Lipschitz pairs of metric subspaces

by

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1. Introduction. Let A and B be two subspaces of the metric space X. Then A and B are called a *Lipschitz pair* if every function Lipschitz on each is necessarily Lipschitz on their union.

Before characterizing Lipschitz pairs let us examine two examples which illustrate how the property can fail. The real-valued function

$$f(x, y) = \begin{cases} x & \text{if} \quad y \geqslant 0, \\ -x & \text{if} \quad y < 0 \end{cases}$$

defined on the Euclidean plane shows that the pairs of subspaces $A_1 = \{(x, y)\colon\ y = 1\}$ and $B_1 = \{(x, y)\colon\ y = -1\}$ and $A_2 = \{(x, y)\colon\ y = x^2\}$ and $B_2 = \{(x, y)\colon\ y = -x^2\}$ both fail to be Lipschitz pairs. The characterization will reveal that both these pairs of subspaces are "too tangential" to be Lipschitz pairs.

To simplify notation, the distance between two points x_1 and x_2 in the domain space X is denoted by x_1x_2. Thus a function $f\colon X \to Y$ is Lipschitz if there exists a constant $K > 0$ such that, for any two points x_1 and x_2 in X, $d\big(f(x_1), f(x_2)\big) \leqslant K x_1 x_2$.

2. Characterization.

THEOREM. *Let X be a metric space and A, B be non-empty subspaces of X. Then the following conditions are equivalent to each other.*

(a) *If Y is any metric space and f any Y-valued function on X, then, if $f|A$ and $f|B$ are Lipschitz, so also is $f|A \cup B$.*

(b) *If f is any real-valued function on X and if $f|A$ and $f|B$ are Lipschitz, so also is $f|A \cup B$.*

(c) Case (i): $A \cap B = \emptyset$. *To every pair of points, a_0 in A and b_0 in B, there corresponds a real number $K = K(a_0, b_0) > 0$ such that for any a in A and b in B,*

$$aa_0 + bb_0 \leqslant K ab\,.$$

Case (ii): $A \cap B \neq \emptyset$. *There is a real number $K > 0$ such that for any a in A and b in B,*

$$a(A \cap B) + b(A \cap B) \leqslant Kab .$$

(d) Case (i): $A \cap B = \emptyset$. *There are points a_0 in A and b_0 in B and a real number $D > 0$ such that for any a in A and b in B*

$$aa_0 + bb_0 \leqslant Dab .$$

Case (ii): $A \cap B \neq \emptyset$. *There is a real number $D > 0$ such that for any a in A and b in B there is a point $c = c(a, b)$ in $A \cap B$ such that*

$$ac + bc \leqslant Dab .$$

Proof. Condition (b) is a special case of condition (a). The proof will be completed by showing that (b)⇒(c)⇒(d)⇒(a).

The proof that (b)⇒(c) is divided into two cases. In case (i) $A \cap B = \emptyset$. Let a_0 be any point in A, b_0 any point in B and define

$$f(x) = \begin{cases} xa_0 & \text{if} \quad x \in A, \\ -xb_0 & \text{if} \quad x \in B, \\ 0 & \text{otherwise}. \end{cases}$$

The functions $f|A$ and $f|B$ are Lipschitz since ([1], p. 120, Theorem 8), if $\emptyset \neq C \subset X$, the real-valued function $h(x) = xC$ is Lipschitz. It follows from condition (b) that f is Lipschitz on $A \cup B$ with some constant K. Hence, for arbitrary a in A and b in B,

$$|f(a) - f(b)| \leqslant Kab .$$

That is,

$$aa_0 + bb_0 \leqslant Kab .$$

In case (ii) $A \cap B |= \emptyset$. After defining

$$f(x) = \begin{cases} x(A \cap B) & \text{if} \quad x \in A, \\ -x(A \cap B) & \text{if} \quad x \in B, \\ 0 & \text{otherwise} \end{cases}$$

the proof continues exactly as in case (i).

The fact that (c)⇒(d) is obvious in case (i). To show that it also holds in case (ii), let a in A and b in B be arbitrary points. If $a = b$, the choice $c = a = b$ gives the condition for arbitrary D. Otherwise $ab \neq 0$ and there exist points c and d in $A \cap B$ satisfying

$$ac \leqslant a(A \cap B) + \tfrac{1}{2}ab \quad \text{and} \quad bd \leqslant b(A \cap B) + \tfrac{1}{2}ab .$$

Adding these inequalities and imposing condition (c), we obtain

$$ac + bd = a(A \cap B) + b(A \cap B) + ab \leqslant (K+1)ab .$$

With the help of the triangle inequality, we obtain successively

$$cd \leqslant ca + ab + bd \leqslant (K+2)ab$$

and

$$ac + cb \leqslant ac + cd + db \leqslant (2K+3)ab .$$

The last inequality is condition (d) with $D = 2K+3$.

Finally let us show that (d)⇒(a). Suppose $f\colon X \to Y$ is Lipschitz on A with constant K_A and Lipschitz on B with constant K_B. It is sufficient to produce a constant K such that for arbitrary a in A and b in B, $d(f(a), f(b)) \leqslant Kab$. For then $\max(K_A, K_B, K)$ will serve as a Lipschitz constant for f on all of $A \cup B$.

In case (i) with $A \cap B = \emptyset$ the first step is to show that A and B are bounded apart. To this end, let $a_0 b_0 = D_0(D+1)$. Then

$$0 < D_0(D+1) = a_0 b_0 \leqslant a_0 a + ab + bb_0 \leqslant (D+1)ab$$

and the distance between A and B is not less than D_0. Now, using the triangle inequality in Y, the Lipschitz properties of f and the metric property of A and B, we estimate

$$\begin{aligned} d(f(a), f(b)) &\leqslant d(f(a), f(a_0)) + d(f(a_0), f(b_0)) + d(f(b_0), f(b)) \\ &\leqslant K_A aa_0 + D_f + K_B b_0 b \\ &\leqslant \max(K_A, K_B)[aa_0 + bb_0] + D_f ab/D_0 \\ &\leqslant [D\max(K_A, K_B) + D_f/D_0]ab . \end{aligned}$$

The proof for case (ii) is even easier. The point c of the hypothesis and the triangle inequality in Y give

$$d(f(a), f(b)) \leqslant d(f(a), f(c)) + d(f(c), f(b)) .$$

Then the Lipschitz properties of f together with the metric property of A and B yield

$$\begin{aligned} d(f(a), f(b)) &\leqslant K_A ac + K_B cb \\ &\leqslant \max(K_A, K_B)[ac + bc] \\ &\leqslant D\max(K_A, K_B)ab . \end{aligned}$$

This completes the proof of the theorem.

If one of the sets, say A, has a finite diameter D_A, then condition (c) case (i) is equivalent to the simpler condition that the sets be bounded apart by some distance D_0. Indeed then

$$aa_0 + bb_0 \leqslant aa_0 + ba + aa_0 + a_0 b_0 \leqslant ba + 2D_A + a_0 b_0 .$$

Since $ab \geqslant D_0$, the constant $K = 1 + (2D_A + a_0 b_0)/D_0$ makes $aa_0 + bb_0 \leqslant Kab$. The pair A_1 and B_1 of the introduction show that "bounded apart" is not sufficient for arbitrary pairs of metric subspaces to form Lipschitz pairs.

3. Lipschitz pairs and continuous pairs. In [2] two subspaces A and B of a topological space X are defined to form a *continuous pair* if every function continuous on each is necessarily continuous on their union. A characterization of such pairs is given in [2]. This characterization reduces to $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ if $A \cap B = \emptyset$ and shows that it is sufficient for A and B to be closed. With these remarks we can compare Lipschitz pairs and continuous pairs of metric subspaces.

The sets A_1 and B_1 of the introduction are closed. Hence disjoint continuous pairs need not be Lipschitz pairs.

The sets A_2 and B_2 of the introduction are closed. Hence intersecting continuous pairs need not be Lipschitz pairs.

If A and B form a disjoint Lipschitz pair then they are bounded apart, $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and they therefore form a continuous pair.

However, an intersecting Lipschitz pair need not be a continuous pair. For example, let A be a closed square in the plane less one corner and let B be a side of the square including the missing corner. Then condition (d) (ii) shows that A and B form a Lipschitz pair. But if the square is set in the positive quadrant of the (x, y) plane with the missing corner at the origin, the function $\tan^{-1}(y/x)$ shows that A and B do not form a continuous pair.

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References

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On restrictive semigroups of continuous functions

by

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1. Introduction and statement of main theorem. Let X be a topological space and let Y be a nonempty subspace of X. The semigroup, under composition, of all continuous selfmaps of X which also carry Y into Y will be referred to as a *restrictive semigroup of continuous functions* and will be denoted by $S(X, Y)$. In case $Y = X$, we use the simpler notation $S(X)$ in place so $S(X, X)$. Such semigroups have been investigated in [4], [7] and [8] and restrictive semigroups of closed functions have been studied in [6]. A function is regarded in [6] as closed if it takes closed subsets into closed subsets. In particular, continuity is not assumed. Other related semigroups have been studied in [9]. Our main purpose here is to prove a result about restrictive semigroups of continuous functions which is somewhat analogous to Theorem (2.17) of [6, p. 1222] and Theorem (3.8) of [9]. Before stating this result, we need to recall the definition of an S^*-space [5]. An S^*-space is any T_1 space X with the property that for each closed subset H of X and each point $p \in X - H$, there exists a continuous selfmap f of X and a point q in X such that $f(x) = q$ for each $x \in H$ and $f(p) \neq q$.

One readily shows that a space X is an S^*-space if and only if it is T_1 and the point-inverses of X (sets of the form $f^{-1}(x)$ where $x \in X$ and f is a continuous selfmap) form a basis for the closed subsets of X. The class of S^*-spaces is rather extensive. For example, Theorems 2 and 3 of [5, p. 296] taken together yield the fact that every 0-dimensional Hausdorff space as well as every completely regular Hausdorff space which contains an arc is an S^*-space. In this paper a 0-dimensional space is one which has a basis of sets which are both closed and open. Also, let us recall that a space is Lindelöf if every open cover has a countable subcover and it is hereditarily Lindelöf if each subspace is Lindelöf.

It is immediate from the previous discussion that if X is an S^*-space and one takes $Y = X$, then there exist S^*-spaces Z such that $S(X, Y)$