algebra with only one unary operation. On the other hand, there exists such an algebra with only one binary operation, which we now define.

Let $A = N \cup G \cup \{a, b\}$ be as above, and again take f and g to be the two components of a bijection from N onto R. We define

$$F(x, y) = \begin{cases} f(x) & \text{if } x \in N \text{ and } y = a; \\ g(y) & \text{if } y \in N \text{ and } x = a; \\ a & \text{if } x \in G \text{ and } y \in G; \\ b & \text{otherwise}. \end{cases}$$

The proof that $\langle A, F\rangle$ is atomic-compact is very similar to the above proof that $\langle A, f, g\rangle$ is atomic-compact, and will therefore be omitted. To see that $\langle A, F\rangle$ is not a retract of a compact topological relational structure, it suffices to show that $\chi(\varphi, \langle A, F\rangle) = \aleph_0$, where

$$\varphi = \exists x \exists y [F(x, x) = a \wedge F(y, a) = x \wedge F(a, y) = x].$$

This is shown by an argument similar to the preceeding one.

Added in proof (February 19, 1971). I stated Problem 1.14 (for graphs), together with a related problem in pure graph theory, as Problem 43 in *Combinatorial Structures and their Applications*, Gordon and Breach, New York, 1970. My note, *Generalized chromatic numbers*, in the same volume, gives some further information on chromatic numbers.

For a syntactic condition equivalent to the conditions of Theorem 3.1 [or of Corollary 5.1], see my paper with G. Fuhrken, *Weakly atomic-compact relational structures*, to appear in J. Symbolic Logic.

I give a positive answer to the question asked here in the last sentence of § 5, in my paper, *Some constructions of compact algebras*, to appear in Annals of Mathematical Logic.

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Retracting fans onto finite fans

by

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1. Introduction. By a *continuum* we mean a compact connected metric space. A continuum which is hereditarily unicoherent and arcwise connected is a *dendroid*. A dendroid which has only one ramification point (a point which is the common part of 3 arcs, and an end point of each) is called a *fan*. A locally connected dendroid is called a *tree* or *dendrite*. A dendroid is *finite* if the set of end points is finite. Clearly, each finite dendroid is a tree and finite fans are the union of a finite collection of arcs, whose common part is a single point. The cone over the Cantor set, on $[0, 1]$ is a planar fan.

It is easy to see that dendroids are hereditarily decomposable and thus one-dimensional. In this paper we will establish that fans have a very strong one-dimensional structure, namely, they can be approximated from within by finite fans. This is the content of Theorem 1, which states that each fan can be retracted onto a finite fan, by a map which does not move points very far. From this it follows that each fan is tree-chainable, indeed is an inverse limit of finite fans, and (in a joint work with C. A. Eberhart) that the product of any collection of fans has the fixed point property.

2. Preliminary results. A *chain*, in a metric space, is a collection $\mathcal{E} = \{E_1, \ldots, E_m\}$ of open sets such that $E_i \cap E_j \neq \emptyset$ iff $|i-j| \leqslant 1$. The elements of $\mathcal{E}$ are *links*; frequently we denote $\mathcal{E}$ by $E(1, m)$ and denote $\bigcup\{E_i\colon 1 \leqslant i \leqslant m\}$ by $E^*(1, m)$ or $\mathcal{E}^*$. If each link of $\mathcal{E}$ has diameter $< \varepsilon$, we call $\mathcal{E}$ an *ε-chain*. A *tree chain* is a finite collection of open sets, no three of which have a point in common and the collection contains no circular chains. We shall often use Z^n to denote the first n positive integers.

The ramification point of a fan is called the *top*. It is shown in [1] that each point of a fan, except the top, lies on a unique arc from the top to an end point. We wish to commence our proof of Theorem 1 by covering each such arc by a chain in which the arc is straight.

DEFINITION. If $[a, b]$ is a an arc and $\mathcal{E} = E(1, m)$ is a chain covering $[a, b]$ then $[a, b]$ is *straight* in $\mathcal{E}$ provided

1. $\mathcal{E}$ is a chain from a to b i.e. $a \in E_1 - \mathrm{Cl}\, E_2$, $b \in E_m - \mathrm{Cl}\, E_{m-1}$,

2. $(\partial E_i) \cap [a, b]$ is a one point set if $i = 1$ or $i = m$ and a two-point set otherwise.

(If $\mathcal{E}$ has only one link, any arc covered by $\mathcal{E}$ is straight in $\mathcal{E}$.)

LEMMA 1. *Suppose X is a dendroid and Y is a finite tree, $Y \subset X$ and $p \in Y$. Let $\mathcal{K} = \{K$: K is a component of $Y - \{p\}\}$. Then for each open set U such that $p \in U$, there is an open set V such that $p \in V \subset U$, and* $\mathrm{card}(Y \cap \partial V) = \mathrm{card}\, \mathcal{K}$.

Proof. $\mathcal{K}$ is a finite set, since each component of $Y - \{p\}$ contains an end point of Y. This follows from the fact that if $K \in \mathcal{K}$, then K is arcwise connected, because Y is locally connected. The end points of Y are precisely the end points of maximal arcs in Y. Since $K \cup \{p\}$ is a tree and K is arcwise connected, then if A is a maximal arc in $K \cup \{p\}$, at least one end point of A is an end point of Y.

Suppose $\mathcal{K} = \{K_1, \ldots, K_n\}$. According to [4], p. 88 there is an set V', open in Y such that $p \in V' \subset U$, and $\partial_Y V'$, the boundary of V' relative to Y, contains exactly n points. Now V' must be connected, since if V'' is the component of V' containing p, then V'' is open in Y and $\partial_Y V'' \subset \partial_Y V'$. Since we may assume that for each i, $K_i \not\subset \mathrm{Cl}\, U$, $\partial_Y V''$ contains a point from each K_i. Since $\partial_Y V'$ contains only n points, $V' = V''$.

Thus $Y - \partial_Y V'$ is the union of two separated sets, one of which is V' and the other contains $Y - U$. There are disjoint sets S and T, open in X, such that $V' \subset S$ and $Y - U \subset T$. Now let $V = U - \mathrm{Cl}\, T$. Then $(\partial V) \cap Y = (\partial T) \cap Y = \partial_Y V'$, an n-point set.

LEMMA 2. *Suppose $[a, b]$ is straight in $\mathcal{E} = E(1, m)$ and W is an open set containing $[a, b]$. Then $[a, b]$ is straight in $\{E_1 \cap W, E_2 \cap W, \ldots, E_m \cap W\}$.*

Proof. It is clear from the definition of straightness that for each i, $\partial(E_i \cap W)$ contains at least as many points of $[a, b]$ as ∂E_i does. Conversely since $\partial(E_i \cap W) \subset (\partial E_i) \cup (\partial W)$ and $[a, b] \subset W$, $\big(\partial(E_i \cap W)\big) \cap [a, b] \subset (\partial E_i) \cap [a, b]$. Thus $\partial(E_i \cap W)$ contains exactly as many points of $[a, b]$ as ∂E_i does. That is, $[a, b]$ is straight in $\{E_1 \cap W, \ldots, E_m \cap W\}$.

We now show that each arc in a dendroid can be covered by chains in which the arc is straight.

PROPOSITION 1. *If $[a, b]$ is an arc in a dendroid X and $\varepsilon > 0$ then there is an ε-chain $\mathcal{E} = E(1, m)$ of sets open in X such that $[a, b]$ straight in $\mathcal{E}$.*

Proof. Suppose, to the contrary, that there is an arc $[a, b]$ in X and an $\varepsilon > 0$ such that $[a, b]$ is not straight in any ε-chain of sets open in X. For this fixed ε, and fixed arc $[a, b]$, we shall say that a subarc $[a', b']$ of $[a, b]$ has *property* P iff $[a', b']$ is not straight in any ε-chain. Clearly $[a, b]$ has property P; we now show that property P is inductive.

Let L be a sequence such that, for each positive integer i, L_i has property P and $L_{i+1} \subset L_i$. We must show that $\bigcap \{L_i$: i is a positive integer$\}$ has property P. If it does not, then $\bigcap \{L_i$: is a positive integer$\}$ is not degenerate, hence it is a subarc $[c, d]$ of $[a, b]$. Since $[a, b]$ has property P, $[c, d] \neq [a, b]$; without loss of generality, we may assume that $a < c < d \leqslant b$, $<$ denoting the usual order from a to b on $[a, b]$. Since $[c, d]$ does not have property P, there is an ε-chain $\mathcal{F} = F(1, n)$ of sets open in X such that $[c, d]$ is straight in $\mathcal{F}$. Let U be an open set such that $c \in U$ and $\mathrm{Cl}\, U \subset F_1 - \mathrm{Cl}\, F_2$. According to Lemma 1, there is an open set V such that $c \in V \subset U$, and $(\partial V) \cap [c, d]$ is degenerate. Similarly, there is an open set R such that $d \in R \subset \mathrm{Cl}\, R \subset F_n - \mathrm{Cl}\, F_{n-1}$, and $(\partial R) \cap [c, d]$ is degenerate. Now $(V \cup [c, d] \cup R) \cap [a, b]$ is open in $[a, b]$ and contains $[c, d] = \bigcap \{L_i$: i is a positive integer$\}$. Hence there is an integer j such that $L_j \subset (V \cup [c, d] \cup R) \cap [a, b]$. If $L_j = [a_j, b_j]$, then we may assume that $a_j \in V$ and $b_j \in R$, since $L_j - [c, d] \subset V \cup R$. Since $V \subset F_1 - \mathrm{Cl}\, F_2$ and $R \subset F_n - \mathrm{Cl}\, F_{n-1}$, $\mathcal{F}$ is a chain from a_j to b_j covering $[a_j, b_j]$. Since, for each i, $(\partial F_i) \cap (V \cup R] = \emptyset$, $\partial F_i \cap [a_j, b_j] = \partial F_i \cap [c, d]$, which is degenerate if F_i is an end link of $\mathcal{F}$ and a two-point set otherwise. Thus $L_j = [a_j, b_j]$ is straight in $\mathcal{F}$; this is impossible, for L_j was assumed to have property P. It follows that $[c, d]$ must have property P, hence that property P is inductive. (Implicit in the above is the assumption that $\mathcal{F}$ has more than one link; if it does not, we can immediately obtain an $L_j \subset F_1$, hence L_j is straight in F_1.)

Since $[a, b]$ has property P, there is a subcontinuum of $[a, b]$ which is irreducible with respect to having property P. This subcontinuum must be non-degenerate; we shall simply assume that $[a, b]$ is irreducible with respect to having property P. Let x be a non-end point of $[a, b]$. Since $[a, x]$ and $[x, b]$ are proper subarcs of $[a, b]$, neither has property P. Hence there are ε-chains $\mathcal{G} = G(1, j)$ and $\mathcal{H} = H(1, k)$ of sets open in X such that $[a, x]$ is straight in $\mathcal{G}$ and $[x, b]$ is straight in $\mathcal{H}$.

Using regularity and Lemma 1, we obtain an open set Q such that $x \in Q \subset \mathrm{Cl}\, Q \subset (G_j - \mathrm{Cl}\, G_{j-1}) \cap (H_1 - \mathrm{Cl}\, H_2)$ and $(\partial Q) \cap [a, b]$ contains exactly two points, one in $[a, x]$, the other in $[x, b]$. (In what follows, we will assume that both $\mathcal{G}$ and $\mathcal{H}$ contain more than one link. If each has only one link, we choose Q so that $\{a, b\} \cap \mathrm{Cl}\, Q = \emptyset$ and $\mathrm{Cl}\, Q \subset G_1 \cap H_1$. If $\mathcal{G}$ has only one link and $\mathcal{H}$ has more than one, we choose Q so that $\{a, b\} \cap \mathrm{Cl}\, Q = \emptyset$ and $\mathrm{Cl}\, Q \subset G_1 \cap (H_1 - \mathrm{Cl}\, H_2)$.) Clearly $[a, x] - Q$ and $[x, b] - Q$ are disjoint closed sets. Suppose there is a continuum N in $X - Q$, which intersects each of $[a, x] - Q$ and $[x, b] - Q$. Then $N \cap [a, b]$ is contained in the union of the separated sets $[a, x] - Q$ and $[x, b] - Q$ and intersects each. Thus $N \cap [a, b]$ is not connected, and this contradicts the hereditary unicoherence of X. It follows that $X - Q$ is the union of two disjoint closed sets A and B, with $[a, x] - Q \subset A$ and $[x, b] - Q \subset B$.

Normality guarantees the existence of open sets S and T such that $A \subset S$, $B \subset T$ and $\mathrm{Cl} S \cap \mathrm{Cl} T = \emptyset$. We define chains $\mathcal{G}' = G'(1, j)$ and $\mathcal{H}' = H'(1, k)$, one-to-one refinements of $\mathcal{G}$ and $\mathcal{H}$ respectively, by $G'_i = G_i \cap (S \cup Q)$, $H'_i = H_i \cap (S \cup Q)$. Lemma 2 shows that $[a, x]$ is straight in $\mathcal{G}'$ and $[x, b]$ is straight in $\mathcal{H}'$. Since the only points in a link of $\mathcal{G}'$ and a link of $\mathcal{H}'$ are those in Q, we may define a chain $\mathcal{E} = E(1, m)$ by

$$E_i = G'_i, \text{ if } 1 \leqslant i \leqslant j; \quad E_i = H'_{i-j}, \text{ if } j+1 \leqslant i \leqslant j+k.$$

We will show that $[a, b]$ is straight in $\mathcal{E}$. Clearly $\mathcal{E}$ is an ε-chain covering $[a, b]$. Furthermore, since $a \in G'_1 - \mathrm{Cl} G'_2 = E_1 - \mathrm{Cl} E_2$ and $b \in H'_k - H'_{k-1} = E_{j+k} - \mathrm{Cl} E_{j+k-1}$, $\mathcal{E}$ is a chain from a to b. It remains to show that for each i, $1 \leqslant i \leqslant j+k$, $\partial E_i \cap [a, b]$ has the proper cardinality. If $1 \leqslant i \leqslant j-1$, then $E_i = G'_i$ and $\partial G'_i \subset \mathrm{Cl} G'_i \subset \mathrm{Cl} S$. Since $\mathrm{Cl} S \cap [x, b] = \emptyset$, $\partial G'_i \cap [a, b] = \partial G'_i \cap [a, x]$. This last set is degenerate if $i = 1$, and a two point set otherwise, since $[a, x]$ is straight in $\mathcal{G}'$. In like fashion, if $j+2 \leqslant i \leqslant j+k$, then $\partial E_i \cap [a, b] = \partial H'_i \cap [x, b]$, which has the proper cardinality, since $[a, b]$ is straight in $\mathcal{H}'$.

Now $(\partial E_j) \cap [a, b] = (\partial G'_j) \cap ([a, x] \cup [x, b]) = \big((\partial G'_j) \cap [a, x]\big) \cup \big((\partial G'_j) \cap [x, b]\big)$. Since $[a, x]$ is straight in $\mathcal{G}'$, $\partial G'_j \cap [a, x]$ is degenerate. Since $G'_j = (S \cup Q) \cap G_j = (S \cap G_j) \cup (Q \cap G_j) = (S \cap G_j) \cup Q$, $\partial G'_j \subset \big(\partial(S \cap G_j)\big) \cup \partial Q$ and hence $(\partial G'_j) \cap [x, b] \subset \Big(\big(\partial(S \cap G_j)\big) \cap [x, b]\Big) \cup \big((\partial Q) \cap [x, b]\big)$. Inasmuch as $\partial(S \cap G_j) \subset \mathrm{Cl} S$ and $\mathrm{Cl} S \cap [x, b] = \emptyset$, $\big(\partial(S \cap G_j)\big) \cap [x, b] = \emptyset$. We chose Q so that $\partial Q \cap [a, b]$ contains two points, exactly one of which is in $[x, b]$. Thus $\partial G'_j \cap [x, b] \subset (\partial Q) \cap [x, b]$, a degenerate set. Hence, $(\partial E_j) \cap [a, b]$ is contained in the union of two degenerate sets, $\partial G'_j \cap [a, x]$ and $\partial Q \cap [x, b]$. Since $\{a, b\} \cap E_j = \emptyset$, ∂E_j contains exactly two points of $[a, b]$.

If $i = j+1$, then $E_i = H'_1$ and a similar argument shows that $\partial E_i \cap [a, b]$ is a two-point set. Thus $[a, b]$ is straight in $\mathcal{E}$, and the proposition is established.

Proposition 1 shows that one can cover each arc from the top of a fan to an end point, by a chain in which the arc is straight, and a finite collection of these chains will cover the fan. However, different chains may intersect very badly. In order to cut them apart, we will need some control over the boundaries of the links. Hence we establish

PROPOSITION 2. *Suppose X is a fan, t is the top of X and W is the set of end points of X. For each $\varepsilon > 0$ and each $w \in W$, there is an ε-chain $\mathcal{E} = E(1, m)$, of sets open in X, such that $[t, w]$ is straight in $\mathcal{E}$ and $\partial E^*(2, m) \subset E_1$.*

Proof. Given $\varepsilon > 0$ and $w \in W$, we use Proposition 1 to obtain an ε-chain $\mathcal{F} = F(1, m)$ of sets open in X such that $[t, w]$ is straight in $\mathcal{F}$. We may assume that $m \geqslant 2$, since if $m = 1$, then we may take $\mathcal{E} = \mathcal{F}$.

Since $t \in F_1 - \mathrm{Cl} F_2$, we may use Lemma 1 to obtain an open set V such that $t \in V \subset \mathrm{Cl} V \subset F_1 - \mathrm{Cl} F_2$, and $\partial V \cap [t, w]$ is a singleton. Since $[t, w] \subset \mathcal{F}^*$, $[t, w] - V$ and $X - \mathcal{F}^*$ are disjoint closed subsets of $X - V$. We show that no continuum in $X - V$ intersects both of these sets. For suppose K is such a continuum. Then K contains an arc K' with these properties. Since w is an end point of X, $K' \cap [t, w] \neq \{w\}$. Since $K' \not\subset [t, w]$ and $t \notin K'$, $K' \cap [t, w]$ contains a ramification point of X distinct from t. This is impossible, hence no such continuum K exists.

It follows that $X - V$ is the union of two disjoint closed sets, one containing $[t, w] - V$, the other containing $X - \mathcal{F}^*$. There are open sets S and T such that $X - V \subset S \cup T$, $\mathrm{Cl} S \cap \mathrm{Cl} T = \emptyset$, $[t, w] - V \subset S$ and $X - \mathcal{F}^* \subset T$. Define an ε-chain $\mathcal{E} = E(1, m)$ by $E_i = F_i \cap (S \cup V)$. According to Lemma 2, $[t, w]$ is straight in $\mathcal{E}$. We shall show that $\partial E^*(2, m) \subset E_1$. Now $E^*(2, m) = F^*(2, m) \cap (S \cup V) = F^*(2, m) \cap S$, since $V \cap F^*(2, m) = \emptyset$. Thus $\partial E^*(2, m) \subset \mathrm{Cl} E^*(2, m) \subset \mathrm{Cl} S \subset S \cup V$. Moreover, since $\mathrm{Cl} E^*(2, m) \subset \mathrm{Cl} F^*(2, m) \subset X - V$, $\partial E^*(2, m) \subset S$. Since $X - \mathcal{F}^* \subset T$, $S \subset \mathcal{F}^* = F^*(1, m)$ and therefore $\partial E^*(2, m) \subset \big(S \cap F^*(2, m)\big) \cup (S \cap F_1) = E^*(2, m) \cup (S \cap F_1)$. Clearly, $E^*(2, m) \cap \partial E^*(2, m) = \emptyset$, thus $\partial E^*(2, m) \subset S \cap F_1 \subset E_1$. This concludes the proof.

Given a fan X, and $\varepsilon > 0$, we want to cover X with an ε-tree chain whose nerve is a triangulation of a finite fan. That is, we want the tree chain to look as does Figure 3. We first show that we can do this for a finite subfan Y of X.

PROPOSITION 3. *Suppose X is a fan, Y is a finite subfan of X, the top of X, t, is the top of Y and each end point w of Y, $w \neq t$, is an end point of X. If $Y = \bigcup \{[t, w_i]: i \in Z^n\}$ and $\delta > 0$, then there is a finite collection $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ such that*

(i) *each $\mathcal{F}_j = F_j(1, r_j) = \{F_{j1}, F_{j2}, \ldots, F_{jr_j}\}$ is a δ-chain with at least 3 links,*

(ii) *for each j, $[t, w_j]$ is straight in $\mathcal{F}_j$,*

(iii) *for each J, $\partial F^*_j(2, r_j) \subset F_{j1}$,*

(iv) *for each j, $F_{j1} = F_{11}$,*

(v) *if $i \neq j$ then $\big([t, w_j] \cup F^*_j(2, r_j)\big) \cap \mathrm{Cl} F^*_i(2, r_i) = \emptyset$.*

Proof. We proceed by induction on n. If $n = 1$ then Y is an arc and Proposition 2 yields the desired chain. Suppose then that the conclusion holds for $n \leqslant k-1$ and Y has k end points, $w_1, \ldots, w_k$. Let $Y' = \bigcup \{[t, w_i]: 1 \leqslant i \leqslant k-1\}$. Clearly Y' is a finite fan satisfying the induction hypothesis and $Y = Y' \cup [t, w_k]$. We obtain a collection $\mathcal{E}_1, \ldots, \mathcal{E}_{k-1}$ of δ-chains which cover Y' and satisfy (i)–(v).

Lemma 1 yields an open set W such that $t \in W$, and ∂W contains exactly k points of Y, one from each arc $[t, w_i]$. Let $[t, w_k] \cap \partial W = \{s\}$. Then $[t, w_k] = [t, s) \cup [s, w_k]$, $[t, s) \subset W$ and $[t, w_k] \cap \mathrm{Cl} W = [t, s]$. We may assume that W was chosen so small that $[t, w_k] \not\subset \mathrm{Cl} W$ and $\mathrm{Cl} W \cap \mathrm{Cl} E_i^*(2, r_i) = \emptyset$, if $1 \leqslant i \leqslant k-1$. We apply Proposition 2 to obtain an δ-chain $\mathcal{G} = G(1, p)$ such that $[t, w_k]$ is straight in $\mathcal{G}$ and $\partial G^*(2, p) \subset G_1$. Let $G(b, p)$ denote the minimal subchain of $\mathcal{G}$ containing $[s, w_k]$. We may assume, without loss of generality, that the mesh of $\mathcal{G}$ was chosen so small that $G(b, p)$ has at least 3 links and that if $1 \leqslant i \leqslant k$ and $G_j \cap W \neq \emptyset$, then $\mathrm{Cl} G_j \cap \mathrm{Cl} E_i^*(2, r_i) = \emptyset$. Since $[s, w_k]$ and Y' are disjoint closed sets, we may further assume that $Y' \cap \mathrm{Cl} G^*(b, p) = \emptyset$.

Let $V = W - \mathrm{Cl} G^*(b+1, p)$. Since $\mathrm{Cl} G^*(b, p) \cap Y' = \emptyset$, $Y' \cap \partial V = Y' \cap \partial W$. By the choice of b, $[s, w_k] \not\subset G^*(b+1, p)$; since $[t, w_k]$ is straight in $\mathcal{G}$, $[t, s] \cap G^*(b+1, p) = \emptyset$. Thus $\mathrm{Cl} G^*(b+1, p) \cap [t, w_k] \cap \partial V$ contains at most one point, namely s. It follows that $[t, w_k] \cap \partial V = \{s\}$. Thus $Y \cap \partial V$ contains exactly k points, one from each arc $[t, w_i]$. Thus V has essentially the same properties as does W, and in addition, $V \cap G^*(b+1, p) = \emptyset$.

We define a chain $\mathcal{E}_k = E_k(1, r_k)$ covering $[t, w_k]$ by $E_{k1} = V \cup G_b$, $E_{ki} = G_{b+i-1}$, if $i > 1$. Since V misses $G^*(b+1, p)$, it is clear that $\mathcal{E}_k$ is a chain covering $[t, w_k]$. We show that $[t, w_k]$ is straight in $\mathcal{E}_k$. If $j \geqslant 2$, then $[t, w_k] \cap \partial E_{kj}$ has the proper cardinality because $[t, w_k]$ is straight in $\mathcal{G}$. Since $[t, w_k] \cap \partial V = \{s\} \subset G_b \subset E_{k1}$, $[t, w_k] \cap \partial E_{k1} = [t, w_k] \cap \partial G_b$. Now ∂G_b contains two points of $[t, w_k]$, one of which is in $[t, s] \subset V \subset E_{k1}$. Thus $[t, w_k] \cap \partial E_{k1}$ is a singleton and $[t, w_k]$ is straight in $\mathcal{E}_k$.

Since $[t, w_k] \subset \mathcal{E}_k^*$, $[t, w_k] - E_{k1}$ and $(X - \mathcal{E}_k^*) \cup (Y' - E_{k1})$ are disjoint closed sets. Inasmuch as t is the only ramification point of X, no continuum in $X - E_{k1}$ intersects both these sets. Thus there are disjoint closed sets A and B such that $X - E_{k1} = A \cup B$, $[t, w_k] - E_{k1} \subset A$ and $(X - \mathcal{E}_k^*) \cup (Y' - E_{k1}) \subset B$. Since $[t, w_k]$ is straight in $\mathcal{E}_k$, $[t, w_k] \cap E_k^*(2, r_k)$ is connected and $[t, w_k] \cap \mathrm{Cl} E_k^*(2, r_k)$ is an arc, $[v, w_k]$. Note that $[t, w_k] - E_{k1} \subset [v, w_k]$. Since $A \cup [v, w_k]$ and $B \cup Y'$ are disjoint closed sets, there are disjoint open sets S and T such that $A \cup [v, w_k] \subset S$ and $B \cup Y' \subset T$. Thus we have $[t, w_k] - E_{k1} \subset [v, w_k] \subset S$ and $Y' \cup (X - \mathcal{E}_k^*) \subset T$.

We now define the chains $\mathcal{F}_j = F_j(1, r_j)$, $1 \leqslant j \leqslant k$.

$$
\begin{aligned}
F_{ji} &= E_{ji} \cap T && \text{if} \quad j \neq k,\ i > 1\,,\\
F_{ki} &= E_{ki} \cap S && \text{if} \quad i > 1\,,\\
F_{j1} &= (E_{11} \cap T) \cup E_{k1} && \text{for each } j\,.
\end{aligned}
$$

(Recall that $E_{11} = E_{j1}$, if $j \neq k$.)

We show that the $\mathcal{F}_j$'s satisfy (i)–(v). Properties (i) and (iv) are immediate consequences of the definitions. If $i \neq k \neq j$, then (v) holds since $\mathcal{F}_j$ refines $\mathcal{E}_j$ and $\mathcal{F}_i$ refines $\mathcal{E}_i$. We shall show that (v) holds for k. If $j \neq k$ then $[t, w_j] \subset Y' \subset Y' \cup B \subset T$ and $F_j^*(2, r_j) \subset T$. Since $F_k^*(2, r_k) \subset S$ and $T \cap \mathrm{Cl} S = \emptyset$, $([t, w_j] \cup F_j^*(2, r_j)) \cap \mathrm{Cl} F_k^*(2, r_k) = \emptyset$. Now $[t, w_k] = [t, v] \cup [v, w_k] \subset E_{k1} \cap S$. Since $F_j^*(2, r_j) \subset T$, $F_j^*(2, r_j) \cap [t, w_k] \subset E_{k1} \cap F_j^*(2, r_j) \subset E_{k1} \cap E_j^*(1, r_j) = \emptyset$. Thus $([t, w_k] \cup F_k^*(2, r_j)) \cap \mathrm{Cl} F_j^*(2, r_j) = \emptyset$, if $j \neq k$ and (v) holds for all i and j.

If $j \neq k$ then $\partial F_j^*(2, r_j) \subset \partial E_j^*(2, r_j) \cup \partial T \subset E_{j1} \cup E_{k1}$, since $\mathcal{E}_j$ satisfies (iii). Now $(\partial F_j^*(2, r_j)) - E_{k1} \subset X - E_{k1} \subset S \cup T$. Since $\partial F_j^*(2, r_j) \subset \mathrm{Cl}\, T$ and $S \cap \mathrm{Cl}\, T = \emptyset$, we have $(\partial F_j^*(2, r_j)) - E_{k1} \subset T$. Thus $(\partial F_j^*(2, r_j)) - E_{k1} \subset E_{j1} \cap T$, and $\partial F_j^*(2, r_j) \subset (E_{j1} \cap T) \cup E_{k1} = F_{j1}$; so (iii) holds for $j \neq k$. In the remaining case $\partial F_k^*(2, r_k) \subset (\partial E_k^*(2, r_k)) \cup \partial S \subset E_{k1} \cup \partial S = E_{k1} \subset F_{k1}$. Thus (iii) holds for each j.

Finally, we demonstrate (ii). If $j \neq k$ then $F_{j1} = E_{11} \cap (T \cup E_{k1})$, since $E_{k1} \subset E_{11} - \mathrm{Cl} E_k^*(2, r_k)$. Then Lemma 2 shows that $[t, w_j]$ is straight in $\mathcal{F}_j$, for each $j \neq k$. If $i > 1$, then $[t, w_k] \cap \partial F_{ki} \subset ([t, w_k] \cap \partial E_{ki}) \cup ([t, w_k] \cap \partial S)$. However, $[t, w_k] \cap \partial F_{ki} \subset [t, w_k] \cap \mathrm{Cl} E_k^*(2, r_k) \subset [v, w_k] \subset S$. Since $S \cap \partial S = \emptyset$, $[t, w_k] \cap \partial F_{ki} \subset [t, w_k] \cap \partial E_{ki}$. Since $[t, w_k]$ is straight in $\mathcal{E}_k$ and $F_{ki} \subset E_{ki}$, $[t, w_k] \cap \partial F_{ki}$ cannot have fewer points than $[t, w_k] \cap \partial E_{ki}$. Thus $[t, w_k] \cap \partial F_{ki} = [t, w_k] \cap \partial E_{ki}$, and this last set has the proper cardinality for straightness. The last step is to show that $[t, w_k] \cap \partial F_{k1}$ is degenerate. Clearly, $[t, w_k] \cap \partial F_{k1} \subset ([t, w_k] \cap \partial (E_{11} \cap T)) \cup ([t, w_k] \cap \partial E_{k1})$. Since $E_{k1} \subset F_{k1}$, $[t, w_k] \cap \partial F_{k1} \subset [t, w_k] - E_{k1} \subset S$. Since $S \cap \mathrm{Cl}\, T = \emptyset$ and $\partial (E_{11} \cap T) \subset \mathrm{Cl}\, T$, $[t, w_k] \cap (\partial F_{k1}) \cap \partial (E_{11} \cap T) = \emptyset$. Thus $[t, w_k] \cap \partial F_{k1} \subset [t, w_k] \cap \partial E_{k1}$, which is a degenerate, because $[t, w_k]$ is straight in $\mathcal{E}_k$. Since $[t, w_k] \cap \partial F_{k1} \neq \emptyset$, $[t, w_k] \cap \partial F_{k1}$ is a point and $[t, w_k]$ is straight in $\mathcal{F}_k$. This concludes the proof of Proposition 3.

Once we have covered the fan X as in Figure 3, we use the cover to construct the retraction. To do this, we will piece together retractions of chains onto straight arcs. We therefore prove

PROPOSITION 4. *Suppose $[a, b]$ is an arc which is straight in an ε-chain $\mathcal{E} = E(1, m)$, $\mathcal{E}^* \subset X$ a compact metric space, $\partial E^*(2, m) \subset E_1$ and $\{p\} = (\partial E_1) \cap [a, b]$. Then there is a continuous function f: $(\mathcal{E}^* - E_1) \to [p, b]$ such that f is a retraction onto $[p, b]$, $f[(\partial E_1) \cap E_2] = p$ and for each $x \in \mathcal{E}^* - E_1$, $d(x, f(x)) < \varepsilon$.*

Proof. Since $\partial E^*(2, m) \subset E_1$, $\mathcal{E}^* - E_1$ is compact and for each i, $2 \leqslant i \leqslant m-1$, ∂E_i is the union of two disjoint closed sets, $(\partial E_i) \cap E_{i-1}$ and $(\partial E_i) \cap E_{i+1}$. Since $[a, b]$ is straight in $\mathcal{E}$, for each i, $1 \leqslant i \leqslant m-1$, $(\partial E_i) \cap E_{i+1} \cap [a, b]$ is a single point, r_i. Then $p = r_1$; let $b = r_m$. Again, straightness guarantees that $p = r_1 < r_2 < \ldots < r_m = b$, where $<$ denotes the usual order from a to b on $[a, b]$.

For each i, $1 \leqslant i \leqslant m-2$, we define a function

$$g_i\colon \big((\partial E_i) \cap E_{i+1}\big) \cup [r_i, r_{i+1}] \cup \big((\partial E_{i+1}) \cap E_{i+2}\big) \to [r_i, r_{i+1}]$$

by

$$
\begin{aligned}
g_i(x) &= r_i && \text{if} \quad x \in (\partial E_i) \cap E_{i+1}\\
g_i(x) &= x && \text{if} \quad x \in [r_i, r_{i+1}]\,,\\
g_i(x) &= r_{i+1} && \text{if} \quad x \in (\partial E_{i+1}) \cap E_{i+2}\,.
\end{aligned}
$$

Then g_{m-1} is defined as above, except that domain g_{m-1} is $((\partial E_{m-1}) \cap \cap E_m) \cup [r_{m-1}, r_m]$. Clearly, each g_i is a continuous retraction onto $[r_i, r_{i+1}]$. Since arcs are absolute retracts, for each i there is a continuous extension h_i of g_i, h_i: $\mathrm{Cl}\, E_{i+1} - E_i \to [r_i, r_{i+1}]$. Notice that each h_i is a retraction which moves each point less than ε, since $\operatorname{diam}(\mathrm{Cl}\, E_{i+1}) < \varepsilon$. Moreover, if $i \neq j$ then domain $h_i \cap$ domain $h_j \neq \emptyset$ if and only if $j = i-1$ or $j = i+1$. Clearly, domain $h_i \cap$ domain $h_{i+1} = (\partial E_{i+1}) \cap E_{i+2}$ and both h_i and h_{i+1} send all points of this set to r_{i+1}. Thus the function $f = h_1 \cup h_2 \cup \ldots \cup h_{m-1}$ is a continuous retraction of $\mathcal{E}^* - E_1$ onto $[p, b]$ moving each point less than ε.

THEOREM 1. *Suppose X is a fan and $\varepsilon > 0$. Then there is a finite fan $Y \subset X$ and a retraction r: $X \to Y$ such that if $x \in X$, then $d(x, r(x)) < \varepsilon$.*

Proof. Let t denote the top of X and let W denote the set of end points of X. Then, as shown in [1], $X = \bigcup \{[t, w]: w \in W\}$. For each $w \in W$, we apply Proposition 2 to obtain $\mathcal{E}_w$, and $\varepsilon/8$-chain, having at least two links, such that $[t, w]$ is straight in $\mathcal{E}_w$ and $\partial(\mathcal{E}_w - E_{w1})^* \subset E_{w1}$. There is a finite subset $W' \subset W$ such that $\{\mathcal{E}_w^*: w \in W'\}$ covers X. If $W' = \{w_1, w_2, \ldots, w_n\}$, let us relabel the corresponding chains $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$. For each $j \in Z^n$, $\mathcal{E}_j = \{E_{j1}, E_{j2}, \ldots, E_{jm_j}\} = E_j(1, m_j)$. We have, then

(1) $\bigcup \{\mathcal{E}_j: j \in Z^n\}$ covers X,

(2) For each $j \in Z^n$, $[t, w_j]$ is straight in $\mathcal{E}_j$ and

(3) For each $j \in Z^n$, $\partial E_j^*(2, m_j) \subset E_{j1}$.

For each $j \in Z^n$, let Y_j be the arc $[t, w_j]$ and let $Y = \bigcup \{Y_j: j \in Z^n\}$. Clearly Y is a finite fan with top t and each w_j is an end point of both Y and X.

Step I. If $i \neq j$ we cut the links of $E_j(2, r_j)$ away from $[t, w_i]$ and we modify the intersection of E_{j1} with $[t, w_i]$. That is, for each $j \in Z^n$, we will obtain an $\varepsilon/8$-chain $\mathcal{G}_j$, a one-to-one refinement of $\mathcal{E}_j$, satisfying (1), (2), (3) and

(4) If $i \neq j$ then $[t, w_i] \cap \mathrm{Cl}\, G_j^*(2, m_j) = \emptyset$.

(5) If $i \neq j$ then $[t, w_i] \cap \mathrm{Cl}\, G_{j1} \subset [t, w_j] \cap G_{i1}$.

(See Figure 1.)

Choose $\delta > 0$, $\delta < \varepsilon/8$ and let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ be a collection of δ-chains satisfying the conclusion of Proposition 3. We may assume that each $\mathcal{F}_j$ refines $\mathcal{E}_j$. Since each $[t, w_j]$ is straight in $\mathcal{E}_j$, $t \notin \mathrm{Cl}\, E_j^*(2, m_j)$. For each $j \in Z^n$, $t \in F_j^*(1, 2)$, hence we may assume that δ was chosen so small that for each $j \in Z^n$, $F_j^*(1, 2) \cap \mathrm{Cl} \bigcup \{E^*(2, m_i): i \in Z^n\} = \emptyset$.

For each j, let $B_j = \mathrm{Cl} \bigcup \{F_i^*(2, r_i): i \in Z^n, i \neq j\}$ and let $B_j' = \mathrm{Cl} \bigcup \{F_i^*(3, r_i): i \in Z^n, i \neq j\}$. Note that Proposition 3 (v) guarantees that

($*$) for each j, $([t, w_j] \cup F_j^*(2, r_j)) \cap B_j = \emptyset$ and Proposition 3 (iii) and (v) shows that

($**$) for each j, $\mathcal{F}_j^* \cap B_j' = \emptyset$, $\mathcal{F}_j^* \cap B_j \subset F_{j1}$.

For each j, we define the chain $\mathcal{G}_j = G_j(1, m_j)$ as follows:

$$G_{jk} = E_{jk} - B_j \quad \text{if } 2 \leqslant k \leqslant m_j\,,$$
$$G_{j1} = E_{j1} - B_j'\,.$$

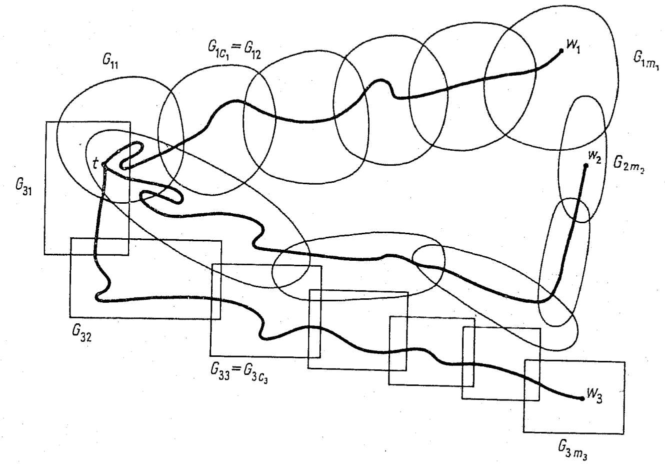


Fig. 1

Clearly, each $\mathcal{G}_j$ is a one-to-one refinement of $\mathcal{E}_j$. The definitions show that for each j, $\mathrm{Cl}\, G_j^*(2, m_j) \cap B_j \subset \partial B_j$. Proposition 3 (iii) shows that $\partial B_j \subset F_{j1}$. The $\mathcal{F}_j$'s were constructed so that $F_{j1} \cap \mathrm{Cl}\, E_j^*(2, m_j) = \emptyset$, thus $F_{j1} \cap \mathrm{Cl}\, G_j^*(2, m_j) = \emptyset$ and we have

($***$) for each j, $\mathrm{Cl}\, G_j^*(2, m_j) \cap B_j = \emptyset$.

We demonstrate that $\bigcup \{\mathcal{G}_j: j \in Z^n\}$ covers X. If $x \in \mathcal{E}_j^* - \mathcal{G}_j^*$, then $x \in B_j$ and there is an i, $i \neq j$, such that $x \in \mathrm{Cl}\, F_i^*(2, r_i)$. Since ($*$) shows that $F_i^*(2, r_i) \cap B_i = \emptyset$, $F_i^*(2, r_i) \subset \mathcal{E}_i^* - B_i \subset \mathcal{G}_i^*$. Thus we may assume that $x \in \partial F_i^*(2, r_i)$. According to Proposition 3 (iii), $x \in F_{i1}$; the construction of $\mathcal{F}_i$ and ($**$) yields $F_{i1} \subset E_{i1} - B_i' = G_{i1}$. Thus $x \in \mathcal{G}_i^*$ and (1) is satisfied.

It follows from the definitions that for each j and k, $\partial G_{jk} \subset (\partial E_{jk}) \cup \cup (\partial B_j)$. Since ($*$) shows that $\partial B_j \cap [t, w_j] = \emptyset$, $(\partial G_{jk}) \cap [t, w_j] \subset (\partial E_{jk}) \cap \cap [t, w_j]$. Since $[t, w_j]$ is straight in $\mathcal{E}_j$, this last set has the proper cardinality, and thus so does $(\partial G_{jk}) \cap [t, w_j]$. Thus $[t, w_j]$ is straight in $\mathcal{G}_j$.

It follows from ($***$) that $\partial G_j^*(2, m_j) \subset \partial E_j^*(2, m_j)$. Since $\mathcal{E}_j$ satisfies (3) $\partial E_j^*(2, m_j) \subset E_{j1}$. Since $B_j' \subset B_j$ and ($***$) holds, $\partial G_j^*(2, m_j) \subset E_{j1} - B_j' = G_{j1}$. Thus each $\mathcal{G}_j$ satisfies (3).

Now, for each i and each $j \neq i$, $[t, w_i] \subset \mathcal{F}_i^* \subset F_{i1} \cup B_j$. Since (⁂) shows that $B_j \cap \mathrm{Cl}\, G_j^*(2, m_j) = \emptyset$, $[t, w_i] \cap \mathrm{Cl}\, G_j^*(2, m_j) \subset F_{i1} \cap \mathrm{Cl}\, G_j^*(2, m_j)$. Each $\mathcal{F}_i$ was constructed so that $F_{i1} \cap \mathrm{Cl}\, E_j^*(2, m_j) = \emptyset$, so $F_{i1} \cap \mathrm{Cl}\, G_j^*(2, m_j) = \emptyset$ and (4) holds.

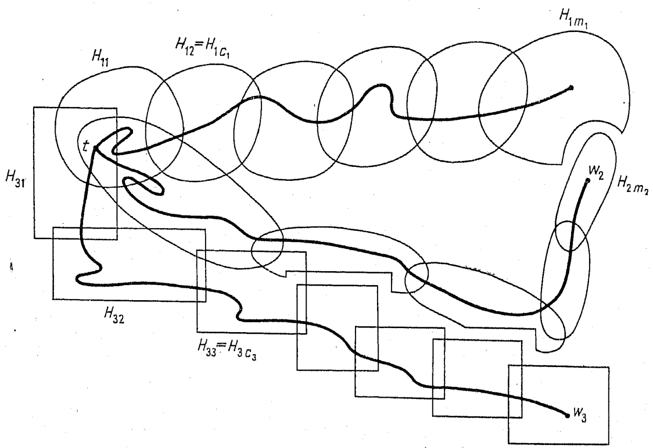
If $i \neq j$ then $[t, w_i] \subset \mathcal{F}_i^* \subset F_i^*(1, 2) \cup \mathrm{Int}\, B_j'$. From the definition of G_{j1} we know that $(\mathrm{Cl}\, G_{j1}) \cap \mathrm{Int}\, B_j' = \emptyset$, thus $\mathrm{Cl}\, G_{j1} \cap [t, w_i] \subset F_i^*(1, 2)$. Now (⁑⁑) guarantees that $F_i^*(1, 2) \cap B_i' = \emptyset$, thus $F_i^*(1, 2) \subset E_{i1} - B_i' = G_{i1}$ and (5) holds.

Step II. If $i \neq j$, we refine $\mathcal{G}_i$ and $\mathcal{G}_j$ so that the refinements intersect at most along initial subchains, which have small diameter. More precisely, for each $j \in Z^n$ we obtain an $\varepsilon/8$-chain $\mathcal{H}_j = H_j(1, m_j)$, a one-to-one refinement of $\mathcal{G}_j$, such that $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ satisfies (1), (2), (4) and (5) as well as

(6) For each j, there is an integer c_j, $2 \leqslant c_j \leqslant m_j$, such that if $i \neq j$ then $H_j^*(c_j+1, m_j) \cap H_i^*(c_i+1, m_i) = \emptyset$.

(7) For each j, $\operatorname{diam} H_j^*(1, c_j) \leqslant \varepsilon/2$.

See Figure 2.



$\mathcal{H}_3 = \mathcal{G}_3$

Fig. 2

Let $V = \bigcup\{G_{j1}: j \in Z^n\}$. For each $j \in Z^n$, let $c_j = \max\{k$: if $i \in Z^k$, then $G_{ji} \cap V \neq \emptyset\}$. Since V is the union of the first links of the $\mathcal{G}_j$'s, it follows that each $c_j \geqslant 2$. If $x \in G_j^*(1, c_j)$, then x belongs to a link of $\mathcal{G}_j$ which

intersects the first link of some $\mathcal{G}_i$. Thus $d(x, t) < \varepsilon/8 + \varepsilon/8 = \varepsilon/4$, and $\operatorname{diam} G_j^*(1, c_j) < \varepsilon/2$.

For each $j \in Z^n$, we define the chain $\mathcal{H}_j = H_j(1, m_j)$ by for each l, $1 \leqslant l \leqslant m_j$,

$$H_{jl} = G_{jl} - \mathrm{Cl} \bigcup \{G_k^*(c_k+1, m_k): k \in Z^n,\ k > j\}.$$

Note that $\mathcal{H}_n = \mathcal{G}_n$. Moreover, since each $H_j^*(1, c_j) \subset G_j^*(1, c_j)$, it follows that $\operatorname{diam} H_j^*(1, c_j) < \varepsilon/2$, and (7) holds.

For each $k \in Z^n$, $c_k \geqslant 2$, thus if $j < k$, then $[t, w_j] \cap \mathrm{Cl}\, G_k^*(c_k+1, m_k) = \emptyset$, since $\mathcal{G}_k$ satisfies (4). Hence $[t, w_j] \cap \mathrm{Cl} \bigcup \{G_k^*(c_k+1, m_k): k \in Z^n, k > j\} = \emptyset$; it follows that each $[t, w_j]$ is straight in $\mathcal{H}_j$. Since the $\mathcal{G}_j$'s satisfy (4) and $\mathcal{H}_j$ is a one-to-one refinement of $\mathcal{G}_j$, the $\mathcal{H}_j$'s satisfy (4).

Since $[t, w_i] \cap \mathrm{Cl} \bigcup \{G_k^*(c_k+1, m_k): k \in Z^n, k > i\} = \emptyset$, $[t, w_i] \cap G_{i1} = [t, w_i] \cap H_{i1}$. If $i \neq j$, then (5) insures that $[t, w_i] \cap \mathrm{Cl}\, G_{j1} \subset [t, w_i] \cap G_{i1} = [t, w_i] \cap H_{i1}$. Since $\mathrm{Cl}\, H_{j1} \subset \mathrm{Cl}\, G_{j1}$, we have $[t, w_i] \cap \mathrm{Cl}\, H_{j1} \subset [t, w_i] \cap H_{i1}$, and the $\mathcal{H}_i$'s satisfy (5). If $i \neq j$, then we may assume that $i < j$. Then $\mathrm{Cl}\, G_j^*(c_j+1, m_j) \subset \mathrm{Cl} \bigcup \{G_k^*(c_k+1, m_k): k > i\}$. The definition of $\mathcal{H}_i$ guarantees that $\mathcal{H}_i^* \cap G_j^*(c_j+1, m_j) = \emptyset$; a fortiori (6) holds.

From the definition, for each $i \in Z^n$, $\mathcal{G}_i^* - \mathcal{H}_i^* \subset \mathrm{Cl} \bigcup \{G_k^*(c_k+1, m_k): k \in Z^n, k > i\}$. For each $k \in Z^n$, since $\mathcal{G}_k$ satisfies (3) $\mathrm{Cl}\, G_k^*(c_k+1, m_k) \subset \mathcal{G}_k^*$; thus $\mathcal{G}_i^* - \mathcal{H}_i^* \subset \bigcup \{\mathcal{G}_k^*: k \in Z^n, k > i\}$. Suppose $x \in X$ and let $a = \max\{i: i \in Z^n, x \in \mathcal{G}_i^*\}$. Thus $x \in \mathcal{G}_a^*$. If $x \notin \mathcal{H}_a^*$, then $a < n$, since $\mathcal{H}_n = \mathcal{G}_n$. Thus $x \in \mathcal{G}_a^* - \mathcal{H}_a^* \subset \bigcup \{\mathcal{G}_i^*: i > a\}$. This contradicts the choice of a, hence $x \in \mathcal{H}_a^*$ and (1) holds.

Step III. We now construct ε-chains $\mathcal{K}_1 = K_1(1, p_1)$, $\mathcal{K}_2 = K_2(1, p_2) \ldots$ $\ldots \mathcal{K}_n = K_n(1, p_n)$ satisfying (1), (2) and

(8) If $j \in Z^n$, then $K_{j1} = K_{11}$,

(9) If $i \neq j$ then $K_i^*(2, p_i) \cap K_j^*(2, p_j) = \emptyset$.

See Figure 3. (That is the $\mathcal{K}_j$'s cover X in much the same way that the $\mathcal{F}_j$'s cover Y in the conclusion of Proposition 3. Here, (1), (2), (8) and (9) are analogous to (i), (ii), (iv) and (v) respectively.) We construct the $\mathcal{K}_j$'s by consolidating all the initial subchains of the $\mathcal{H}_j$'s. For each j, let $K_{j1} = \bigcup \{H_i^*(1, c_i): i \in Z^n\}$. If $l > 1$ and K_{jc_j} is not the last link of $\mathcal{K}_j$, let $K_{jl} = H_{j, c_j+l-1}$. Condition (6) insures that each $\mathcal{K}_j$ is a chain. For each j, $\mathcal{H}_j^* \subset \mathcal{K}_j^*$, thus (1) holds.

We show that each arc $[t, w_j]$ is straight in $\mathcal{K}_j$. If $l > 1$, then $[t, w_j] \cap \partial K_{jl} = [t, w_j] \cap \partial H_{j, c_j+l-1}$. Since $[t, w_j]$ is straight in $\mathcal{H}_j$, $[t, w_j] \cap \partial H_{j, c_j+l-1}$ is either a one or two-point set, depending on whether H_{j, c_j+1} is or is not the last link of $\mathcal{H}_j$. In either case, $\partial K_{jl} \cap [t, w_j]$ has the proper cardinality. If $l = 1$, then $(\partial K_{jl}) \cap [t, w_j] = \big(\partial \bigcup \{H_i^*(1, c_i): i \in Z^n\}\big) \cap [t, w_j]$. If $i \neq j$ then (4) shows that $\big(\mathrm{Cl}\, H_i^*(2, m_i)\big) \cap [t, w_j] = \emptyset$ and (5) shows that $\mathrm{Cl}\, H_{i1} \cap [t, w_j] \subset H_{j1}$. Thus $(\partial K_{j1}) \cap [t, w_j] \subset \partial H_j^*(1, c_j)$. Since

$[t, w_j]$ is straight in $\mathcal{K}_j$, $(\partial H_j^*(1, c_j)) \cap [t, w_j]$ is a one-point set, unless H_{j,c_j} is the last link of $\mathcal{K}_j$, in which case this set is empty. Thus, for each l, $(\partial K_{jl}) \cap [t, w_j]$ has the proper cardinality and (2) holds.

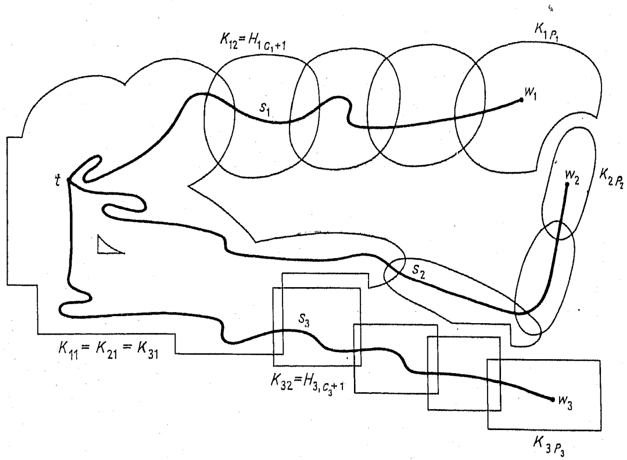


Fig. 3

The definition of $\mathcal{K}_j$ yields (8). If $i \neq j$ then $K_i^*(2, p_i) \cap K_j^*(2, p_j) = H_i^*(c_i+1, m_i) \cap H_j^*(c_j+1, m_j) = \emptyset$, from (6). Thus (9) is satisfied.

Step IV. We now construct the map r of X onto $Y = \bigcup\{[t, w_j]: j \in Z^n\}$. We will assume that each $\mathcal{K}_j$ has more than one link; if this is not true, the needed modifications in the definition of r are obvious.

For each $j \in Z^n$, there is a point $s_j \in [t, w_j]$ such that $(\partial K_{j1}) \cap [t, w_j] = (\partial K_{j1}) \cap Y \cap K_{j2} = \{s_j\}$. Since each $[t, w_j]$ is straight in the ε-chain $\mathcal{K}_j$, we apply Proposition 4 to obtain a retraction f_j: $(\mathcal{K}_j^* - K_{j1}) \to [s_j, w_j]$ such that $f_j[(\partial K_{j1}) \cap K_{j2}] = \{s_j\}$ and f_j moves each point less than ε. (Note that ∂K_{j1} is the union of the disjoint closed sets $(\partial K_{j1}) \cap K_{12}$, $(\partial K_{j1}) \cap K_{22}, \ldots, (\partial K_{j1}) \cap K_{n2}$. If $i \neq j$ then $(\text{domain } f_i) \cap (\text{domain } f_j) \subset K_i^*(2, p_i) \cap K_j^*(2, p_j) = \emptyset$, from (9). Thus we may define $f = \bigcup\{f_j: j \in Z^n\}$. Clearly f is an retraction of $X - K_{11}$ onto $\bigcup\{[s_i, w_i]: i \in Z^n\}$ moving each point less than ε.

Now $Y \cap K_{11} = \bigcup\{[t, s_i): i \in Z^n\}$ and $(\mathrm{Cl}\,K_{11}) \cap Y = \bigcup\{[t, s_i]: i \in Z^n\}$, since each $[t, w_i]$ is straight in $\mathcal{K}_i$. We define g: $(\partial K_{11}) \cup ((\mathrm{Cl}\,K_{11}) \cap Y) \to (\mathrm{Cl}\,K_{11}) \cap Y$ by $g(x) = x$, if $x \in Y$ and $g(x) = s_i$, if $x \in (\partial K_{11}) \cap K_{i2}$. Since $(\mathrm{Cl}\,K_{11}) \cap Y$ is a tree, it is an absolute retract. Hence g can be extended to a map h: $\mathrm{Cl}\,K_{11} \to (\mathrm{Cl}\,K_{11}) \cap Y$. Since $\mathrm{diam}(\mathrm{Cl}\,K_{11}) < \varepsilon$, h moves each point less than ε. Finally, let $r = h \cup f$. Since f and h agree on the intersection of their domains, ∂K_{11}, r is well-defined and continuous. Obviously, r is a retraction of X onto Y. Since neither h nor f moves any point as much as ε, neither does r.

THEOREM 2. *Each fan is an inverse limit of a sequence of finite fans.*

Proof. Clearly, each finite fan is a polyhedron. If $\mathfrak{L}$ denotes the class of all finite fans then Theorem 1 establishes that for each fan X and each $\varepsilon > 0$ there is a map of X onto a member of $\mathfrak{L}$ which is an ε-map; i.e. the inverse image of each of each point has diameter $< \varepsilon$. It follows from [3] that X is homeomorphic to an inverse limit of a sequence of members of $\mathfrak{L}$.

THEOREM 3. *Each fan is tree-chainable.*

Proof. The collection $\mathcal{K}_1, \ldots, \mathcal{K}_n$ of Step III in Theorem 1 is a tree-chain.

For completeness, we state the following theorem, whose proof may be found in [2].

THEOREM 4. *If X is a compact connected Hausdorff space which is a product of fans, then X has the fixed point property.*

Proof. We can establish this by showing that X can be retracted onto Y, a finite product of finite fans, by a small retraction. Since each finite fan is an AR, so is Y. Thus X is an "approximate AR". By [2], Corollary 1.2, X has the fixed point property. Details may be found in [2].

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