3. Lipschitz pairs and continuous pairs. In [2] two subspaces A and B of a topological space X are defined to form a *continuous pair* if every function continuous on each is necessarily continuous on their union. A characterization of such pairs is given in [2]. This characterization reduces to $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ if $A \cap B = \emptyset$ and shows that it is sufficient for A and B to be closed. With these remarks we can compare Lipschitz pairs and continuous pairs of metric subspaces.

The sets A_1 and B_1 of the introduction are closed. Hence disjoint continuous pairs need not be Lipschitz pairs.

The sets A_2 and B_2 of the introduction are closed. Hence intersecting continuous pairs need not be Lipschitz pairs.

If A and B form a disjoint Lipschitz pair then they are bounded apart, $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and they therefore form a continuous pair.

However, an intersecting Lipschitz pair need not be a continuous pair. For example, let A be a closed square in the plane less one corner and let B be a side of the square including the missing corner. Then condition (d) (ii) shows that A and B form a Lipschitz pair. But if the square is set in the positive quadrant of the (x, y) plane with the missing corner at the origin, the function $\tan^{-1}(y/x)$ shows that A and B do not form a continuous pair.

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On restrictive semigroups of continuous functions

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1. Introduction and statement of main theorem. Let X be a topological space and let Y be a nonempty subspace of X. The semigroup, under composition, of all continuous selfmaps of X which also carry Y into Y will be referred to as a *restrictive semigroup of continuous functions* and will be denoted by $S(X, Y)$. In case $Y = X$, we use the simpler notation $S(X)$ in place so $S(X, X)$. Such semigroups have been investigated in [4], [7] and [8] and restrictive semigroups of closed functions have been studied in [6]. A function is regarded in [6] as closed if it takes closed subsets into closed subsets. In particular, continuity is not assumed. Other related semigroups have been studied in [9]. Our main purpose here is to prove a result about restrictive semigroups of continuous functions which is somewhat analogous to Theorem (2.17) of [6, p. 1222] and Theorem (3.8) of [9]. Before stating this result, we need to recall the definition of an S^*-space [5]. An S^*-space is any T_1 space X with the property that for each closed subset H of X and each point $p \in X - H$, there exists a continuous selfmap f of X and a point q in X such that $f(x) = q$ for each $x \in H$ and $f(p) \neq q$.

One readily shows that a space X is an S^*-space if and only if it is T_1 and the point-inverses of X (sets of the form $f^{-1}(x)$ where $x \in X$ and f is a continuous selfmap) form a basis for the closed subsets of X. The class of S^*-spaces is rather extensive. For example, Theorems 2 and 3 of [5, p. 296] taken together yield the fact that every 0-dimensional Hausdorff space as well as every completely regular Hausdorff space which contains an arc is an S^*-space. In this paper a 0-dimensional space is one which has a basis of sets which are both closed and open. Also, let us recall that a space is Lindelöf if every open cover has a countable subcover and it is hereditarily Lindelöf if each subspace is Lindelöf.

It is immediate from the previous discussion that if X is an S^*-space and one takes $Y = X$, then there exist S^*-spaces Z such that $S(X, Y)$

and $S(Z)$ are isomorphic. One need only take Z to be any homeomorphic copy of Y. Our main result states that if X happens to be a first countable (each point has a countable base) 0-dimensional hereditarily Lindelöf Hausdorff space, then these are the only circumstances under which $S(X, Y)$ can be isomorphic to $S(Z)$ for some S^*-space Z. We formally state this as the

MAIN THEOREM. *Let X be a first countable 0-dimensional hereditarily Lindelöf Hausdorff space and let Y be a nonempty subspace of X. Furthermore, suppose there exists an S^*-space Z such that $S(X, Y)$ and $S(Z)$ are isomorphic. Then Z is homeomorphic to Y and $Y = X$.*

2. Proof of main theorem, some supporting results and some related results. Perhaps a few words are in order about the techniques which will be used here. The result on restrictive semigroups of closed functions which is analogous to the Main Theorem in this paper is, as we mentioned previously, Theorem (2.17) of [6, p. 1222]. There, the proof consisted of examining the ideal structure of the semigroups involved and repeated use was made of the fact that any function with a finite range is a closed function (all spaces were assumed to be T_1). Since this is not true for continuous functions unless X happens to be discrete, the techniques used in [6] simply do not carry over to restrictive semigroups of continuous functions. In fact, the techniques we use here are completely different. Among other things, various tools from the theory of E-compact spaces, introduced by Engleking and Mrówka in [1] and further developed by Mrówka in [10] and [11] play a very essential role here. Before discussing E-compact spaces further, we recall two definitions and a theorem from [8] and we use this result to get a "first approximation" to the proof of the Main Theorem.

DEFINITION (2.1). A permissible pair (X, Y) is a Hausdorff space X together with a subspace Y such that the following conditions are satisfied:

(2.1.1) *For every closed subset F of X and every point $p \in X-F$, there exists a function $f \in S(X, Y)$ and a point $q \in Y$ such that $f(x) = q$ for $x \in F$ and $f(p) \neq q$.*

(2.1.2) *For every quadruple p, q, r, s of points of Y with $p \neq q$, there exists a continuous function f in $S(X, Y)$ such that $f(p) = r$ and $f(q) = s$.*

DEFINITION (2.2). A subspace Y of X is *S-embedded in X* if every continuous selfmap of Y can be extended to a continuous selfmap of X.

This next result appears in [8] as Theorem (3.3).

THEOREM (2.3). *Let (X, Y) be a permissible pair and let Z be an S^*-space. Then $S(X, Y)$ and $S(Z)$ are isomorphic if and only if Z is homeomorphic to Y and Y is a dense S-embedded subset of X.*

Now, let X, Y and Z satisfy the hypothesis of the Main Theorem. We first show that *the conclusion follows quickly if Y consists of one point* which we denote by p. In this case, $S(X, Y)$ has a two-sided zero, namely, the constant function which maps everything into p. Since $S(X, Y)$ is isomorphic to $S(Z)$, the semigroup $S(Z)$ must also have a two-sided zero and this can happen only when Z consists precisely of one point. Otherwise, $S(Z)$ contains at least two distinct constant functions and these are both left zeros. But then, $S(Z)$ must be the one-element semigroup. It follows that $Y = X$ and X consists of one point for an assumption to the contrary results in the contradiction that $S(X, Y)$ has at least two distinct elements, the identity function and the constant function which maps everything into the point p.

Therefore, we need only devote our attention to the case where Y contains more than one point. First of all, it is not difficult to show that the pair (X, Y) is permissible in this case. In fact, this follows from Proposition (2.7) of [8] which states that if X is a 0-dimensional Hausdorff space and Y is any subset containing more than one point, then the pair (X, Y) is permissible. Thus, Theorem (2.3) can be applied and we get the "first approximation" to the proof which we spoke of earlier.

LEMMA (2.4). *Let X, Y, Z, $S(X, Y)$ and $S(Z)$ satisfy the hypothesis of the Main Theorem and, in addition, let Y have more than one point. Then Z is homeomorphic to Y and Y is a dense S-embedded subset of X.*

It is now evident that our task is to show that the only dense S-embedded subset of X is X itself and this is where the theory of E-compact spaces is used. It is appropriate at this point to mention that a comprehensive treatment of this topic is given in [11]. We recall some definitions and results which are needed here.

In what follows, the symbol $\mathcal{R}$ will be used to denote the space of real numbers, $\mathcal{I}$ will denote the closed unit interval, $\mathcal{N}$ will denote the countably infinite discrete space and $\mathcal{D}$ will denote the two-element discrete space. Let E be any Hausdorff space. A space is defined in [11, p. 161] to be E-completely regular if it is homeomorphic to a subspace of some power of E and it is E-compact if it is homeomorphic to a closed subspace of some power of E. One easily sees that a space is completely regular in the usual sense if and only if it is $\mathcal{R}$-completely regular or, equivalently, $\mathcal{I}$-completely regular. Furthermore, a space is compact in the usual sense if and only if it is $\mathcal{I}$-compact and it is realcompact if and only if it is $\mathcal{R}$-compact. For an extensive treatment of realcompact spaces, one should consult [2]. In the case of the space $\mathcal{D}$, it follows rather easily from Theorem 2.1 of [11, p. 165] that the $\mathcal{D}$-completely regular spaces are precisely the 0-dimensional spaces and the $\mathcal{D}$-compact spaces are the compact 0-dimensional spaces. It also follows in a similar manner

that the $\mathcal{N}$-completely regular spaces are precisely the 0-dimensional spaces. We will also need the following result from [10, 2.1, p. 598] which we state here as

THEOREM (2.5) (Mrówka). *Every 0-dimensional Lindelöf Hausdorff space is $\mathcal{N}$-compact.*

In general, we will refer to a space Z as an E-compactification of X if it is E-compact and contains X as a dense subspace. We note that this forces an E-compactification to be Hausdorff since E is understood to be Hausdorff. We need the following important result from the theory of E-compact spaces [1, Theorem 4, p. 433], [11, Theorem 4.14, p. 177].

THEOREM (2.6) (Engleking and Mrówka). *Every E-completely regular space X has an E-compactification $\beta_E X$ which satisfies the following two conditions:*

(2.6.1) *Every continuous function mapping X into E can be extended to a continuous function which maps $\beta_E X$ into E.*

(2.6.2) *Every continuous function mapping X into an E-compact spaces Y can be extended to a continuous function which mapps $\beta_E X$ into Y.*

Furthermore, $\beta_E X$ is unique in the sense that if $\alpha_E X$ is any E-compactification satisfying either of the two conditions, then there exists a homeomorphism from $\alpha_E X$ into $\beta_E X$ whose restriction to X is the identity map.

When there exists a homeomorphism between two E-compactifications $\alpha_E X$ and $\gamma_E X$ of X which is the identity when restricted X, we regard them as equivalent and do not distinguish between them. If there exists a continuous function mapping $\gamma_E X$ onto $\alpha_E X$ which is the identity on X, we write $\alpha_E X \leqslant \gamma_E X$. It follows easily from (2.6.2) that if E is compact in the usual sense, then $\alpha_E X \leqslant \beta_E X$ for each E-compactification $\alpha_E X$ of X.

Our next result relates the concept of S-embeddedness to spaces of the form $\beta_E X$.

THEOREM (2.7). *Let X be an E-compact space and let Y be any subspace containing a copy of E which is closed in X. Then Y is a dense S-embedded subspace of X if and only if $X = \beta_E Y$.*

Proof. First of all, if $X = \beta_E Y$, it follows from Theorem (2.6) that every continuous function mapping Y into Y can be extended to a continuous function which maps X into X.

Suppose, on the other hand, that Y is dense in X and S-embedded as well. To prove that $X = \beta_E Y$, it is sufficient, according to Theorem (2.6) to show that if f is any continuous function from Y into E, then f can be continuously extended to a function which maps X into E. By hypothesis, there exists a homeomorphism k from E onto a subset H of Y which is closed in X. Then $k \circ f$ is a continuous mapping from Y into Y and since Y is S-embedded in X, $k \circ f$ has a continuous extension g which maps X into X. Since H is closed in X, we have

$$g[X] = g[\mathrm{cl}_X Y] \subset \mathrm{cl}_X g[Y] = \mathrm{cl}_X k \circ f[Y] \subset H .$$

That is, the range of g is a subset of H. It follows that $k^{-1} \circ g$ is a continuous extension of f which maps X into E. Consequently, X must be the space $\beta_E Y$ which is described in Theorem (2.6).

We digress for a bit in order to mention that the following two corollaries which appear in [8] as Propositions (4.2) and (4.3) are immediate consequences of Theorem (2.7) and the observations proceeding Theorem (2.5).

COROLLARY (2.8). *Let X be a compact space and let Y be a subspace which contains an arc. Then Y is a dense S-embedded subspace of X if and only if X is the Stone–Čech compactification of Y.*

COROLLARY (2.9). *Let X be a realcompact space and suppose Y is a subspace of X which contains a copy of the real line which is closed in X. Then Y is a dense S-embedded subspace of X if and only if X is the Hewitt realcompactification of Y.*

At this point, we have collected most of the formal machinery we need for completing the proof of the Main Theorem. We consider two cases depending upon whether or not X is compact. Since we have taken care of the case where Y consists of one point, we may also assume that Y has at least two points, that is Y contains a copy of the space $\mathfrak{D}$. By Lemma (2.4), Y is dense and S-embedded in X. It therefore follows from Theorem (2.7) and the observations preceeding Theorem (2.5) that $X = \beta_{\mathfrak{D}} Y$. Now $\beta_{\mathfrak{D}} Y$ is a compactification of Y in the usual sense and by a well known property of βY the Stone–Čech compactification of Y, we have $\beta_{\mathfrak{D}} Y \leqslant \beta Y$. Although the definition of dimension zero given in [2, p. 246] differs from that given here, the two definitions do coincide for Lindelöf spaces [2, Theorem 16.17, p. 247] and, of course, X is hereditarily Lindelöf. Thus, it follows from Theorem 16.11 of [2, p. 245] that βY is 0-dimensional. Consequently, βY is a $\mathfrak{D}$-compactification of Y and, as we observed in the discussion following Theorem (2.6), we have $\beta Y \leqslant \beta_{\mathfrak{D}} Y$. Therefore $X = \beta_{\mathfrak{D}} Y$ is, in fact, the Stone–Čech compactification of Y. But no point of $\beta Y - Y = X - Y$ has a countable base [2, 9.7, p. 123] and since X is a first countable space, it follows that Y must coincide with X.

Now we take care of the remaining case where X is not compact. Since every Lindelöf space is realcompact, it follows that X is not pseudocompact [2, 5H-2, p. 79]. Thus, there exists an unbounded continuous function f mapping X into the real line. There is no loss in generality

in assuming that f takes on only positive values. Choose any $y_1 \in Y$ and let

$$G_2 = \{r \in \mathcal{R}: r > f(y_1)+1\}.$$

Since f is unbounded, $f^{-1}[G_2] \neq \emptyset$ and since Y is dense in X, there exists a point $y_2 \in Y \cap f^{-1}[G_2]$. Similarly, let

$$G_3 = \{r \in \mathcal{R}: r > f(y_2)+1\}$$

and choose a point $y_3 \in Y \cap f^{-1}[G_3]$. By continuing in this manner, we get an infinite sequence $\{y_n\}_{n=1}^{\infty}$ of distinct points of Y with the property that $\{f(y_n)\}_{n=1}^{\infty}$ is a strictly increasing sequence of real numbers, such that $f(y_{n+1})-f(y_n) > 1$ for each positive integer n. It follows from this that the subset $\{y_n\}_{n=1}^{\infty}$ of Y has no limit points in X. Thus, Y contains a copy of $\mathcal{N}$ which is closed in X. By Theorem (2.5), X is an $\mathcal{N}$-compactification of Y and since Y is S-embedded in X, Theorem (2.7) implies that $X = \beta_{\mathcal{N}} Y$. As we noted in the discussion preceeding Theorem (2.5), the class of $\mathfrak{D}$-completely regular spaces is identical to the class of $\mathcal{N}$-completely regular spaces. They are all 0-dimensional spaces. Therefore, since $\mathfrak{D}$ is $\mathcal{N}$-compact, it follows from Theorem 4.17 of [11, p. 178] that $X = \beta_{\mathcal{N}} Y$ can actually be regarded as a subspace of $\beta_{\mathfrak{D}} Y$ which, by the same reasons that were given in the previous case, coincides with βY. Thus, we have $Y \subset X \subset \beta Y$. Again, it follows from the fact that no point of $\beta Y - Y$ has a countable base that Y must coincide with X. Therefore Y must coincide with X in both cases and this fact, together with Lemma (2.4) proves the theorem.

We conclude this paper with an example that shows that it is not possible to prove the theorem without making some restrictions on the spaces involved. De Groot in [3] has proven the existence of a proper dense subspace Y of the Euclidean plane W such that the only continuous selfmaps of Y are the constant maps and the identity map. Although, $Y \neq W$, it follows readily that $S(W, Y)$ is isomorphic to $S(Y)$. We note that these spaces fail to satisfy the hypothesis of the Main Theorem in that W is not 0-dimensional and Y is not an S^*-space. The remainder of the hypothesis is satisfied.

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