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Subdirect decomposition of distributive quasilattices

by

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Following Płonka [3], we define a *quasilattice* to be a nonempty set with binary operations $\wedge$ and $\vee$ which are idempotent, commutative, and associative, and a *distributive* quasilattice to be one which obeys the laws

$$x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)\,, \qquad x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)\,.$$

It is easily checked that the tables

$\wedge$	0	1	∞
0	0	0	∞
1	0	1	∞
∞	∞	∞	∞

$\vee$	0	1	∞
0	0	1	∞
1	1	1	∞
∞	∞	∞	∞

define a distributive quasilattice, $\mathfrak{X}$ say. Let $\mathfrak{L}$ and $\mathfrak{S}$ be the sub-quasilattices of $\mathfrak{X}$ with underlying sets $\{0, 1\}$ and $\{0, \infty\}$ respectively; $\mathfrak{L}$ is a lattice, and $\mathfrak{S}$ is essentially a semilattice (it obeys the law $x\wedge y = x\vee y$). The object of this paper is to prove the following

THEOREM. *A distributive quasilattice with more than one element is isomorphic to a subdirect product of copies of $\mathfrak{X}$, $\mathfrak{L}$, and $\mathfrak{S}$.*

This extends Birkhoff's subdirect decomposition theorem for distributive lattices ([1], p. 193, Theorem 15, Corollary 1), and also contains a similar theorem for semilattices.

In any quasilattice an identity element for $\wedge$ (resp. $\vee$), if it exists, is unique, and will be denoted by I (resp. O) (cf. [1], p. 63, ex. 7, and [2], but note that the free distributive quasilattice with O, I, and one generator has five, not seven, elements).

LEMMA 1. *Let Q be a distributive quasilattice with O and I. Then, for all x and y in Q,*

(i) $x\wedge O = O$ *if and only if* $x\vee I = I$;

(ii) $x\wedge y = I$ *if and only if* $x = y = I$; *and*

(iii) $x\wedge y\wedge O = O$ *if and only if* $x\wedge O = y\wedge O = O$.

Also, we may define a congruence relation B on Q by setting xBy if and only if $x = y$ or $x\wedge O \neq O$ and $y\wedge O \neq O$.

Proof. (i) If $x\wedge O = O$ then

$$I = O\vee I = (x\wedge O)\vee I = (x\vee I)\wedge(O\vee I) = (x\vee I)\wedge I = x\vee I,$$

and dually.

(ii) If $x\wedge y = I$ then $x = x\wedge I = x\wedge(x\wedge y) = x\wedge y = I$, and similarly $y = I$. The converse is trivial.

(iii) If $x\wedge y\wedge O = O$ then, by (i), $I = (x\wedge y)\vee I = (x\vee I)\wedge(y\vee I)$, hence $x\vee I = y\vee I = I$ by (ii), and hence $x\wedge O = y\wedge O = O$ by (i). The converse is trivial.

B is obviously an equivalence relation, and is selfdual by (i). If xBy then, by (iii) and its dual, $(x\wedge z)B(y\wedge z)$ and $(x\vee z)B(y\vee z)$ for all z. This completes the proof.

LEMMA 2. *Let Q be a distributive quasilattice, and let $a\in Q$. Then*

(i) we may define congruence relations C_a, D_a on Q by setting xC_ay if and only if $x\wedge a = y\wedge a$, and xD_ay if and only if $x\vee a = y\vee a$;

(ii) $x(C_a\cap D_a)y$ if and only if $x\wedge(x\vee a) = y\wedge(y\vee a)$; and

(iii) $C_a = O$ if and only if Q has an I and $a = I$, and $D_a = O$ if and only if Q has an O and $a = O$.

Proof. (i) is easily verified. To prove (ii), we note first that if $x(C_a\cap D_a)y$ then

$$\begin{aligned} x\wedge(x\vee a) &= x\wedge(y\vee a) = (x\wedge y)\vee(x\wedge a) \\ &= (y\wedge x)\vee(y\wedge a) = y\wedge(x\vee a) = y\wedge(y\vee a). \end{aligned}$$

Conversely, if $x\wedge(x\vee a) = y\wedge(y\vee a)$, then

$$\begin{aligned} x\vee a &= (x\vee a)\wedge\big((x\vee a)\vee a\big) = \big(x\wedge(x\vee a)\big)\vee a \\ &= \big(y\wedge(y\vee a)\big)\vee a = (y\vee a)\wedge\big((y\vee a)\vee a\big) = y\vee a, \end{aligned}$$

whence xD_ay; moreover the condition $x\wedge(x\vee a) = y\wedge(y\vee a)$ is equivalent to its dual, hence, by duality, xC_ay. This proves (ii). By duality, it will be sufficient to prove the first part of (iii). It is clear that if Q has an I then $C_I = O$; conversely, if $C_a = O$, then, since $(x\wedge a)\wedge a = x\wedge a$ for all x in Q, we have $x\wedge a = x$ for all x, and thus a is an I. This completes the proof.

LEMMA 3. *Let Q be a subdirectly irreducible distributive quasilattice. Then*

(i) Q possesses elements O and I (not necessarily distinct); and

(ii) $a\wedge O = O$ if and only if $a = O$ or $a = I$.

Proof. (i) Let $C = \bigcap_{a\in Q} C_a$. Then if xCy we have xC_xy and xC_yy, i.e. $x\wedge x = y\wedge x$ and $x\wedge y = y\wedge y$, whence $x = y$. Thus $C = O$. Since Q is subdirectly irreducible it follows that $C_a = O$ for some a in Q, and $a = I$ by Lemma 2 (iii). Dually, Q has an O.

(ii) If $a\wedge O = O$ then $C_a\cap D_a = O$: for

$$x = x\vee O = x\vee(a\wedge O) = (x\vee a)\wedge(x\vee O) = x\wedge(x\vee a)$$

for all x, and hence, by Lemma 2 (ii), if $x(C_a\cap D_a)y$ then $x = y$. Since Q is subdirectly irreducible, it follows that $C_a = O$ or $D_a = O$ and hence, by Lemma 2 (iii), that $a = O$ or $a = I$. The converse of (ii) is trivial.

LEMMA 4. *A subdirectly irreducible distributive quasilattice Q with more than one element is isomorphic to $\mathfrak{X}$, $\mathfrak{L}$, or $\mathfrak{S}$.*

Proof. Let $P = \{x\in Q\colon x\wedge O \neq O\} = Q\setminus\{O, I\}$ (cf. Lemma 3 (ii)). If $P = \emptyset$ then $Q\cong\mathfrak{L}$. Suppose therefore that $P\neq\emptyset$ and let $E = B\cap\big(\bigcap_{a\in P}(C_a\cap D_a)\big)$, where B is defined as in Lemma 1. We show that $E = O$. We wish to prove that if xEy then $x = y$, and we may assume that $x\in P$ and $y\in P$ for otherwise $x = y$ since xBy. But then $x(C_x\cap D_x)y$, and hence, by Lemma 2 (ii), $x\wedge(x\vee x) = y\wedge(y\vee x)$, i.e. $x = y\wedge(y\vee x)$, whence $y\wedge x = x$; similarly, since $x(C_y\cap D_y)y$, $x\wedge y = y$; and thus $x = y$. This proves that $E = O$, and, since $C_a\neq O$ and $D_a\neq O$ for all a in P by Lemma 2 (iii), it follows, since Q is subdirectly irreducible, that $B = O$. Hence P has just one element, and $Q\cong\mathfrak{S}$ or $Q\cong\mathfrak{X}$ according as $O = I$ or $O\neq I$.

The theorem stated in the first paragraph follows from Lemma 4 and Birkhoff's general subdirect decomposition theorem ([1], p. 193, Theorem 15).

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