On the topology of curves III

by

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From topological point of view, the arc seems to be the simplest example of a curve and there is an interest in examining curves that are close to arcs. Roughly speaking, there exist two possible ways of approximating curves by arcs: one from the outside and the other from the inside. While the former materialized in many results concerning mainly the notion of a chainable curve, the latter inspired fewer works. Some boundary properties of a curve occur when its structure is investigated relative to distribution of arcs in it. We [1] return to an old concept of Karl Menger [13] and we describe such a property in this paper with an attention paid to the class of acyclic curves (see § 1). We also introduce two classes of curves which we call radial and strictly radial, respectively (see § 3). To some extent they are reminiscent of the cone construction as it appears in topology under several circumstances. These ideas enable us to construct some paradoxical examples of curves (see § 4) one of which is an enlargement of a rational curve constructed by Zygmunt Janiszewski [7]. Incidentally, a characterization of rational curves is obtained (see § 2) which deals with mappings of the Cantor set onto a curve.

§ 1. Rim-types of rational curves. By the *rim-type* of a rational curve X we mean the minimum ordinal α such that X admits an open basis consisting of sets with countable boundaries whose α-th derivatives are empty (compare [13], p. 294). Thus rim-types of rational curves are countable ordinals; curves of rim-type 1 coincide with regular curves. The rim-types will be shown to affect the existence of arcs in acyclic curves. Let S denote the unit circle, i.e. the set of complex numbers with module one, having the natural topology inherited from the plane. A curve X is called *acyclic* provided each continuous mapping of X into S is homotopic to a constant mapping. It is known that all tree-like curves

(1) The second author was participating in an exchange of scientists administrated jointly by the Polish Academy of Sciences and the National Academy of Sciences (U.S.A.).

are acyclic but not conversely (see [3], pp. 74 and 81). It is also known that all acyclic curves are unicoherent (see [10], p. 437). Since curves are 1-dimensional, continuous mappings of a subcurve of a curve X into S admit extensions over X (ibidem, p. 354). It then follows from the classical theory of S. Eilenberg that all subcurves of an acyclic curve are acyclic (ibidem, pp. 407 and 421–427). Hence all acyclic curves are hereditarily unicoherent. Moreover, by same theory, all hereditarily unicoherent and hereditarily decomposable curves are acyclic (see [4], p. 216). Let us mention that, by a recent theorem of H. Cook [6], the last statement can be strengthened by replacing "acyclic" by "tree-like". Since rational curves are hereditarily decomposable, we conclude that all hereditarily unicoherent rational curves are tree-like and acyclic.

1.1. Lemma. *If $A\subset X$ is a subset of an acyclic curve X, $U\subset X$ is an open subset and $a_j, a_j', p\in X$ are points such* (²) *that*

$$a_j\in A\cap \mathrm{cl}U, \quad a_j'\in A\setminus U, \quad Q(A, a_j)=Q(A, a_j')$$

for $j=1,2,\ldots$ and

$$p=\lim_{j\to\infty} a_j=\lim_{j\to\infty} a_j', \quad \dim_p(\mathrm{cl}U\setminus U)=0,$$

then $p\in \mathrm{cl}[A\cap(\mathrm{cl}U\setminus U)]$.

Proof. Let $B=\mathrm{cl}[A\cap(\mathrm{cl}U\setminus U)]$ and let us suppose on the contrary that $p\notin B$. Observe that p and B lie in the boundary $\mathrm{cl}U\setminus U$ of U. Since $\mathrm{cl}U\setminus U$ is 0-dimensional at the point p, there exist closed subsets B_0, $B_1\subset X$ such that

$$p\in B_0, \quad B\subset B_1, \quad B_0\cap B_1=\emptyset, \quad B_0\cup B_1=\mathrm{cl}U\setminus U.$$

Denote by $\gamma\colon \mathrm{cl}U\setminus U\to I$ the real-valued function which sends B_0 and B_1 into 0 and 1, respectively. Let $\psi\colon \mathrm{cl}U\to I$ and $\psi'\colon X\setminus U\to I$ be continuous extensions of γ such that $B_0=\psi^{-1}(0)=\psi'^{-1}(0)$. We define a continuous mapping $f\colon X\to S$ by means of the formula

$$f(x)=\begin{cases} e^{\pi i\psi(x)} & \text{for} \quad x\in \mathrm{cl}U, \\ e^{-\pi i\psi'(x)} & \text{for} \quad x\in X\setminus U. \end{cases}$$

It follows from the acyclicity of X that f is homotopic to a constant mapping. Then there exists a real-valued continuous function $\varphi\colon X\to R$ such that $f(x)=e^{2\pi i\varphi(x)}$ for $x\in X$ (see [10], p. 427). We have $A\cap B_0=\emptyset$ and $B_0=\psi^{-1}(0)$, whence the set $\psi(A\cap \mathrm{cl}U)$ does not contain 0. Similarly, the set $\psi'(A\setminus U)$ does not contain 0, and we conclude that the set

$$f(A)=f(A\cap \mathrm{cl}U)\cup f(A\setminus U)$$

(²) We maintain the terminology of [11] in using the symbol $Q(A, a)$ to denote the quasi-component of a space A at a point $a\in A$. Thus $Q(A, a)$ is the intersection of all closed-open subsets of A that contain a.

does not contain the point $e^0=1$. Consequently, the set $\varphi(A)$ contains no integer. But since both points a_j and a_j' belong to a quasi-component of A, the set $\varphi(A)$ contains the segment with end points $\varphi(a_j)$ and $\varphi(a_j')$; it follows that there exists an integer n_j such that

$$n_j<\varphi(a_j)<n_j+1, \quad n_j<\varphi(a_j')<n_j+1.$$

for $j=1,2,\ldots$ Since $a_j\in \mathrm{cl}U$ and $a_j'\in X\setminus U$, the points

$$f(a_j)=e^{\pi i\psi(a_j)}, \quad f(a_j')=e^{-\pi i\psi'(a_j')}$$

belong to the upper and the lower closed half-planes, respectively, which have the real line as a common boundary. Hence

$$n_j<\varphi(a_j)\leqslant n_j+\tfrac{1}{2}, \quad n_j+\tfrac{1}{2}\leqslant\varphi(a_j')<n_j+1$$

for $j=1,2,\ldots$ On the other hand, we have $p\in B_0$, whence $\psi(p)=\psi'(p)=\gamma(p)=0$ and $f(p)=1$. Thus $\varphi(p)$ is an integer. But the sequences of points a_j and a_j' being convergent to p, the numbers $\varphi(a_j)$ and $\varphi(a_j')$ must converge to $\varphi(p)$. Because n_j and $\varphi(p)$ are integers, there must be indexes j_1 and j_2 such that $n_j=\varphi(p)$ for $j>j_1$ and $n_j+1=\varphi(p)$ for $j>j_2$. Taking k greater than j_1 and j_2, we get $n_k=n_k+1$, a contradiction which completes the proof of 1.1.

1.2. Theorem. *If X is a hereditarily unicoherent rational curve of finite rim-type, then each subcurve of X contains an arc.*

Proof. Let n be the rim-type of X and let $\boldsymbol{G}$ be an open basis in X such that the n-th derivative of $\mathrm{cl}G\setminus G$ is empty for every set $G\in\boldsymbol{G}$. Suppose on the contrary that there exists a curve $C_1\subset X$ such that C_1 contains no arc. Thus $1<n<\infty$. Let $J_1\subset C_1$ be a curve such that J_1 is an irreducible continuum and let $g_1\colon J_1\to I$ denote the finest monotone continuous mapping of J_1 onto the unit segment I (see [10], p. 199). Since J_1 is not an arc, there exists a number $t_1\in I$ such that $g_1^{-1}(t_1)$ is non-degenerate. Since J_1 is rational, thus hereditarily decomposable, the set $g_1^{-1}(t_1)$ has void interior in J_1 (ibidem, p. 216). This means that the union of the sets

$$A_1=\{x\in J_1\colon\ 0\leqslant g_1(x)<t_1\}, \quad B_1=\{x\in J_1\colon\ t_1<g_1(x)\leqslant 1\}$$

is dense in J_1. The sets A_1 and B_1 are composants of the irreducible continua $\mathrm{cl}A_1$ and $\mathrm{cl}B_1$, respectively (ibidem, pp. 202 and 209). Thus $\mathrm{cl}A_1\setminus A_1$ and $\mathrm{cl}B_1\setminus B_1$ are continua (ibidem, p. 210) whose union is $g_1^{-1}(t_1)$. Consequently, at least one of these two continua is non-degenerate. In view of the symmetrical role of A_1 and B_1 (relative to the orientation of I) we can assume that $\mathrm{cl}A_1\setminus A_1$ is non-degenerate. Let us denote $X_1=\mathrm{cl}A_1$, $C_2=X_1\setminus A_1$, and apply to C_2 the same procedure as that just applied to C_1. Repeating this procedure n times, we successively get curves C_k

$(k=1,\ldots,n+1)$ such that $C_{k+1}=X_k\setminus A_k$ where $X_k=\operatorname{cl}A_k$ and $A_k\subset C_k$ is a connected set for $k=1,\ldots,n$. Hence $X_{k+1}\subset C_{k+1}\subset X_k$ $(k=1,\ldots,n-1)$. Since C_{n+1} is non-degenerate, there exist distinct points c, $c'\in C_{n+1}$ and let $G_0\in \boldsymbol{G}$ be an element of the basis such that $c\in G_0$ and $c'\notin \operatorname{cl}G_0$. We define sets $U_1,\ldots,U_n$ inductively by the formulas

$$U_1=X_1\setminus\operatorname{cl}[X_1\setminus\operatorname{cl}(X_1\cap G_0)],$$
$$U_{k+1}=X_{k+1}\setminus\operatorname{cl}[X_{k+1}\setminus\operatorname{cl}(X_{k+1}\cap U_k)]$$

for $k=1,\ldots,n-1$. Thus U_k is an open subset of X_k with the boundary $F_k=\operatorname{cl}U_k\setminus U_k$ $(k=1,\ldots,n)$. Moreover, we have the inclusions

$$X_1\cap G_0\subset U_1\subset\operatorname{cl}(X_1\cap G_0),$$
$$X_{k+1}\cap U_k\subset U_{k+1}\subset\operatorname{cl}(X_{k+1}\cap U_k)$$

from which we conclude that $X_k\cap G_0\subset U_k$ $(k=1,\ldots,n)$ and

$$F_1=\operatorname{cl}U_1\setminus U_1\subset\operatorname{cl}(X_1\cap G_0)\setminus(X_1\cap G_0)\subset\operatorname{cl}G_0\setminus G_0,$$
$$F_{k+1}=\operatorname{cl}U_{k+1}\setminus U_{k+1}\subset\operatorname{cl}(X_{k+1}\cap U_k)\setminus(X_{k+1}\cap U_k)\subset\operatorname{cl}U_k\setminus U_k=F_k$$

for $k=1,\ldots,n-1$. But $c\in C_{n+1}\subset X_n$ and $c\in G_0$, whence $c\in X_n\cap G_0\subset U_n$. On the other hand, the closure $\operatorname{cl}U_n$ is contained in $\operatorname{cl}G_0$, and therefore $c'\notin\operatorname{cl}U_n$. Because C_{n+1} is a continuum joining c and c' in X_n, the boundary of U_n must intersect C_{n+1}, i.e. the set $F_{n+1}=C_{n+1}\cap F_n$ is non-empty.

Since we have the inclusions

$$F_{n+1}\subset F_n\subset\ldots\subset F_1\subset\operatorname{cl}G_0\setminus G_0$$

and the n-th derivative of the boundary of G_0 is empty, a contradiction completing the proof of 1.2 will be found out if we prove that F_{n+1} lies in the n-th derivative of $\operatorname{cl}G_0\setminus G_0$.

To do this, let us take any integer $k=1,\ldots,n$ and a point $p\in F_{k+1}$; thus $p\in X_k\setminus A_k$ and $p\in\operatorname{cl}U_k$. The set A_k is dense in X_k and the set U_k is open in X_k. Hence $\operatorname{cl}U_k=\operatorname{cl}(A_k\cap U_k)$, and so $p\in\operatorname{cl}(A_k\cap U_k)$. Given any open neighbourhood V of p in X_k, let us suppose that $A_k\cap V\subset U_k$. Then again, by the density of A_k in X_k, we should have $\operatorname{cl}V=\operatorname{cl}(A_k\cap V)$, and so $p\in V\subset\operatorname{cl}U_k$. However, it follows from the definition of U_k that each open subset of X_k is contained in U_k provided it is contained in $\operatorname{cl}U_k$. Consequently, we should have $p\in U_k$ contrary to the assumption that $p\in F_{k+1}\subset F_k$. We have therefore shown that $A_k\cap V$ is not contained in U_k, which implies $p\in\operatorname{cl}(A_k\setminus U_k)$. Since the point p belongs to the closures of both sets $A_k\cap U_k$ and $A_k\setminus U_k$, there exist points $a_j\in A_k\cap U_k$ and $a'_j\in A_k\setminus U_k$ $(j=1,2,\ldots)$ such that a_j and a'_j converge to p. The set $A_k\subset X_k\subset X$ is connected, and X being hereditarily unicoherent and rational, X_k is an acyclic curve. The boundary F_k of U_k in X_k is contained in the boundary of G_0 in X which is countable, hence 0-dimensional. Thus we can apply 1.1 to A_k, X_k and U_k. As a result we obtain $p\in\operatorname{cl}(A_k\cap F_k)$.

Since p was an arbitrary point of F_{k+1} and $F_{k+1}\subset C_{k+1}=X_k\setminus A_k$, we conclude that

$$F_{k+1}\subset\operatorname{cl}(A_k\cap F_k)\subset\operatorname{cl}(F_k\setminus F_{k+1})\subset\operatorname{cl}G_0\setminus G_0$$

for $k=1,\ldots,n$. It follows that F_{n+1} is, indeed, contained in the n-th derivative of $\operatorname{cl}G_0\setminus G_0$, and the proof of 1.2 is completed.

Remark. There exist hereditarily unicoherent rational curves of rim-type ω which contain no arcs (³). Historically first construction of such a curve was given by Z. Janiszewski [7], but a detailed description of his curve was never published. Later some other authors (see [1], [2], [8] and [14], for instance) gave constructions which could serve as examples of curves of rim-type ω and containing no arcs. An example which seems very likely to resemble the Janiszewski curve is due to O. V. Lokucievskiĭ [12], and its construction utilizes an inverse limit procedure. As an effect of this procedure one gets a chainable rational curve of rim-type ω which contains no arc. On the other hand, each subcurve of a rational curve of rim-type 1 is regular, and therefore arcwise connected. The latter statement has, by 1.2, an analogue for acyclic rational curves of higher rim-types. That this analogue seems best possible is indicated by the existence of the Janiszewski curve as well as of the example from 1.3 below.

By the *arc-component* of a topological space X at a point $x\in X$ we mean the union of $\{x\}$ and of all arcs contained in X that contain x.

1.3. EXAMPLE. *There exists a chainable rational curve X of rim-type 2 such that X has uncountably many arc-components.*

Proof. Let T denote the Cantor ternary set in the unit segment I and let $I_1, I_2,\ldots$ be the segments which are closures of components of $I\setminus T$; we take the curve

$$C=\operatorname{cl}\{(x,y):\ 0<|x|<\pi^{-1},\quad y=\sin x^{-1}\}$$

and a homeomorphism h_i of C into the plane circular disk whose diameter is I_i $(i=1,2,\ldots)$. We require that the points $h_i((\pm\pi^{-1},0))$ are end points of I_i. Then X is defined by the formula

$$X=\operatorname{cl}\bigcup_{i=1}^{\infty}h_i(C)$$

and, clearly, no arc contained in X meets the set $T\setminus(I_1\cup I_2\cup\ldots)$ which is uncountable.

(³) Here ω stands for the minimum infinite ordinal.

§ 2. A characterization of rational curves. There are known characterizations of rational curves by means of real-valued continuous functions defined upon them (see [10], p. 289). It turns out that this method can be reversed, in a sense, to produce an alternative characterization.

2.1. LEMMA. *If X is a rational curve, then there exists an infinite sequence $\mathbf{C}_1, \mathbf{C}_2, \ldots$ of finite closed covers of X such that, for $n = 1, 2, \ldots$, the following conditions are fulfilled:*

(i)$_n$ $\mathbf{C}_{n+1}$ *is a refinement of* $\mathbf{C}_n$,

(ii)$_n$ $\mathbf{C}_{n+1}(C) = \{C' \in \mathbf{C}_{n+1}\colon C' \subset C\}$ *is a cover of C for every set* $C \in \mathbf{C}_n$,

(iii)$_n$ $C \cap C' \cap C'' = \emptyset$ *for every triple of different sets* $C, C', C'' \in \mathbf{C}_n$,

(iv)$_n$ $\operatorname{card} C \cap C' \leqslant \aleph_0$ *for every couple of different sets* $C, C' \in \mathbf{C}_n$,

(v)$_n$ $\operatorname{diam} C \leqslant n^{-1}$ *for every set* $C \in \mathbf{C}_n$.

Proof. A well-known decomposition theorem (see [10], p. 288) guarantees the existence of a finite closed cover $\mathbf{C}_1$ of X which satisfies conditions (iii)$_1$-(v)$_1$. Let us assume that we are given a finite closed cover $\mathbf{C}_n$ of X satisfying (iii)$_n$-(v)$_n$. To prove 2.1, it is enough to find a finite closed cover $\mathbf{C}_{n+1}$ of X such that conditions (i)$_n$, (ii)$_n$ and (iii)$_{n+1}$-(v)$_{n+1}$ are satisfied. Since X is one-dimensional, there exists a finite open cover $\{G_1, \ldots, G_m\}$ of X such that

(1) $$\operatorname{diam} G_i < (n+1)^{-1}, \qquad G_i \cap G_j \cap G_k = \emptyset$$

for different indexes $i, j, k = 1, \ldots, m$. Let E be the union of all sets $C \cap C'$ where C, C' run over all couples of different sets of $\mathbf{C}_n$. By (iv)$_n$, the set E is countable, therefore zero-dimensional. Since $\mathbf{C}_n$ is finite, E is closed. Then there exists an open cover $\{G_1', \ldots, G_m'\}$ of X such that

(2) $$G_i' \subset G_i, \qquad E \cap G_i' \cap G_j' = \emptyset$$

for different indexes $i, j = 1, \ldots, m$ (see [9], p. 296). Since X is rational, there exists a closed cover $\{F_1, \ldots, F_m\}$ of X such that

(3) $$F_i \subset G_i', \qquad \operatorname{card} F_i \cap F_j \leqslant \aleph_0$$

for different indexes $i, j = 1, \ldots, m$ (see [10], p. 287). Putting

$$\mathbf{C}_{n+1} = \{C \cap F_i\colon C \in \mathbf{C}_n,\ i = 1, \ldots, m\},$$

we readily see that (i)$_n$, (ii)$_n$ and (v)$_{n+1}$ are true. Let us consider any triple $C \cap F_i$, $C' \cap F_j$, $C'' \cap F_k$ of different sets of $\mathbf{C}_{n+1}$. Let A and B be the intersections of the three and the first two of these sets, respectively.

If $C = C' = C''$, then the indexes i, j, k are different and

$$A \subset F_i \cap F_j \cap F_k \subset G_i \cap G_j \cap G_k = \emptyset,$$

by (1), (2) and (3). If exactly two of the sets C, C', C'' are different, then we can assume that $C \neq C' = C''$, whence $j \neq k$ and

$$A \subset (C \cap C') \cap F_j \cap F_k \subset E \cap G_j' \cap G_k' = \emptyset,$$

by (2) and (3). If all the sets C, C', C'' are different, then

$$A \subset C \cap C' \cap C'' = \emptyset,$$

by (iii)$_n$. Thus we have proved (iii)$_{n+1}$. Finally, observe that $C = C'$ implies $i \neq j$, whence

$$\operatorname{card} B \leqslant \operatorname{card} F_i \cap F_j \leqslant \aleph_0,$$

by (3). On the other hand, the inequality $C \neq C'$ implies the inequalities

$$\operatorname{card} B \leqslant \operatorname{card} C \cap C' \leqslant \aleph_0,$$

by (iv)$_n$. Thus we have also proved (iv)$_{n+1}$ which completes the inductive proof of 2.1.

2.2. LEMMA. *If X is a rational curve, then there exists an infinite sequence $\mathbf{C}_1, \mathbf{C}_2, \ldots$ of finite closed covers of X such that, for $n = 1, 2, \ldots$, conditions* (i)$_n$–(v)$_n$ *of 2.1 are fulfilled and*

(vi) $$C = \operatorname{cl}[X \setminus \operatorname{cl}(X \setminus C)] \neq \emptyset$$

for $C \in \mathbf{C}_n$, $n = 1, 2, \ldots$

Proof. We prove 2.2 by modifying what has been done in 2.1. Let $\mathbf{C}_1, \mathbf{C}_2, \ldots$ be the covers of X as given by 2.1. We shall prove that the required modification will be achieved if we define

$$\mathbf{D}_n = \{\operatorname{cl}[X \setminus \operatorname{cl}(X \setminus C)]\colon C \in \mathbf{C}_n,\ X \neq \operatorname{cl}(X \setminus C)\}$$

for $n = 1, 2, \ldots$ Clearly, each element C of $\mathbf{D}_n$ satisfies condition (vi). We want to prove that $\mathbf{D}_n$ is a cover of X such that conditions (i)$_n$–(v)$_n$ hold with $\mathbf{C}$ replaced by $\mathbf{D}$. Observe that $\mathbf{D}_n$ consists of the closures of interiors in X of those sets of $\mathbf{C}_n$ whose interiors are non-empty. It follows that (i)$_n$ and (iii)$_n$–(v)$_n$ hold with $\mathbf{C}$ replaced by $\mathbf{D}$. Let $C \in \mathbf{C}_n$ be a set satisfying the inequality $X \neq \operatorname{cl}(X \setminus C)$ and let $U = X \setminus \operatorname{cl}(X \setminus C)$. If $p \in \operatorname{cl} U$ is any point and V is an open neighbourhood of p in X, the set $U \cap V$ is open and non-empty. This set is contained in C and $\mathbf{C}_{n+1}(C)$ is a finite closed cover of C, by (ii)$_n$. Consequently, there exists a set $C' \in \mathbf{C}_{n+1}(C)$ whose interior U' in X meets $U \cap V$, and since V is an arbitrary neighbourhood of p, we can assume that $p \in \operatorname{cl} U'$. But $C' \subset C$ implies $\operatorname{cl} U' \subset \operatorname{cl} U$, and we have $\operatorname{cl} U' \in \mathbf{D}_{n+1}$. Hence $\operatorname{cl} U$ is the union of a subcollection of $\mathbf{D}_{n+1}$. Thus (ii)$_n$ holds with $\mathbf{C}$ replaced by $\mathbf{D}$. A quite similar argument, when applied to $C = X$, shows that the collections $\mathbf{D}_n$ are covers of X $(n = 1, 2, \ldots)$, and 2.2 is proved.

2.3. THEOREM. *In order that a curve X be rational it is necessary and sufficient that there exist a continuous mapping $f\colon T \to X$ of the Cantor ternary set T onto X and a countable set $Q \subset X$ such that $f^{-1}(x)$ is a two-point set for $x \in Q$ and $f^{-1}(x)$ is degenerate for $x \in X \setminus Q$.*

Proof. If such a mapping $f\colon T\to X$ exists, then $f|f^{-1}(X\setminus Q)$ is a homeomorphism of the zero-dimensional set $f^{-1}(X\setminus Q)\subset T$ onto $X\setminus Q$, and X is rational (see [10], p. 285). Thus the condition is sufficient. To prove that it is also necessary, let us take the infinite sequence $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots$ of finite closed covers of X as constructed in 2.1 and modified in 2.2. We are going to define inductively an infinite sequence $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots$ of finite collections of closed segments of the real line such that the segments of $\boldsymbol{I}_n$ are pairwise disjoint of length less than n^{-1} and there exists a one-to-one function $\Phi_n\colon \boldsymbol{I}_n\to\boldsymbol{C}_n$ mapping $\boldsymbol{I}_n$ onto $\boldsymbol{C}_n$ $(n=1,2,\ldots)$. Let $\boldsymbol{I}_1$ be any collection of pairwise disjoint closed segments of length less than 1 such that $\operatorname{card}\boldsymbol{I}_1=\operatorname{card}\boldsymbol{C}_1$, and let Φ_1 be any one-to-one correspondence between the segments of $\boldsymbol{I}_1$ and the sets of $\boldsymbol{C}_1$. Assuming $\boldsymbol{I}_n$ and Φ_n are given, we define $\boldsymbol{I}_{n+1}$ and Φ_{n+1} as follows. First of all, observe that no set of $\boldsymbol{C}_{n+1}$ is countable, by (vi). Thus each set of $\boldsymbol{C}_{n+1}$ is contained in exactly one set of $\boldsymbol{C}_n$, by $(\mathrm{i})_n$ and $(\mathrm{iv})_n$. Consequently, the collection $\boldsymbol{C}_{n+1}$ is the union of pairwise disjoint subcollections $\boldsymbol{C}_{n+1}(C)$, where $C\in\boldsymbol{C}_n$, according to $(\mathrm{ii})_n$. For $I\in\boldsymbol{I}_n$, let us find pairwise disjoint subsegments of I of length less than $(n+1)^{-1}$ and a one-to-one correspondence between them and the sets of $\boldsymbol{C}_{n+1}(\Phi_n(I))$. Let $\boldsymbol{I}_{n+1}$ be the collection of all these subsegments where $I\in\boldsymbol{I}_n$. So defined correspondence is a one-to-one function Φ_{n+1} which maps $\boldsymbol{I}_{n+1}$ onto $\boldsymbol{C}_{n+1}$. Moreover, for $I\in\boldsymbol{I}_n$ and $I'\in\boldsymbol{I}_{n+1}$, we have $I'\subset I$ if and only if $\Phi_{n+1}(I')\subset\Phi_n(I)$.

By (vi), no set $C\in\boldsymbol{C}_n$ is degenerate. It follows from $(\mathrm{ii})_n$ and $(\mathrm{v})_n$ that, for m sufficiently large, the set $C\in\boldsymbol{C}_n$ contains at least two sets of $\boldsymbol{C}_m$. Then the segment $\Phi_n^{-1}(C)$ contains at least two segments of $\boldsymbol{I}_m$. This guarantees that the intersection of the decreasing sequence $|\boldsymbol{I}_1|\supset|\boldsymbol{I}_2|\supset\ldots$ of the unions of segments of $\boldsymbol{I}_n$ is a compact perfect subset of the real line. Since the lengths of segments of $\boldsymbol{I}_n$ converge to zero when n tends to the infinity, the intersection is zero-dimensional and thus it is a Cantor set. We can assume that

$$T=\bigcap_{n=1}^{\infty}|\boldsymbol{I}_n|\,,$$

and let us denote by $I_n(t)$, for $t\in T$, the segment of $\boldsymbol{I}_n$ which contains t $(n=1,2,\ldots)$. Hence $I_{n+1}(t)\subset I_n(t)$ which yields

$$\Phi_{n+1}\big(I_{n+1}(t)\big)\subset\Phi_n\big(I_n(t)\big)\in\boldsymbol{C}_n$$

for $t\in T$ and $n=1,2,\ldots$ We define a mapping $f\colon T\to X$ by the formula

$$\{f(t)\}=\bigcap_{n=1}^{\infty}\Phi_n\big(I_n(t)\big)$$

for $t\in T$, according to $(\mathrm{v})_n$. The sets $I\cap T$ are open in T for $I\in\boldsymbol{I}_n$ $(n=1,2,\ldots)$, which implies that f is continuous, by $(\mathrm{v})_n$. Since $\boldsymbol{C}_n$ is a cover of X and Φ_n is a mapping of $\boldsymbol{I}_n$ onto $\boldsymbol{C}_n$, it follows from $(\mathrm{ii})_n$ that f is a mapping of T onto X. For $x\in X$, at most two sets of $\boldsymbol{C}_n$ can contain x $(n=1,2,\ldots)$, by $(\mathrm{iii})_n$. Since Φ_n is one-to-one, the set $f^{-1}(x)$ is contained in the union of two segments of $\boldsymbol{I}_n$, for $n=1,2,\ldots$ Thus $\operatorname{card}f^{-1}(x)\leqslant 2$.

Finally, let us define

$$Q=\bigcup_{n=1}^{\infty}\bigcup_{\substack{C,C'\in\boldsymbol{C}_n\\ C\neq C'}}(C\cap C')$$

and notice Q is countable, by $(\mathrm{iv})_n$. For $x\in X$, $\operatorname{card}f^{-1}(x)=2$ if and only if there exists a positive integer n_0 and segments $I_1, I_2\in\boldsymbol{I}_{n_0}$ such that $I_1\neq I_2$ and $x\in\Phi_{n_0}(I_1)\cap\Phi_{n_0}(I_2)$. The latter statement is equivalent to $x\in Q$ because Φ_{n_0} is a one-to-one mapping of $\boldsymbol{I}_{n_0}$ onto $\boldsymbol{C}_{n_0}$.

§ 3. Radial and strictly radial curves.

A curve X will be called *radial* provided there exist a point $o\in X$ and a collection $\boldsymbol{A}$ of arcs such that $|\boldsymbol{A}|=X$, the point o is an end point of any arc from $\boldsymbol{A}$, and $A_1\cap A_2=\{o\}$ for $A_1, A_2\in\boldsymbol{A}$ and $A_1\neq A_2$. If, in addition, each arc contained in X is contained in the union of two arcs from $\boldsymbol{A}$, then X will be called *strictly radial*. Clearly, each radial curve is arcwise connected.

3.1. *If $f\colon R\to X$ is a one-to-one continuous mapping of the real line R into a strictly radial curve X, then the closure of $f(R)$ in X is an arc.*

Proof. It suffices to show that $f(R)$ is contained in an arc. If there exists an arc $A\in\boldsymbol{A}$ containing $f(R)$, we are done. If this is not so, we have two numbers $t_1, t_2\in R$ and two different arcs $A_1, A_2\in\boldsymbol{A}$ such that $o\neq f(t_1)\in A_1$ and $o\neq f(t_2)\in A_2$. Given any closed segment $J\subset R$ containing t_1 and t_2, the image $f(J)\subset X$ is an arc which meets $A_1\setminus\{o\}$ and $A_2\setminus\{o\}$. But since X is strictly radial, the arc $f(J)$ is contained in the union of two arcs from $\boldsymbol{A}$. Thus $f(J)$ cannot meet any set $A\setminus\{o\}$ where $A\in\boldsymbol{A}$ and $A_1\neq A\neq A_2$. It follows that $f(J)\subset A_1\cup A_2$, whence also $f(R)\subset A_1\cup A_2$ and 3.1 is proved.

3.2. *Each strictly radial curve has the fixed point property.*

Proof. It is known that if the closure of the image of the real line under a one-to-one continuous mapping into a curve X is an arc, then X has the fixed point property (see [15], p. 493). Therefore 3.2 follows from 3.1.

§ 4. Embeddings in radial curves.

The classes of radial and strictly radial curves are not so narrow as they look at first sight. Here we investigate their properties related to acyclicity. For instance, a strictly radial curve need not be hereditarily unicoherent, and a radial curve can even contain a simple closed curve. In constructions of the examples which follow, we identify the point $(x_1,\ldots,x_n)$ of the Euclidean n-space R^n with the point $(x_1,\ldots,x_n,0,0,\ldots)$ of the Hilbert space. Thus we have

$R = R^1 \subset R^2 \subset \ldots$ We mean by a *1-parallel line* any homeomorphic image $h(R)$ of R in R^n such that there exist numbers $t_0, x_{1i}, x_{2i} \in R$ $(i = 1, \ldots, n)$ satisfying the condition

$$\{h(t)\colon\ t_0 \leqslant |t|\} = \{(x_{11}+t, x_{12}, \ldots, x_{1n})\colon\ 0 \leqslant t\} \cup \cup \{(x_{21}+t, x_{22}, \ldots, x_{2n})\colon\ 0 \leqslant t\};$$

and then the set $h(R^+)$ is called a *1-parallel closed half-line*, the symbol R^+ denoting a closed half-line of the real line R. We also denote

$$H(t) = \{(x_1, x_2, \ldots)\colon\ t \leqslant x_1\}.$$

4.1. EXAMPLE. *There exists a radial hereditarily decomposable curve X such that the circle is embedable in X.*

Proof. A collection $\boldsymbol{K}_n$ of 1-parallel closed half-lines in R^{2n+2} and a countable collection $\boldsymbol{L}_n$ of 1-parallel lines in R^{2n+2} will be defined by induction on $n = 1, 2, \ldots$ The sets belonging to the union $\boldsymbol{K}_n \cup \boldsymbol{L}_n$ will be pairwise disjoint $(n = 1, 2, \ldots)$. Let $\varphi\colon T \to I$ be the standard continuous mapping of the Cantor ternary set T onto the unit segment I, defined by the formula

$$\varphi\Big(\sum_{i=1}^{\infty} 3^{-i} t_i\Big) = \sum_{i=1}^{\infty} 2^{-i-1} t_i$$

where $t_i = 0, 2$ for $i = 1, 2, \ldots$ Let $f\colon T \to S$ be the mapping of T onto the unit circle S, defined by $f(t) = e^{-2\pi i\varphi(t)}$ for $t \in T$. Let us locate S and T in R^4 on the plane $x_1 = x_2 = 0$ and on the line $x_1 = x_3 = x_4 = 0$, respectively. Observe that then the union of the straight segments $\overline{pf(p)}$ with end points p and $f(p)$, where $p \in T$, is homeomorphic to the mapping cylinder of f. Moreover, by the definition of f, there exists a countable set $Q \subset S$ such that $f^{-1}(s)$ is a two-point set for $s \in Q$ and $f^{-1}(s)$ is degenerate for $s \in S \setminus Q$. Thus we can write

$$f^{-1}(s) = \{p_1(s), p_2(s)\}, \qquad p_i(s) = \big(0, x_{i2}(s), 0, 0\big)$$

$(i = 1, 2)$ for $s \in Q$, and

$$f^{-1}(s) = \{p(s)\}, \qquad p(s) = \big(0, x_2(s), 0, 0\big)$$

for $s \in S \setminus Q$. Let us denote

$$K_i(s) = \big\{\big(t, x_{i2}(s), 0, 0\big)\colon\ 0 \leqslant t\big\},$$

$$K(s) = \big\{\big(t, x_2(s), 0, 0\big)\colon\ 0 \leqslant t\big\},$$

and define the collections $\boldsymbol{K}_1$ and $\boldsymbol{L}_1$ by the formulas

$$\boldsymbol{K}_1 = \{K(s) \cup \overline{p(s)s}\colon\ s \in S \setminus Q\},$$

$$\boldsymbol{L}_1 = \{K_1(s) \cup \overline{p_1(s)s} \cup \overline{sp_2(s)} \cup K_2(s)\colon\ s \in Q\}.$$

Notice that $|\boldsymbol{K}_1| \cup |\boldsymbol{L}_1| \subset H(0)$.

Assume now the collections $\boldsymbol{K}_n$ and $\boldsymbol{L}_n$ are already defined. We shall construct suitable collections $\boldsymbol{K}_{n+1}$ and $\boldsymbol{L}_{n+1}$. Since $\boldsymbol{L}_n$ is countable, we have $\boldsymbol{L}_n = \{L_1, L_2, \ldots\}$ where L_j is a 1-parallel line in R^{2n+2} for $j = 1, 2, \ldots$ By the definition of 1-parallel lines, there exists a point $q_j = (x_{j1}, \ldots, x_{j2n+2})$ such that the closure K_j of a component of $L_j \setminus \{q_j\}$ is a 1-parallel closed half-line in R^{2n+2} and

$$L_j \setminus K_j = \{(x_{j1}+t, x_{j2}, \ldots, x_{j2n+2})\colon\ 0 < t\}$$

$(j = 1, 2, \ldots)$. Moreover, we can assume that $n \leqslant x_{j1}$; otherwise, we could increase x_{j1} by choosing another point of L_j to be q_j. Let J_j be the straight segment

$$J_j = \{(x_{j1}+t, x_{j2}, \ldots, x_{j2n+2})\colon\ 0 \leqslant t \leqslant 1\}$$

and let C_j be the closed half-circle in R^{2n+4}, defined by the formula

$$C_j = \{(x_{j1}+1, x_{j2}, \ldots, x_{j2n+2}, y, z)\colon\ y^2 + (z - j^{-1})^2 = j^{-2},\quad 0 \leqslant y\}$$

$(j = 1, 2, \ldots)$. Thus the intersection $C_j \cap J_j$ consists of only one point $p_j' = (x_{j1}+1, x_{j2}, \ldots, x_{j2n+2}, 0, 0)$ which is an end point of both C_j and J_j. Let p_j'' denote the end point of C_j different from p_j'. Observe that each 2-plane in R^{2n+4} which contains J_j has at most two points in common with the half-circle C_j and one of these points is p_j'. Let T_j be the Cantor ternary set located on C_j such that $p_j', p_j'' \in T_j$. Let $f_j\colon T_j \to J_j$ denote the standard continuous mapping of T_j onto J_j such that $\{p_j'\} = f_j^{-1}(q_j)$ and $\{p_j''\} = f_j^{-1}(p_j')$. Then there exists a countable set $Q_j \subset J_j$ such that $f_j^{-1}(r)$ is a two-point set for $r \in Q_j$ and $f_j^{-1}(r)$ is degenerate for $r \in J_j \setminus Q_j$.

We write

$$f_j^{-1}(r) = \{p_{1j}(r), p_{2j}(r)\}, \qquad p_{ij}(r) = \big(x_{j1}+1, x_{j2}, \ldots, x_{j2n+2}, y_i(r), z_i(r)\big)$$

$(i = 1, 2)$ for $r \in Q_j$, and

$$f_j^{-1}(r) = \{p_j(r)\}, \qquad p_j(r) = \big(x_{j1}+1, x_{j2}, \ldots, x_{j2n+2}, y(r), z(r)\big)$$

for $r \in J_j \setminus Q_j$. Hence $q_j, p_j' \in J_j \setminus Q_j$ and $p_j(q_j) = p_j'$, $p_j(p_j') = p_j''$. Let us denote

$$K_{ij}(r) = \big\{\big(x_{j1}+1+t, x_{j2}, \ldots, x_{j2n+2}, y_i(r), z_i(r)\big)\colon\ 0 \leqslant t\big\},$$

$$K_j(r) = \big\{\big(x_{j1}+1+t, x_{j2}, \ldots, x_{j2n+2}, y(r), z(r)\big)\colon\ 0 \leqslant t\big\},$$

and define the collections $\boldsymbol{K}_{n+1}$ and $\boldsymbol{L}_{n+1}$ by the formulas

$$\boldsymbol{K}_{n+1} = \{K_j\colon\ j = 1, 2, \ldots\} \cup \bigcup_{j=1}^{\infty} \{K_j(r) \cup \overline{p_j(r)r}\colon\ r \in J_j \setminus Q_j,\quad p_j' \neq r \neq q_j\},$$

$$\boldsymbol{L}_{n+1} = \{(L_j \setminus K_j \setminus J_j) \cup \overline{p_j'p_j''} \cup K_j(p_j')\colon\ j = 1, 2, \ldots\} \cup \cup \bigcup_{j=1}^{\infty} \{K_{1j}(r) \cup \overline{p_{1j}(r)r} \cup \overline{rp_{2j}(r)} \cup K_{2j}(r)\colon\ r \in Q_j\}.$$

Notice that $|\boldsymbol{L}_n| \subset |\boldsymbol{K}_{n+1}| \cup |\boldsymbol{L}_{n+1}|$, $|\boldsymbol{K}_{n+1}| \subset |\boldsymbol{L}_n| \cup H(n)$ and $|\boldsymbol{L}_{n+1}| \subset H(n)$, whence we conclude that the collections $\boldsymbol{K}_n$ and $\boldsymbol{L}_n$ just defined satisfy

(4) $$|\boldsymbol{K}_n| \cup |\boldsymbol{L}_n| \subset H(n-2)$$

for $n = 1, 2, \ldots$ Also, the union

(5) $$Y_n = \bigcup_{k=1}^{n} (|\boldsymbol{K}_k| \cup |\boldsymbol{L}_k|) = |\boldsymbol{L}_n| \cup \bigcup_{k=1}^{n} |\boldsymbol{K}_k|$$

is a closed subset of R^{2n+2}. It follows that Y_n is locally compact for $n = 1, 2, \ldots$ According to (4) and (5), we have

(6) $$Y = \bigcup_{n=1}^{\infty} Y_n = \bigcup_{n=1}^{\infty} (|\boldsymbol{K}_n| \cup |\boldsymbol{L}_n|) = \bigcup_{n=1}^{\infty} |\boldsymbol{K}_n|$$

because the decreasing sequence of the sets $H(n)$ $(n = 1, 2, \ldots)$ has an empty intersection. By the same reason, the sets $Y \backslash H(n)$ form an open cover of Y. But

$$Y \backslash H(n) = \bigcup_{k=1}^{n+1} (|\boldsymbol{K}_k| \cup |\boldsymbol{L}_k|) \backslash H(n) = Y_{n+1} \backslash H(n),$$

by (4), (5) and (6). Thus Y is a locally compact subset of the Hilbert space. We claim the one-point compactification $X = Y \cup \{o\}$ of Y is an example of a curve possessing all the features promised by 4.1.

Indeed, the set Y is connected and one-dimensional, and therefore X is a curve. Take the collection $\boldsymbol{A}$ of all the sets $K \cup \{o\}$ where $K \in \boldsymbol{K}_n$ and $n = 1, 2, \ldots$ Since these sets K are pairwise disjoint 1-parallel closed half-lines, elements of $\boldsymbol{A}$ are arcs having the point o as one of their end points and the only point in common. Hence X is a radial curve which contains the circle S, and it remains to prove that X is hereditarily decomposable. Suppose on the contrary that there exists an indecomposable subcurve $C \subset X$. Then C has uncountably many pairwise disjoint composants, and since the set $Q \cup \{o\}$ is countable, there exists a composant C' of C such that C' does not meet $Q \cup \{o\}$. Let K' be the closure of the component of the set $X \backslash (Q \cup \{o\})$ which contains C'. The composant C' being dense in C, we have $C \subset K'$. By (6) and the definition of $\boldsymbol{K}_1$ and $\boldsymbol{L}_1$, the components of

$$X \backslash (Q \cup \{o\}) = Y \backslash Q$$

are either elements of $\boldsymbol{K}_1$ or parts of elements of $\boldsymbol{L}_1$ augmented with some closed subsets of $|\boldsymbol{K}_k| \cup |\boldsymbol{L}_k|$ where $k \geqslant 2$. The elements of $\boldsymbol{K}_1$ and $\boldsymbol{L}_1$ being half-lines and lines, respectively, we conclude that the indecomposable continuum C must be contained in the closure of the union of $|\boldsymbol{K}_k| \cup |\boldsymbol{L}_k|$ where $k \geqslant 2$. A similar argument applied to Q_j in lieu of Q yields the inductive step in a proof of the inclusion

$$C \subset \operatorname{cl} \bigcup_{k=n}^{\infty} (|\boldsymbol{K}_k| \cup |\boldsymbol{L}_k|)$$

for $n = 1, 2, \ldots$ Consequently, we obtain

$$C \subset \bigcap_{n=1}^{\infty} \operatorname{cl} \bigcup_{k=n}^{\infty} (|\boldsymbol{K}_k| \cup |\boldsymbol{L}_k|) \subset \bigcap_{n=1}^{\infty} \operatorname{cl}[H(n-2) \cap X] = \{o\},$$

by (4), a contradiction which completes the proof of 4.1.

4.2. THEOREM. *Each rational curve containing no arc is embedable in a strictly radial hereditarily decomposable curve.*

Proof. The proof is analogous to that of 4.1. We only point out the places in which it should be modified. Let Z be a rational curve containing no arc. The construction of the radial curve X in 4.1 makes use of some standard continuous mappings of the Cantor ternary set T onto S and J_j. What is essentially needed is only the requirement that the inverses of points under those mappings either consist of two points or are degenerate, depending on whether the point belongs to a countable set or to its complement, respectively. By 2.3, we can replace all the sets S and J_j in the above construction by topological copies of Z. As a result we get again a radial curve X, and the proof of the hereditary decomposability of X is a replica of that from 4.1. Moreover, the fact that Z contains no arc now guarantees that each arc contained in Y must be contained in a half-line $K \in \boldsymbol{K}_n$ where $n = 1, 2, \ldots$ It follows that each arc contained in X is contained in the union of two arcs from $\boldsymbol{A}$, the collection defined in the same manner as in 4.1. Thus X now appears to be strictly radial.

4.3. EXAMPLE. *There exists a strictly radial hereditarily decomposable curve X such that X is not a dendroid.*

Proof. Let Z be the Janiszewski curve or a curve constructed by O. V. Lokucievskiĭ [12]. Then Z is a rational curve containing no arc. By 4.2, the curve Z can be embedded into a strictly radial hereditarily decomposable curve X. Since each subcurve of a dendroid is a dendroid, X and Z are not dendroids.

Remark. According to 3.1, the radial curve described in 4.1 is not strictly radial. By 3.1 and 4.3, there exist arcwise connected hereditarily decomposable curves which are not dendroids but still have the property that the union of an increasing sequence of arcs is contained in an arc; this answers some questions of J. J. Charatonik [5].

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Subdirect decomposition of distributive quasilattices

by

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Following Płonka [3], we define a *quasilattice* to be a nonempty set with binary operations $\wedge$ and $\vee$ which are idempotent, commutative, and associative, and a *distributive* quasilattice to be one which obeys the laws

$$x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)\,, \qquad x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)\,.$$

It is easily checked that the tables

$\wedge$	0	1	∞
0	0	0	∞
1	0	1	∞
∞	∞	∞	∞

$\vee$	0	1	∞
0	0	1	∞
1	1	1	∞
∞	∞	∞	∞

define a distributive quasilattice, $\mathfrak{X}$ say. Let $\mathfrak{L}$ and $\mathfrak{S}$ be the sub-quasilattices of $\mathfrak{X}$ with underlying sets $\{0,1\}$ and $\{0,\infty\}$ respectively; $\mathfrak{L}$ is a lattice, and $\mathfrak{S}$ is essentially a semilattice (it obeys the law $x\wedge y = x\vee y$). The object of this paper is to prove the following

THEOREM. *A distributive quasilattice with more than one element is isomorphic to a subdirect product of copies of* $\mathfrak{X}$, $\mathfrak{L}$, *and* $\mathfrak{S}$.

This extends Birkhoff's subdirect decomposition theorem for distributive lattices ([1], p. 193, Theorem 15, Corollary 1), and also contains a similar theorem for semilattices.

In any quasilattice an identity element for $\wedge$ (resp. $\vee$), if it exists, is unique, and will be denoted by I (resp. O) (cf. [1], p. 63, ex. 7, and [2], but note that the free distributive quasilattice with O, I, and one generator has five, not seven, elements).

LEMMA 1. *Let Q be a distributive quasilattice with O and I. Then, for all x and y in Q,*

(i) $x\wedge O = O$ *if and only if* $x\vee I = I$;
(ii) $x\wedge y = I$ *if and only if* $x = y = I$; *and*
(iii) $x\wedge y\wedge O = O$ *if and only if* $x\wedge O = y\wedge O = O$.