σ-ideals and related Baire systems

by

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Suppose S is a metric space with metric d, R is a proper σ-ideal of subsets of S and G is the collection of all real functions defined on S which are continuous almost everywhere with respect to R. Let $B_0(G)$ be G and for each ordinal number α, $0 < \alpha < \Omega$, let $B_\alpha(G)$ be the collection of all pointwise limits or sequences taken from the collection $\sum_{\gamma<\alpha} B_\alpha(G)$.

In this paper, the collections $B_\alpha(G)$, the analytic representable functions or Baire functions of class α generated by G, are characterized in terms of an associated collection of Baire type sets (Theorem 1). These Baire type sets are characterized by a relation to the classical Baire sets (Theorems 2a, b, and c). In Theorem 3, the collections $B_\alpha(G)$, $\alpha > 0$ are characterized by a relation to Baire's class α. Finally, in case the space S is separable, a theorem of T. Traczyk is used to give another characterization of the collections $B_\alpha(G)$, $\alpha > 0$, (Theorem 4).

Notation. The collection of all sets of the form $(a < f < b)$, where (a, b) is a number segment and f is in G, is denoted by D. If L is a collection of subsets of S, then $W_0(L)$ denotes L and for each ordinal number α, $0 < \alpha < \Omega$, $W_\alpha(L)$ denotes the collection to which X belongs if and only if $X = \sum_{n=1}^{\infty} \big(\prod_{p=1}^{\infty} X'_{np}\big)$ where, for each np, there is some $\xi_{np} < \alpha$ such that X_{np} is in $W_{\xi_{np}}(L)$ and X'_{np} is the complement of X_{np}.

THEOREM 1. *Suppose α is an ordinal number, $0 \leqslant \alpha < \Omega$. A real function f on S is in $B_\alpha(G)$ if and only if for each number segment (a, b), the set $(a < f < b)$ is in $W_\alpha(D)$.*

Indication. It is true that G is a linear lattice of real functions on S containing the constant real functions on S. Also, if f is in G and U is a continuous real function on the range of f, then $U[f]$ is in G. Also, it is true that $G = \mathrm{US}G \cdot \mathrm{LS}G$, where $\mathrm{US}G$ ($\mathrm{LS}G$) is the collection of all limits of nonincreasing (nondecreasing) sequences from G. Using these facts, Theorem 1 for the case $\alpha = 0$ follows from Theorem 11 of [5] and the cases $0 < \alpha < \Omega$ follow from Theorem 9 of [5].

As was pointed out in [5], the collection G is a complete ordinary function system as defined by F. Hausdorff [1, Chapter 9] and we have the following relationships between the method presented here and the method of F. Hausdorff. The functions in $B_\xi(G)$ are the functions f^ξ, if $0 \leqslant \xi < \omega$ and are the function $f^{\xi+1}$, if $\omega \leqslant \xi < \Omega$. Also, the sets in $W_\xi(D)$ are the sets M^ξ, if $0 \leqslant \xi < \omega$ and the sets $M^{\xi+1}$, if $\omega \leqslant \xi < \Omega$.

In case G is C, the continuous functions on S, then $B_\alpha(C)$ is Φ_α, the Baire functions or analytic representable functions of class α as described by K. Kuratowski in [2, p. 392] and the collection D is G_0, the collection of all open sets. For each α, $0 \leqslant \alpha < \Omega$, let $B_\alpha = B_\alpha(C)$ and let $W_\alpha = W_\alpha(G_0)$. It can be shown by transfinite induction that we have the following relationship between the collections W_α, the analytic representable sets of class α and the Borel sets of class α, as defined in [2, p. 345]:

$$W_\alpha = \begin{cases} G_\alpha, & \alpha \text{ is even and finite}, \\ F_\alpha, & \alpha \text{ is odd and finite}, \\ F_{\alpha+1}, & \alpha \text{ is even and infinite}, \\ G_{\alpha+1}, & \alpha \text{ is odd and infinite}. \end{cases}$$

Theorem 2 characterizes the collections $W_\alpha(D)$, in the general case, in terms of the collections W_α.

THEOREM 2a. *A subset X of S is in the collection $W_0(D)$ if and only if X is a subset of an F_σ set in the σ-ideal R, X is in W_0, or X is the sum of a set in W_0 and a subset of an F_σ set in R.*

Proof. Suppose X is in $D = W_0(D)$. Let f be a function in $G = B_0(G)$ and (a, b) a segment such that $(a < f < b)$ is X. For each n, let H_n be the set of all points p such that the discontinuity of f at p is $\geqslant 1/n$; H_n is a closed set in the σ-ideal R.

Suppose f is continuous at some point of $(a < f < b)$. For each point p of continuity of f in $(a < f < b)$, let S_p be an open set containing p such that S_p is a subset of $(a < f < b)$. Let K be the sum of all the S_p's. The set K is an open set and is a subset of $(a < f < b)$. The set $X = (a < f < b)$ is K or $X = (a < f < b) = K + (a < f < b) \cdot \sum_{n=1}^{\infty} H_n$. So, if f is continuous at some point of $(a < f < b)$, then X is an open set or X is the sum of an open set and a subset of an F_σ in the σ-ideal R.

If f is not continuous at any point of $(a < f < b)$, then $X = (a < f < b) \cdot \sum_{n=1}^{\infty} H_n$ and X is a subset of an F_σ set in R.

Now, suppose that X is an open set. Let f be the function defined as follows:

$$f(p) = \begin{cases} 1, & \text{if} \quad d(p, S-X) \geqslant 1, \\ d(p, S-X), & \text{if} \quad d(p, S-X) < 1, \end{cases}$$

where $d(p, S-X)$ means the distance from the point p to the set $S-X$. The function f is continuous on S and the set X is $(0 < f < 2)$ and so X is in $W_0(D) = D$.

Suppose $X = K + H$, where K is an open set and H is a subset of $\sum_{n=1}^{\infty} H_n$, where each H_n is a closed set in the σ-ideal R. For each p, let $M_p = (S-K) \cdot H_p$; M_p is closed and in R and $X = K + (X-K) \cdot H = K + \sum_{p=1}^{\infty} M_p \cdot [(X-K) \cdot H]$. Let f be the function on S, defined as follows:

$$f(p) = \begin{cases} 1, & \text{if } p \text{ is in } K \text{ and } d(p, S-K) \geqslant 1, \\ d(p, S-K), & \text{if } p \text{ is in } K \text{ and } d(p, S-K) < 1, \\ 1/n, & \text{if } p \text{ is in } (X-K) \cdot H \text{ and } M_n \text{ is the first term of the sequence } M_1, M_2, M_3, \ldots \text{ which contains } p, \\ -1/n, & \text{if } p \text{ is in } \left(\sum_{p=1}^{\infty} M_p\right) - (X-K) \cdot H \text{ and } M_n \text{ is the first term of the sequence } M_1, M_2, M_3, \ldots \text{ which contains } p, \\ 0, & \text{if } p \text{ is in } S - \left(K + \sum_{p=1}^{\infty} M_p\right). \end{cases}$$

The function f is continuous at each point of $S - \sum_{p=1}^{\infty} M_p$, f is in $B_0(G) = G$ and $(0 < f < 2)$ is X. The set X is in $D = W_0(D)$. There is a similar argument to show that, if X is a subset of an F_σ set in R, then X is in $W_0(D)$.

This completes Theorem 2a.

THEOREM 2b. *A subset X of S is in the collection $W_1(D)$ if and only if there is an F_σ set, K in R, a set A in W_1 and a subset B of K such that $X = A \cdot K' + B$.*

Proof. Suppose X is in $W_1(D)$. Then $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} X'_{np}$, where for each n, p, X_{np} is in $D = W_0(D)$. Then $X = \sum_{n=1}^{\infty} \left(\sum_{p=1}^{\infty} X_{np}\right)'$. But, since the collection $W_0(D)$ is countably additive, $X = \sum_{n=1}^{\infty} X'_n$, where for each n, X_n is in $W_0(D)$. For each n, $X_n = A_n + B_n$, where A_n is an open set and B_n is a subset of an F_σ set in the σ-ideal R. $X = \sum_{n=1}^{\infty} X'_n = \left(\prod_{n=1}^{\infty} (A_n + B_n)\right)'$.

So, $X = \left(\left(\prod_{n=1}^{\infty} A_n\right) + C\right)'$, where C is a subset of an F_σ set, K, in R.

So, $X = (\prod_{n=1}^{\infty} A_n)' \cdot C' = (\prod_{n=1}^{\infty} A_n)' \cdot (K' + (K - C))$. Letting $A = (\prod_{n=1}^{\infty} A_n)'$ and $B = A \cdot (K - C)$ we have, $X = A \cdot K' + B$, where A is in W_1, since A is an F_σ set and B is a subset of K, an F_σ set in the σ-ideal R.

Now, suppose $X = A \cdot K' + B$, where A is in W_1, K is an F_σ set in the σ-ideal R and B is a subset of K. Since W_1 is an additive class, $E = A + K$ is in W_1 and $E' = \prod_{n=1}^{\infty} A_n$, where for each n, A_n is open. So, $X = A \cdot K' + B = E \cdot K' + E \cdot B = E \cdot (K' + B)$. Let $C = K \cdot B'$. Then $X = E \cdot C' = (E')' \cdot C' = (E' + C)'$. $X = ((\prod_{n=1}^{\infty} A_n) + C)' = (\prod_{n=1}^{\infty} (A_n + C))' = \sum_{n=1}^{\infty} (A_n + C)'$. It follows from Theorem 2a, that for each n, $A_n + C$ is in $W_0(D)$ and it follows from the definition of $W_1(D)$ that X is in $W_1(D)$. This completes Theorem 2b.

THEOREM 2c. *Suppose $1 < \alpha < \Omega$. A subset X of S is in $W_\alpha(D)$ if and only if $X = A + B$, where A is in W_α and B is a subset of an F_σ set in the σ-ideal R.*

Proof for $\alpha = 2$. Suppose X is in $W_2(D)$. Then

(1) $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} X'_{np}$, where for each n, p, X_{np} is in $W_1(D)$. Noting Theorem 2b, for each np, let $X_{np} = A_{np} \cdot K'_{np} + B_{np}$, where A_{np} is in W_1 and B_{np} is a subset of K_{np}, an F_σ set in R. For each n, p:

$$X'_{np} = (A_{np} \cdot K'_{np} + B_{np})' = (A'_{np} + K_{np}) \cdot B'_{np}$$

and

$$B'_{np} = K'_{np} + (K_{np} - B_{np}).$$

So, $X'_{np} = (A'_{np} + K_{np}) \cdot (K'_{np} + (K_{np} - B_{np}))$;

(2) $X'_{np} = A'_{np} \cdot K'_{np} + (A'_{np} + K_{np}) \cdot (K_{np} - B_{np})$. Using (2) in (1) we have $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} [A'_{np} \cdot K'_{np} + (A'_{np} + K_{np}) \cdot (K_{np} - B_{np})]$ and expanding this we have that $X = (\sum_{n=1}^{\infty} \prod_{p=1}^{\infty} A'_{np} \cdot K'_{np}) + B$, where B is a subset of an F_σ set in R.

For each n, p let $T_{np} = A_{np} + K_{np}$. The set A_{np} is in W_1 and K_{np} is an F_σ set. So, K_{np} is in W_1, and since W_1 is finitely additive, T_{np} is in W_1.

Since $T'_{np} = A'_{np} \cdot K'_{np}$, for each np, we have $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} T'_{np} + B$. The set $A = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} T'_{np}$ is in W_2.

Now, suppose $X = A + B$, where A is in W_2 and B is a subset of $\sum_{p=1}^{\infty} M_p$, each M_p is a closed set in R.

Since A is in W_2, $A = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} A'_{np}$, where for each n, p A_{np} is in W_1. So,

$$X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} A'_{np} + \sum_{n=1}^{\infty} B \cdot M_n;$$

$$X = \sum_{n=1}^{\infty} (\prod_{p=1}^{\infty} A'_{np} + B \cdot M_n) = \sum_{n=1}^{\infty} (\prod_{p=1}^{\infty} (A'_{np} + B \cdot M_n)).$$

But, for each n,

$$B \cdot M_n = (M'_n + (M_n - B \cdot M_n))';$$

so that for each p,

$$A'_{np} + B \cdot M_n = A'_{np} + (M'_n + (M_n - B \cdot M_n))'$$
$$= (A_{np} \cdot M'_n + A_{np} \cdot (M_n - B \cdot M_n))'.$$

Since for each n, M_n is closed, M'_n is in W_0 and since W_1 contains W_0 and W_1 is finitely multiplicative we have that for each p, $A_{np} \cdot M'_n$ is in W_1. It follows from Theorem 2b that $X_{np} = A_{np} \cdot M'_n + A_{np} \cdot (M_n - B \cdot M_n)$ is in $W_1(D)$. So, $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} X'_{np}$, is in the collection $W_2(D)$. This completes the argument for Theorem 2c for the case $\alpha = 2$.

There are arguments for the cases $\alpha > 2$ similar to the argument given here for the case $\alpha = 2$.

Theorems 2a, b, and c give a characterization of each collection $W_\alpha(D)$ in terms of W_α. From these theorems, we see that if α is a countable ordinal number, other than 1, then X is in $W_\alpha(D)$ if and only if there is a set A in W_α and a set B, which is a subset of an F_σ set in the σ-ideal R such that $X = A + B$.

Theorem 3 characterizes each collection $B_\alpha(G)$, the analytic representable functions or Baire functions of class α generated by G, in terms of B_α, the Baire function of class α.

THEOREM 3. *Suppose f is a function on S and $0 < \alpha < \Omega$. The function f is in $B_\alpha(G)$ if and only if there is a function g in B_α and an inner limiting set E such that $f_E = g_E$ and $S - E$ belongs to the σ-ideal R.*

Proof. Suppose f is in $B_1(G)$. Let $f_1, f_2, f_3, \ldots$ be a sequence from $B_0(G) = G$ converging to f. For each n, let H_n be the set of all point of discontinuity of f_n and let $H = \sum_{n=1}^{\infty} H_n$; H is in R and H is an F_σ set. Let $E = S - H$; E is an inner limiting set. For each n, f_{nE}, the partial function of f_n over E is in $C(E)$, the collection of all continuous functions over E.

So, f_E is in $B_1(C(E))$. It follows by a theorem of K. Kuratowski [2, p. 434] that f_E can be extended to S without changing its class. So, there is a function g in B_1 such that $g_E = f_E$.

Now, suppose E is an inner limiting set, $S-E$ is in R, f is a function of S and there is a function g in B_1 such that $f_E = g_E$. Let $g_1, g_2, g_3, \ldots$ be a sequence from $B_0 = C$ converging to g and let $S-E = \sum_{p=1}^{\infty} K_p$, where each K_p is closed.

For each n, let

$$f_n(x) = \begin{cases} g_n(x), & \text{if } x \text{ is in } S-(K_1+\ldots+K_n)\,, \\ f(x), & \text{if } x \text{ is in } K_1+K_2+\ldots+K_n\,. \end{cases}$$

For each n, f_n is continuous at each point of $S-(K_1+\ldots+K_n)$. For each n, f_n is in $B_0(G) = G$ and the sequence $f_1, f_2, f_3, \ldots$ converges to f; f is in $B_1(G)$. This shows that Theorem 3 is true for the case $a = 1$.

Suppose $a > 1$ and Theorem 3 is true for all cases ξ, $1 \leqslant \xi < a$.

Suppose f is in $B_a(G)$. Let $f_1, f_2, f_3, \ldots$ be a sequence converging to f such that for each n, f_n belongs to B_{γ_n} and E_n an inner limiting set such that $S-E_n$ is in R and $g_{nE_n} = f_{nE_n}$. The set E is an inner limiting set and $S-E$ is in R. The function f_E is in $B_a(C(E))$. Again using a theorem of Kuratowski [2, p. 434], it follows that there is a function g in B_a such that $f_E = g_E$. Now, suppose f is a function on S and there is an inner limiting set E and a function g in B_a such that $S-E$ is in R and $f_E = g_E$. Let $g_1, g_2, g_3, \ldots$ be a sequence of functions converging to g such that for each n, g_n is in B_{γ_n}, where $\gamma_n < a$, and let $S-E = \sum_{p=1}^{\infty} K_p$, where each K_p is closed.

For each n, let

$$f_n(x) = \begin{cases} g_n(x), & \text{if } x \text{ is in } S-(K_1+K_2+\ldots+K_n)\,, \\ f(x), & \text{if } x \text{ is in } K_1+\ldots+K_n\,. \end{cases}$$

For each n, $f_{nS-(K_1+\ldots+K_n)} = g_{nS-(K_1+\ldots+K_n)}$ and $S-(K_1+\ldots+K_n)$ is an inner limiting set such that $K_1+\ldots+K_n$ is in the σ-ideal R. So, for each n, f_n is in $B_{\gamma_n}(G)$ and f is in $B_a(G)$. Theorem 3 follows by transfinite induction.

In case S is a separable metric space we can obtain another characterization of the collections $B_a(G)$, $a > 0$ from a theorem of T. Traczyk [7]. In [7], Traczyk makes use of the following definition.

DEFINITION. Suppose S is a metric space, I is σ-ideal of subset of S, D is a metric space and f is a mapping from S into D. The function f has property D_a at the point x_0 of S if for every $\varepsilon > 0$, there is a neighborhood R of x_0, a mapping g of Baire class B_a and a set A in I such that $|f(x)-g(x)| < \varepsilon$, for every x in $A'\cdot R$.

Traczyk gives the following theorem in [7]:

Suppose S is a separable metric space, D is a separable and complete metric space and f is a mapping from S into D. If $a > 0$ and for each closed subset F of S, the mapping f_F has property D_a with respect to F at some point of F, then there is a mapping g in Baire's class B_a and a set A in I such that if x is in $S-A$, then $f(x) = g(x)$.

This theorem is a generalization of some earlier results of G. Lederer [3] and later Lederer generalized this result [4, Theorem III].

Before using this theorem of Traczyk, consider the following situation. The space S is the real numbers and R is the collection of all sets of Lebesgue measure 0. Suppose we let R be I, the σ-ideal of Traczyk's definition. As can be seen from Theorem 3, if f is in $B_a(G)$ and $a > 0$ then f satisfies the hypothesis of Traczyk's theorem. However, every measurable function satisfies the hypothesis of Traczyk's theorem for $a = 2$. But, the Baire system generated by G, the collection of all functions continuous almost everywhere does not contain all the measureable function see [6, Theorem 3]. So, it does not suffice to let $R = I$.

In order to get a characterization of $B_a(G)$ using Traczyk's theorem we do the following. Noting that if f is continuous almost everywhere (with respect to R), then it is continuous except for an F_σ set in R, we let R' be the collection of all sets in R which are subsets of F_σ sets in R. R' is a σ-ideal. Let R' be the σ-ideal I of Traczyk's definition stated above. Then Theorem 4 follows easily from Traczyk's theorem.

THEOREM 4. *Suppose the metric space S is separable and $0 < a$. A function f is in $B_a(G)$ if and only if for each closed subset F of S, the mapping f_F has property D_a (where the σ-ideal I is R') with respect to F at some point of F.*

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