all permutations on a countably infinite set would be isomorphic to a subgroup of $A(G)$, contradicting the residual finiteness of $A(G)$ (see [9], Lemma 1 and Main Theorem). Hence, both $dG[p]$ and $R[p]$ are finite and so is $G[p]$. This implies that $G[p]$ is a bounded group of automorphisms. We have derived (7) from (6).

Case 2. $pG \neq pG$. Then, by Lemma 7, the centralizer of $\Sigma(G)/G[p], G[p]$ induces in $G[p]$ a cyclic group of automorphisms of order $p - 1$. Hence, by (a), $a^p = a^{p - 1}$ induces the identical automorphism in $G[p]$ for every $a \in A(G)$, i.e. $A(G)$ induces in $G[p]$ a bounded group of automorphisms. We have derived (7) from (6).

It remains to show that (7) implies (1). So, let us finally assume the existence of an integer $n \geq 1$ such that

\[ a^{p^n} = 1 \quad \text{for every} \quad a \in A(G) \, . \]

Let $G = D \oplus R$ where $D$ is divisible and $R$ is reduced. Clearly, every automorphism of $D[p]$ is induced by some automorphism of $G$, and the automorphism group of an infinite elementary abelian group is not bounded. Hence, $D[p]$ is finite and $D$ is artinian.

It remains to show that $R$ is finite.

Let us assume, by way of contradiction, that $R$ is infinite. Then there exist elements $a_i \in R$ of order $p^i \neq 1$ such that

\[ R = \bigoplus_{i=1}^{\infty} \{a_i\} \oplus H, \quad a_1 \leq a_2 \leq \ldots \leq a_{i+1} \, . \]
It is easy to check that $G$ possesses an automorphism $a$ such that
\begin{align}
  z_i a = z_i + p^{e_{z_i}} - t_{z_i+1}, \quad \text{for} \quad 1 \leq i \leq n,
  \\
  \sigma_i \in \sigma_i \ominus \sigma_i \ominus \sigma_i \ominus \sigma_i.
\end{align}
and by complete induction one verifies that
\begin{equation}
  z_i a^p = z_i + \sum_{i=1}^{p} (p^{e_{z_i}} - t_{z_i+1}).
\end{equation}
By (b), we have $(p^{e_{z_i}} - t_{z_i}) a^p = p^{e_{z_i}} - t_{z_i}$, and hence, using (e) and the direct decomposition (d),
\begin{equation}
  p^{e_{z_i}} - t_{z_i+1} = p^{e_{z_i+1}} - t_{z_i+1} = 0,
\end{equation}
which contradicts $p^{e_{z_i}}$ to be the order of $z_{i+1}$. This shows the finiteness of $K$, and because of (c), $G = D \oplus R$ is artinian.

The proof of Theorem 8 is completed.

**Lemma 9.** Let $G$ be a reduced abelian $p$-group. Then $G$ is finite or elementary abelian if and only if $A(\sigma)/c_\sigma$ is finite for every primary normal subgroup $\Gamma$ of $A(\sigma)$.

**Proof.** Clearly, if $G$ is finite its automorphism group has the stated property. So, let us assume that $G$ is infinite and elementary abelian. Let $\Gamma$ be a normal $p$-subgroup of $A(\sigma)$, for $p$ a prime. If $p = p$, then $\Gamma \subseteq A(\sigma)$ according to Lemma 2. If $p = p$, then $\Gamma = 1$, as it is easy to verify (cf. [14]). Hence, $1 = A(\sigma)/c_\sigma$ is finite for every primary normal subgroup $\Gamma$ of $A(\sigma)$.

If conversely, $A(\sigma)$ has the property stated above, then in particular $A(\sigma)/c_\sigma A(\sigma)/c_\sigma$ is finite. Since $G$ is reduced, either $pG$ or $pG$ is finite. In the latter case we apply Lemma 7 and obtain, that $A(\sigma)$ induces a bounded group of automorphisms in $G[p]$. By Theorem 8 then $G$ is artinian and consequently finite.

This completes the proof of the lemma.

**Lemma 10.** A reduced abelian $p$-group $G$ is finite if and only if $A(\sigma)/c_\sigma$ is finite for every primary normal torsion subgroup $\Gamma$ of $A(\sigma)$.

**Proof.** In view of Lemma 9 the only thing that remains showing is that an infinite elementary abelian $p$-group $F$ possesses a torsion group $A$ of automorphisms such that $A$ is normal in $A(F)$ and $A(F)/c_\sigma A(F)$ is infinite. It is easily seen that the set of all $a A(F)$ which induce the identity automorphism in a subgroup of $F$ of finite index forms such a group. Lemma 9 therefore implies the corollary.

\section*{Abelian torsion groups}

In order to generalize our results to arbitrary abelian torsion groups we need the following

**Lemma 11.** If all primary components of the abelian torsion group $T$ are artinian, then $A(T)$ is residually finite.

**Proof.** According to Lemma 9 in [9] the automorphism group of an artinian $p$-group is residually finite and the property of residual finiteness is inherited by cartesian products ([9], Lemma 6). Therefore, $A(T)$ is residually finite.

We are now ready to prove our

**Main Theorem.** Every primary component of the abelian torsion group $T$ is artinian if and only if $A(T)$ is residually finite and $A(T)/c_\sigma$ is finite for every primary normal subgroup $\Gamma$ of $A(T)$.

**Proof.** First, let us assume that all primary components of $T$ are artinian. Lemma 11 then implies that $A(T)$ is residually finite. Let $\Gamma$ be a normal $p$-subgroup of $A(T)$ for some prime $p$. Clearly, for every prime $p$, $\Gamma$ induces a group $\Gamma_p$ of automorphisms in the $p$-component $T_p$ of $T$. Since every automorphism of $T_p$ is induced by an automorphism of $T$ it follows, that $\Gamma_p$ is a normal $p$-subgroup of $A(T_p)$. Hence, by Theorem 8,

\begin{equation}
  A(T_p)/c_\sigma A(T_p) \Gamma_p \text{ is finite for all } p.
\end{equation}

But according to Theorem 6 we have $\Gamma_p \subseteq \sigma A(T_p)$ for every $p \neq q$ and $p > 3$, and consequently

\begin{equation}
  A(T_p)/c_\sigma A(T_p) \Gamma_p = 1 \quad \text{for} \quad 3 < p \neq q.
\end{equation}

Let us, as usual, identify $A(T_p)$ with the group of automorphisms of $T$ fixing $\sum_{p'} T_{p'}$ elementwise. Then $\Gamma_p \subseteq A(T) = \prod_p A(T_p)$ and $\Gamma_p \subseteq \prod_p \Gamma_p$. Clearly, we have

\begin{equation}
  c\Gamma = c\left[ \prod_p \Gamma_p \right] = \prod_p (c_\sigma A(T_p) \Gamma_p),
\end{equation}

and therefore

\begin{equation}
  A(T)/c_\sigma A(T) \Gamma_p = \left[ \prod_p A(T_p) \right]/\left[ \prod_p (c_\sigma A(T_p) \Gamma_p) \right],
\end{equation}

\begin{equation}
  \cong \prod_p [A(T_p)/c_\sigma A(T_p) \Gamma_p].
\end{equation}

This, together with (2), implies

\begin{equation}
  A(T)/c_\sigma A(T) \Gamma_p \cong \prod_p [A(T_p)/c_\sigma A(T_p) \Gamma_p],
\end{equation}

\begin{equation}
  \cong \prod_p [A(T_p)/c_\sigma A(T_p) \Gamma_p].
\end{equation}
and hence, we obtain from (1) that
\[ A(T)/\text{c}[^1 \Gamma] \text{ is finite} . \]

Using (3), the finiteness of \( A(T)/\text{c}[^1 \Gamma] \) follows.

Let, on the other hand, \( T \) be an abelian torsion group whose automorphism group has the properties stated above. Since subgroups of residually finite groups are residually finite (cf. [9], Lemma 1), \( A(T_p) \) is residually finite for every prime \( p \). And the finiteness of \( A(T)/\text{c}[^1 \Gamma] \) for every primary normal subgroup \( \Gamma \) of \( A(T) \) implies in particular the finiteness of \( A(T_p)/\text{c}_{A(T_p)} \Gamma_p \) for every primary normal subgroup \( \Gamma_p \) of \( A(T_p) \). Hence, for every prime \( p \), \( A(T_p) \) satisfies condition (5) of Theorem 8, and it follows, that all primary components of \( T \) are artinian.

The proof of the Main Theorem is completed.

Closely related to this result is the following

**Theorem.** Let \( T \) be a reduced abelian torsion group. Then every primary component of \( T \) is either finite or elementary abelian if and only if \( A(T)/\text{c}[^1 \Gamma] \) is finite for every primary normal subgroup \( \Gamma \) of \( A(T) \).

Proof. Let first \( T \) have the stated property and let \( \Gamma \) be a normal \( q \)-subgroup of \( A(T) \) for \( q \) a prime. Then, as shown in the proof of the Main Theorem,
\[ A(T)/\text{c}[^1 \Gamma] = \prod_{p \in \Lambda} A(T_p)/\text{c}_{A(T_p)} \Gamma_p , \]
where \( \Gamma_p \) is the group of automorphisms of \( T_p \) induced by \( \Gamma \). Since \( T_p \) is either finite or elementary abelian, Lemma 9 implies the finiteness of \( A(T_p)/\text{c}_{A(T_p)} \Gamma_p \) for every \( p \). Hence \( A(T)/\text{c}[^1 \Gamma] \) is finite.

If, conversely, \( A(T) \) satisfies the stated condition, then, for every prime \( p \), \( A(T_p)/\text{c}_{A(T_p)} \Gamma_p \) is finite for every primary normal subgroup \( \Gamma_p \) of \( A(T_p) \). Since \( T_p \) is reduced, Lemma 9 implies that \( \Gamma_p \) is either finite or elementary abelian.

This completes the proof of the theorem.

**References**


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