Atomic compactness and elementary equivalence

by

Walter Taylor (Boulder, Colo.)

In this paper we continue the study, begun in [6], of the relationship of (generalized) chromatic numbers to atomic compactness and weak atomic compactness. Our main result (Theorem 3.1) is that a relational structure $\mathfrak{A}$ is an elementary substructure of some weakly atomic-compact relational structure if and only if the class of all chromatic numbers of structures elementarily equivalent to $\mathfrak{A}$ is bounded from above by some cardinal.

In [6] we showed that a relational structure $\mathfrak{A}$ is a retract of a compact topological relational structure if and only if $\mathfrak{A}$ is atomic-compact and all chromatic numbers of $\mathfrak{A}$ are finite. Now we show (Corollary 4.3) that if $\mathfrak{A}$ is a retract of a compact topological structure $\mathfrak{B}$, then there exists such a $\mathfrak{B}$ which is a direct product of $\mathfrak{m}$ finite structures, where $\mathfrak{m}$ is the power of the similarity type of $\mathfrak{A}$ plus that of the universe of $\mathfrak{A}$. Thus our $\mathfrak{B}$ has power at most $2^{\mathfrak{m}}$.

This paper may be read independently of [6]. In particular, we develop the theory of (generalized) chromatic numbers from the beginning. This development improves that of [6], although strictly speaking the two are equivalent (see Theorem 4.1).

§ 0 contains the preliminaries. In § 1 we discuss chromatic number and boundedness of chromatic number in an elementary equivalence class. § 2 contains some lemmas (concerning weak atomic compactness). § 3 contains the main result. In § 4 we discuss our two different definitions of chromatic number and extend the main result of [6] as indicated above. In § 5 we study pure closures of relational structures. In § 6 we correct a proof and refine an example from [5].

0. Preliminaries. Notation will be the same as in [6], except where we deal with chromatic numbers, which will be redefined in § 1. Throughout this paper, the letters A and B will denote the universes of structures $\mathfrak{A}$ and $\mathfrak{B}$, respectively. If $\mathfrak{A} = \langle A, R_t \rangle_{t \in T}$ is a *relational structure* and $X \subseteq A$, then $\mathfrak{A}_X$ denotes the structure $\langle A, R_t, c \rangle_{t \in T, c \in X}$. The reader is referred to [8] for the basic notions associated with the theory of atomic compactness and weak atomic compactness. Briefly, a *structure* $\mathfrak{A}$ *is weakly atomic-compact* [or *atomic-compact*] if and only if every set of atomic formulas in the language of $\mathfrak{A}$ [or in the language of $\mathfrak{A}_A$] which is finitely satisfiable in $\mathfrak{A}$ [or in $\mathfrak{A}_A$] is satisfiable in $\mathfrak{A}$ [or in $\mathfrak{A}_A$].

If $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$, and $S \subseteq T$, then $\mathfrak{A}' = \langle A, R_t\rangle_{t\in S}$ is called a *reduct* of $\mathfrak{A}$. If S is finite, then $\mathfrak{A}'$ is a *finite reduct* of $\mathfrak{A}$. If $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$ and $\mathfrak{B} = \langle B, S_t\rangle_{t\in T}$ are similar relational structures, then a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a mapping $f\colon A \to B$, such that for each $t \in T$ and each $\langle a_1, \ldots, a_{n(t)}\rangle \in R_t$, we have $\langle f(a_1), \ldots, f(a_{n(t)})\rangle \in S_t$. If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, and there is a homomorphism $f\colon \mathfrak{B} \to \mathfrak{A}$ such that $f\upharpoonright A$ is the identity on A, then $\mathfrak{A}$ is a *retract* of $\mathfrak{B}$.

If the universe A of $\mathfrak{A}$ is at the same time a topological space, and each relation R_t of $\mathfrak{A}$ is a closed subset of $A^{n(t)}$ in the product topology, then $\mathfrak{A}$ is a *topological relational structure*. If $\mathfrak{A}$ is any relational structure, then $\beta\mathfrak{A}$ denotes the topological relational structure $\langle \beta A, \bar{R}_t\rangle_{t\in T}$, where $\beta\mathfrak{A}$ is the Stone–Čech compactification of A (considered as a discrete space), and $\bar{R}_t$ is the closure of R_t in $(\beta A)^{n(t)}$.

A structure $\mathfrak{A} = \langle A, R\rangle$ is called a *graph* if R is binary, symmetric and antireflexive.

$\mathfrak{A} \equiv \mathfrak{B}$ denotes *elementary equivalence* of the structures $\mathfrak{A}$ and $\mathfrak{B}$.

An $(\exists, \wedge)$-sentence is a sentence in some first-order language, having only the connectives $\exists$ and $\wedge$.

$|X|$ denotes the cardinal of a set X. As in [6], we adopt a symbol ∞ with the convention that $\mathfrak{n} < \infty$ for every cardinal $\mathfrak{n}$. A class $\boldsymbol{K}$ of cardinals is called bounded if there is a cardinal $\mathfrak{n}$ such that $\mathfrak{m} < \mathfrak{n}$ for each $\mathfrak{m} \in \boldsymbol{K}$, i.e. if $\boldsymbol{K}$ is a set.

1. Elementary boundedness of chromatic number.

DEFINITION 1.1. Let φ be an $(\exists, \wedge)$-sentence in the language of $\mathfrak{A}$. The *φ-chromatic number* of $\mathfrak{A}$, denoted $\chi(\varphi, \mathfrak{A})$, is the least power of any structure $\mathfrak{B}$ satisfying the following conditions:

(i) $\mathfrak{B} \models \neg\varphi$;

(ii) there exists a homomorphism $f\colon \mathfrak{A} \to \mathfrak{B}$.

If no such $\mathfrak{B}$ exists (i.e. $\mathfrak{A} \models \varphi$), we put $\chi(\varphi, \mathfrak{A}) = \infty$.

If $\mathfrak{A} = \langle A, R\rangle$ is a graph and $\varphi = \exists xRxx$, then $\chi(\varphi, \mathfrak{A})$ is the ordinary chromatic number of $\mathfrak{A}$. (Definition 1.1 differs from the definition of chromatic number given in [6]. We feel that 1.1 is more natural and easier to work with; in § 4 we show that the two definitions are equivalent in a certain sense.) We first notice the following easy propositions. For the rest of this section φ denotes an $(\exists, \wedge)$-sentence.

PROPOSITION 1.2. *The following conditions are equivalent*:

(i) $\chi(\varphi, \mathfrak{A}) < \infty$;

(ii) $\chi(\varphi, \mathfrak{A}) \leqslant |A|$;

(iii) $\mathfrak{A} \models \neg\varphi$.

PROPOSITION 1.3. *If there is a homomorphism $f\colon \mathfrak{A} \to \mathfrak{B}$, then $\chi(\varphi, \mathfrak{A}) \leqslant \chi(\varphi, \mathfrak{B})$.*

COROLLARY 1.4. *If $\mathfrak{A}$ is the direct product of the relational structures $\mathfrak{A}_i$ $(i \in I)$, then $\chi(\varphi, \mathfrak{A}) \leqslant \min\{\chi(\varphi, \mathfrak{A}_i)\colon i \in I\}$.*

PROPOSITION 1.5. *Let $\mathfrak{A}_0$ be the reduct of $\mathfrak{A}$ whose only relations and constants are those names appear in φ; then*

$$\chi(\varphi, \mathfrak{A}_0) = \chi(\varphi, \mathfrak{A}) .$$

The next result is a generalization of the Theorem of de Bruijn and Erdös [1], and will be used in 1.8 below.

THEOREM 1.6. *If for each finite substructure $\mathfrak{C}$ of $\mathfrak{A}$, $\chi(\varphi, \mathfrak{C}) \leqslant k < \aleph_0$, then $\chi(\varphi, \mathfrak{A}) \leqslant k$.*

Proof. Let $\mathcal{F}$ be an ultrafilter on the set M of finite substructures of $\mathfrak{A}$ such that for each $\mathfrak{K} \in M$, the set $\{\mathfrak{H} \in M\colon \mathfrak{H} \supseteq \mathfrak{K}\}$ is in $\mathcal{F}$. Clearly there is a homomorphism from $\mathfrak{A}$ into the ultraproduct $\mathfrak{B} = \boldsymbol{P}_{\mathfrak{K}\in M}\mathfrak{K}/\mathcal{F}$, and a homomorphism from $\mathfrak{B}$ into an ultraproduct of structures each of power $\leqslant k$ and each satisfying $\neg\varphi$. Clearly this last ultraproduct has power $\leqslant k$ and satisfies $\neg\varphi$. QED

DEFINITION 1.7. *$\boldsymbol{K}(\varphi, \mathfrak{A})$ is the class of cardinals*

$$\{\chi(\varphi, \mathfrak{B})\colon \mathfrak{B} \equiv \mathfrak{A}\} .$$

Remark 1.8. If $\chi(\varphi, \mathfrak{A}) = k < \aleph_0$, then $\boldsymbol{K}(\varphi, \mathfrak{A})$ is bounded (by k). For if $\mathfrak{B} \equiv \mathfrak{A}$, then any finite reduct of a finite substructure of $\mathfrak{B}$ is isomorphic to the corresponding reduct of some finite substructure of $\mathfrak{A}$. Thus by 1.5 and 1.6, $\chi(\varphi, \mathfrak{B}) \leqslant k$. In fact, by reversing the rôles of $\mathfrak{A}$ and $\mathfrak{B}$ in this argument, one sees that $\chi(\varphi, \mathfrak{B}) = k$.

Remark 1.9. If $\mathfrak{A}$ is weakly atomic-compact and $\mathfrak{A} \models \neg\varphi$, then $\boldsymbol{K}(\varphi, \mathfrak{A})$ is bounded. For if $\mathfrak{B} \equiv \mathfrak{A}$, then there is a homomorphism $f\colon \mathfrak{B} \to \mathfrak{A}$. Hence by 1.3, $\boldsymbol{K}(\varphi, \mathfrak{A})$ is bounded by $\chi(\varphi, \mathfrak{A})$, and $\chi(\varphi, \mathfrak{A}) < \infty$ by 1.2.

Remark 1.10. If $\mathfrak{A}$ is a graph without quadrilaterals, then $\boldsymbol{K}(\exists xRxx, \mathfrak{A})$ is bounded. For if $\mathfrak{B} \equiv \mathfrak{A}$, then $\mathfrak{B}$ is a graph without quadrilaterals, and hence, by a result of Erdös and Hajnal (cf. [3], Theorem 5.6), has countable chromatic number.

Remark 1.11. If $\mathfrak{A}$ is an infinite complete graph, then $\boldsymbol{K}(\exists xRxx, \mathfrak{A})$ is not bounded. The graphs described in [2], Theorems 6 and 7, provide more interesting examples of unbounded classes of this kind.

THEOREM 1.12. *If $\mathfrak{A}_0$ is the reduct of $\mathfrak{A}$ whose only relations and constants are those whose names appear in φ, then $\boldsymbol{K}(\varphi, \mathfrak{A})$ is a cofinal subset of*

$\boldsymbol{K}(\varphi, \mathfrak{A}_0)$. *Thus* $\boldsymbol{K}(\varphi, \mathfrak{A})$ *is bounded iff* $\boldsymbol{K}(\varphi, \mathfrak{A}_0)$ *is bounded and* $\sup \boldsymbol{K}(\varphi, \mathfrak{A}) = \sup \boldsymbol{K}(\varphi, \mathfrak{A}_0)$.

Proof. Let $\mathfrak{n} \in \boldsymbol{K}(\varphi, \mathfrak{A}_0)$; thus $\mathfrak{n} = \chi(\varphi, \mathfrak{B})$, where $\mathfrak{B} \equiv \mathfrak{A}_0$. By Frayne's Theorem (Theorem 2.12 of [4]), we imbed $\mathfrak{B}$ in an ultrapower $\mathfrak{A}_0^I/\mathcal{F}$ of $\mathfrak{A}_0$, which is a reduct of the ultrapower $\mathfrak{A}^I/\mathcal{F}$. Thus by 1.5, $\mathfrak{m} = \chi(\varphi, \mathfrak{A}^I/\mathcal{F}) \geqslant \mathfrak{n}$, and $\mathfrak{m} \in \boldsymbol{K}(\varphi, \mathfrak{A})$. QED

The following theorem will not be applied in this paper, but it leads to an interesting open problem. It states the existence of a kind of Hanf number for $\chi(\varphi, \mathfrak{A})$.

THEOREM 1.13. *There is a cardinal* $\mathfrak{m}$ *such that if* $\boldsymbol{K}(\varphi, \mathfrak{A})$ *is bounded, then* $\chi(\varphi, \mathfrak{A}) < \mathfrak{m}$.

Proof.[1] Let C be the set of cardinals $\mathfrak{n}$ such that $\boldsymbol{K}(\varphi, \mathfrak{A})$ is bounded by $\mathfrak{n}$, where $\mathfrak{A}$ ranges over all elementary equivalence classes of similarity types having only those relations and constants whose names appear in φ. By 1.5, one may take $\mathfrak{m}$ to be the successor of the supremum of the set C. QED

PROBLEM 1.14. *What is the least* $\mathfrak{m}$ *satisfying Theorem* 1.13*? In particular, what is the least* $\mathfrak{m}_0$ *such that every graph of chromatic number* $\geqslant \mathfrak{m}_0$ *is elementarily equivalent to graphs of arbitrarily high chromatic number?* [2]

If $\mathfrak{A}$ is the graph defined in [5] (Definition 4.6), then $\mathfrak{A}$ has countably infinite chromatic number, and yet $\boldsymbol{K}(\exists x Rxx, \mathfrak{A})$ is bounded, by 1.9. Thus $\mathfrak{m}$ and $\mathfrak{m}_0$ of Problem 1.14 are $\geqslant \aleph_1$.

2. Finite satisfiability. The only new idea in this section is in Definition 2.2. The lemmas are obvious in the light of § 2 of [8].

LEMMA 2.1. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be similar relational structures. The following conditions are equivalent*:

(i) *any finite set of atomic formulas which is satisfiable in* $\mathfrak{B}$ *is satisfiable in* $\mathfrak{A}$;

(ii) *for every* $(\exists, \wedge)$*-sentence* φ, *if* $\mathfrak{B} \models \varphi$, *then* $\mathfrak{A} \models \varphi$;

(iii) *there exists a structure* $\mathfrak{C}$ *satisfying only those* $(\exists, \wedge)$*-sentences which are true in* $\mathfrak{A}$, *and a homomorphism* $f\colon \mathfrak{B} \to \mathfrak{C}$;

(iv) *there exists a structure* $\mathfrak{C} \equiv \mathfrak{A}$ *and a homomorphism* $f\colon \mathfrak{B} \to \mathfrak{C}$;

(v) *there exists an ultrapower* $\mathfrak{C}$ *of* $\mathfrak{A}$ *and a homomorphism* $f\colon \mathfrak{B} \to \mathfrak{C}$.

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[1] This proof is essentially the same as Hanf's unpublished proof that for every language L of a certain large class of languages, there exists a cardinal $\mathfrak{m}$ such that if a theory expressed in L has a model of power $\mathfrak{m}$, then the cardinals of its models are unbounded.

[2] I announced in [7] that the least such $\mathfrak{m}$ is either $\aleph_1$ or $\aleph_2$, but I later found a mistake in my proof.

DEFINITION 2.2. $\mathfrak{B}$ is said to be $\mathfrak{A}$*-pure* iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy any (and hence all) of the conditions of Lemma 2.1.

The following result shows the relation of Definition 2.2 to the notion of *pure extension* of [8].

PROPOSITION 2.3. *Let* $\mathfrak{A}$ *be a substructure of* $\mathfrak{B}$. *Then* $\mathfrak{B}$ *is a pure extension of* $\mathfrak{A}$ *iff* $\mathfrak{B}_A$ *is* $\mathfrak{A}_A$*-pure.*

PROPOSITION 2.4 *The following conditions are equivalent*:

(i) $\mathfrak{A}$ *is weakly atomic-compact*;

(ii) *if* $\mathfrak{B}$ *is* $\mathfrak{A}$*-pure, then there is a homomorphism* $f\colon \mathfrak{B} \to \mathfrak{A}$;

(iii) *if* $\mathfrak{C} \equiv \mathfrak{A}$, *then there is a homomorphism* $f\colon \mathfrak{C} \to \mathfrak{A}$;

(iv) *if* $\mathfrak{C}$ *is an ultrapower of* $\mathfrak{A}$, *then there is a homomorphism* $f\colon \mathfrak{C} \to \mathfrak{A}$.

PROPOSITION 2.5. *If* $\boldsymbol{K}(\varphi, \mathfrak{A})$ *is bounded, then the class*

$$\{\chi(\varphi, \mathfrak{B})\colon \mathfrak{B} \text{ is } \mathfrak{A}\text{-pure}\}$$

is also bounded.

3. Weak compactification. We now state the principal theorem of this paper.

THEOREM 3.1. *The following conditions are equivalent*:

(i) $\mathfrak{A}$ *is elementarily equivalent to some weakly atomic-compact structure*;

(ii) $\mathfrak{A}$ *is an elementary substructure of some weakly atomic-compact structure*;

(iii) $\mathfrak{A}$ *is a pure substructure of some weakly atomic-compact structure*;

(iv) *for every* $(\exists, \wedge)$*-sentence* φ, *either* $\boldsymbol{K}(\varphi, \mathfrak{A})$ *is bounded or* $\mathfrak{A} \models \varphi$.

Proof. The equivalence of conditions (i) and (ii) follows immediately from the theorem of Frayne (used above in the proof of 1.12). The equivalence of conditions (ii) and (iii) follows immediately from Lemma 2.2 of [8]. The proof that (i) implies (iv) is the same as the argument in 1.9. Conversely, suppose that (iv) holds; we will prove (i). For each $(\exists, \wedge)$-sentence φ with $\mathfrak{A} \models \neg\varphi$, we let $\mathfrak{n}(\varphi)$ be an upper bound on the φ-chromatic number of all $\mathfrak{A}$-pure structures. We let $\mathfrak{n} = \Pi \mathfrak{n}(\varphi)$. We take $\mathcal{B}$ to be a set of $\mathfrak{A}$-pure structures containing one isomorph of every $\mathfrak{A}$-pure structure $\mathfrak{B}_1$ with $|\mathfrak{B}_1| \leqslant \mathfrak{n}$. We then let $\mathfrak{B} = \bigcup \mathcal{B}$, amalgamated on the constants of the language of $\mathfrak{A}$. It is easy to check that $\mathfrak{B}$ is $\mathfrak{A}$-pure, and hence, by 2.1, that there is a structure $\mathfrak{C} \equiv \mathfrak{A}$, and a homomorphism $f\colon \mathfrak{B} \to \mathfrak{C}$.

We wish to prove that $\mathfrak{C}$ is weakly atomic-compact. Suppose that $\mathfrak{A}_1$ is $\mathfrak{C}$-pure; clearly $\mathfrak{A}_1$ is also $\mathfrak{A}$-pure. Thus for each φ as above, we have a homomorphism $g_\varphi\colon \mathfrak{A}_1 \to \mathfrak{B}_\varphi$, where $|\mathfrak{B}_\varphi| \leqslant \mathfrak{n}(\varphi)$, and $\mathfrak{B}_\varphi \models \neg\varphi$. $\mathfrak{B}_1$ will be taken to be isomorphic to the direct product $\boldsymbol{P}\mathfrak{B}_\varphi$. Clearly $|\mathfrak{B}_1| \leqslant \mathfrak{n}$, and so we may take $\mathfrak{B}_1 \in \mathcal{B}$, by 2.1. Clearly there is a homomorphism

$g\colon \mathfrak{A}_1 \to \mathfrak{B}_1$, and so $f \circ g$ is a homomorphism from $\mathfrak{A}_1$ into $\mathfrak{C}$. Thus by 2.4, $\mathfrak{C}$ is weakly atomic-compact. QED

COROLLARY 3.2. *If every finite reduct of $\mathfrak{A}$ is elementarily equivalent to some weakly atomic-compact structure, then the same is true of $\mathfrak{A}$.*

COROLLARY 3.3. *If every relation of $\mathfrak{A}$ is unary, then $\mathfrak{A}$ is elementarily equivalent to some weakly atomic-compact structure.*

Remark 3.4. It is obvious that Theorem 3.1 remains true for *algebraic structures* (structures with both relations and operations) if our definition of chromatic numbers is extended as follows. If $\mathfrak{A}$ is an algebraic structure, then there exists a relational structure $\mathfrak{A}^*$, with the same universe as $\mathfrak{A}$, and having as relations the relations of $\mathfrak{A}$ together with the graphs of the operations of $\mathfrak{A}$. For each $(\exists, \wedge)$-sentence φ in the language of $\mathfrak{A}$, there is obviously an $(\exists, \wedge)$-sentence φ^* in the language of $\mathfrak{A}^*$, which is equivalent to φ in the following sense: for every structure $\mathfrak{B}$ similar to $\mathfrak{A}$, $\mathfrak{B} \models \varphi$ iff $\mathfrak{B}^* \models \varphi^*$. Now define $\chi(\varphi, \mathfrak{A})$ to be $\chi(\varphi^*, \mathfrak{A}^*)$.

4. Finite chromatic number. In this section we show how the chromatic numbers defined in § 1 above are related to the chromatic numbers defined in [6]; we assume (only for Theorem 4.1) that the reader is familiar with [6].

THEOREM 4.1. *For every n-ary derived relation R of $\mathfrak{A}$, and every equivalence relation ϱ on $\{1, \dots, n\}$, there is an $(\exists, \wedge)$-sentence φ in the language of $\mathfrak{A}_A$ such that $\chi_\varrho(R) = \chi(\varphi, \mathfrak{A}_A)$. Conversely, for every such φ, there exist such R and ϱ, with the property that $\chi_\varrho(R) = \chi(\varphi, \mathfrak{A}_A)$.*

Proof. Given R and ϱ, we define the $(\exists, \wedge)$-sentence $\varphi = \Phi(R, \varrho)$ to be the sentence which asserts that $\chi_\varrho(R) = \infty$. We claim that $\chi_\varrho(R) = \chi(\varphi, \mathfrak{A}_A)$. It is easy to check that $\chi_\varrho(R) \leqslant \chi(\varphi, \mathfrak{A}_A)$. To prove the reverse inequality, assume that $\chi_\varrho(R) = \mathfrak{n}$; thus there is a homomorphism $f\colon \langle A, R\rangle \to \mathfrak{S}(\varrho, \mathfrak{n}) = \langle B, S\rangle$, where $|B| = \mathfrak{n}$. For each m-ary relation R of A, we define the m-ary relation S_t on B as the least relation making f a homomorphism. The definition of $\mathfrak{S}(\varrho, \mathfrak{n})$ ensures that $\neg\varphi$ holds in $\mathfrak{B} = \langle B, S_t\rangle_{t\in T}$. Thus $\chi(\varphi, \mathfrak{A}_A) \leqslant \mathfrak{n}$.

To complete the proof, we show that for each $(\exists, \wedge)$-sentence φ, there exists R and ϱ as above such that $\Phi(R, \varrho)$ is logically equivalent to φ. Clearly any such φ is logically equivalent to a sentence $\exists x_1 \dots \exists x_n(\psi \wedge \theta)$, where ψ is a conjunction of atomic formulas in the language of $\mathfrak{A}_A$ with no variable occurring twice in ψ, and θ is a conjunction of equalities between variables. Clearly ψ defines an n-ary derived relation R of $\mathfrak{A}$, and θ defines an equivalence relation ϱ on the set $\{1, \dots, n\}$. It is easy to check that φ is logically equivalent to $\Phi(R, \varrho)$. QED

In Theorem 4.2 which follows, the equivalence of conditions (i), (iii) and (iv) follows immediately from Theorem 2.10 of [6] and Theorem 4.1 above. Similarly, in Corollary 4.3, the equivalence of (i), (iii) and (iv) follows from Theorem 3.6 of [6] and Theorem 4.1 above. Our proofs of these results, however, are direct, and depend neither on [6] nor on Theorem 4.1 above.

THEOREM 4.2. *Let $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$, and let $\mathfrak{n} = |A| + |T|$. The following conditions are equivalent:*

(i) *$\mathfrak{A}$ is a pure substructure of $\beta\mathfrak{A}$;*

(ii) *$\mathfrak{A}$ is a pure substructure of some product of $\mathfrak{n}$ finite relational structures;*

(iii) *$\mathfrak{A}$ is a pure substructure of a compact topological relation structure;*

(iv) *for every $(\exists, \wedge)$-sentence φ in the language of $\mathfrak{A}_A$, either $\mathfrak{A}_A \models \varphi$, or $\chi(\varphi, \mathfrak{A}_A)$ is finite.*

Proof. Clearly conditions (i) and (ii) each imply condition (iii). Suppose that condition (iii) holds; we will prove condition (iv). $\mathfrak{A}$ is a pure substructure of the compact topological relational structure $\mathfrak{B}$. Suppose that $\chi(\varphi, \mathfrak{A}_A) \geqslant \aleph_0$; we need to show that $\mathfrak{A}_A \models \varphi$. By 1.3, $\chi(\varphi, \mathfrak{B}_A) \geqslant \aleph_0$. Of course φ is logically equivalent to some $(\exists, \wedge)$-sentence $\exists x_1 \dots \exists x_n \psi$, where ψ contains neither any quantifiers nor the equality symbol.(3) Let Θ be the directed set of finite partitions of B, ordered by refinement. For each $\theta \in \Theta$, there exist $b_{1\theta}, \dots, b_{n\theta} \in B$ whose θ-equivalence classes satisfy ψ in $\mathfrak{B}/\theta$. Since B^n is compact, we may take $\langle b_1, \dots, b_n\rangle$ to be the limit in B^n of a subnet of the net $\langle\langle b_{1\theta}, \dots, b_{n\theta}\rangle\colon \theta \in \Theta\rangle$. It is easy to check that $\mathfrak{B}_A \models \psi(b_1, \dots, b_n)$. Thus $\mathfrak{B}_A \models \varphi$ and thus, since $\mathfrak{A}$ is pure in $\mathfrak{B}$, $\mathfrak{A}_A \models \varphi$. Thus condition (iv) holds.

Finally, assuming condition (iv), we will prove (i) and (ii). To see (i), clearly $\mathfrak{A}$ is a substructure of $\beta\mathfrak{A}$. Suppose that $\mathfrak{A}_A \models \neg\varphi$ for the $(\exists, \wedge)$-sentence φ in the language of $\mathfrak{A}_A$. By (iv), there is a homomorphism $f\colon \mathfrak{A}_A \to \mathfrak{B}$, where $\mathfrak{B} \models \neg\varphi$ and $\mathfrak{B}$ is finite, and hence compact Hausdorff. Thus f can be extended to a homomorphism $f\colon \beta\mathfrak{A}_A \to \mathfrak{B}$. Clearly then $\beta\mathfrak{A}_A \models \neg\varphi$. Thus $\mathfrak{A}$ is a pure substructure of $\beta\mathfrak{A}$. To see (ii), notice that whenever $\mathfrak{A}_A \models \neg\varphi$, there is a homomorphism $f_\varphi\colon \mathfrak{A}_A \to \mathfrak{B}_\varphi$, where $\mathfrak{B}_\varphi$ is finite and $\mathfrak{B}_\varphi \models \neg\varphi$. Clearly the direct product $\mathfrak{B} = \boldsymbol{P}\mathfrak{B}_\varphi$ is of the desired kind. Let the homomorphism $f\colon \mathfrak{A}_A \to \mathfrak{B}$ be defined componentwise by the homomorphisms f_φ. It is easy to check that f is an isomorphism of $\mathfrak{A}$ onto a substructure of $\mathfrak{B}$. $\mathfrak{A}$ is pure in $\mathfrak{B}$ by 2.1 and 2.3. QED

COROLLARY 4.3. *Let $\mathfrak{n}$ be as in 4.2. The following conditions are equivalent:*

(i) *$\mathfrak{A}$ is a retract of $\beta\mathfrak{A}$;*

(ii) *$\mathfrak{A}$ is a retract of a product of $\mathfrak{n}$ finite relational structures;*

(3) We exclude the equality symbol from ψ, because if B is not Hausdorff, then equality will not be a closed relation of B.

(iii) $\mathfrak{A}$ *is a retract of a compact topological relational structure*;

(iv) $\mathfrak{A}$ *is atomic-compact, and for every* $(\exists, \wedge)$*-sentence in the language of* $\mathfrak{A}_A$*, either* $\mathfrak{A}_A \models \varphi$*, or* $\chi(\varphi, \mathfrak{A}_A)$ *is finite.*

Proof. By 4.2 and Theorem 2.3 of [8].

Remark 4.4. If $|T| \leqslant |A|$ and $\mathfrak{A}$ satisfies the equivalent conditions of Theorem 4.2, then by 4.2 (ii), $\mathfrak{A}$ is a pure substructure of a compact topological structure of power $2^{|A|}$. Thus our new condition (ii) lowers the power of the compact structure whose existence is asserted in 4.2 (iii). (Recall that the power of βA is $2^{2^{|A|}}$).

Remark 4.5. If $|T| \leqslant |A| = \aleph_0$, then $2^{\aleph_0}$ is the smallest possible power in 4.4. Indeed, let Z be the additive group of integers (with addition expressed as a ternary relation). Z is easily seen to be a pure substructure of a compact topological relational structure $\mathfrak{B}$; for example, one may take $\mathfrak{B} = \boldsymbol{P} Z_n$, where for each integer $n \geqslant 2$, Z_n is the (finite) additive group of integers modulo n. We will show that the cardinality of any such $\mathfrak{B}$ must be at least the cardinality of the continuum. Let P be the set of primes of Z, and let $f\colon P \to Z$ be any function. By the Chinese Remainder Theorem, the congruences $\{x \equiv f(p) (\mathrm{mod}\, p)\}$ are finitely satisfiable in Z, and hence, by compactness, satisfiable in $\mathfrak{B}$. By purity, distinct functions f yield distinct solutions in $\mathfrak{B}$. Thus $\mathfrak{B}$ has continuum many distinct elements.

Remark 4.6. Suppose that $\mathfrak{A} = \langle A, R\rangle$ is *algebraic* in the sense that R is the graph of an (at least binary) operation on A, and that $\mathfrak{A}$ satisfies the equivalent conditions of 4.2. We do not know whether $\mathfrak{A}$ must also satisfy (iii'): $\mathfrak{A}$ is a pure substructure of a compact topological structure $\mathfrak{B} = \langle B, S\rangle$, where S is the graph of an operation on B. Clearly $\beta\mathfrak{A}$ is not always such a structure (e.g. if R is the ternary relation corresponding to addition of integers). Further, if R is the ternary relation corresponding to addition in the set Q of rational numbers, then no product of finite algebras is such a $\mathfrak{B}$. For the reader may easily check that $\langle Q, +\rangle$ is not a subalgebra of any product of finite algebras, since there exists no non-trivial divisible finite Abelian group. But $\langle Q, +\rangle$ satisfies the conditions of 4.2, since $\langle Q, +\rangle$ is a direct summand of the group of rotations of a circle.

5. Pure closure. As in [9], an extension $\mathfrak{B}$ of $\mathfrak{A}$ is called a *closure* of $\mathfrak{A}$ if and only if $\mathfrak{B}_A$ is weakly atomic-compact. If moreover $\mathfrak{B}$ is a pure [elementary] extension of $\mathfrak{A}$, then we call $\mathfrak{B}$ a *pure* [*elementary*] *closure* of $\mathfrak{A}$. If we apply Theorem 3.1 to the structure $\mathfrak{A}_A$, we have the following corollary.

Corollary 5.1. *The following conditions are equivalent*:

(i) $\mathfrak{A}$ *has a pure closure*;

(ii) $\mathfrak{A}$ *has an elementary closure*;

(iii) *for every* $(\exists, \wedge)$*-sentence* φ *in the language of* $\mathfrak{A}_A$*, either* $\mathbf{K}(\varphi, \mathfrak{A}_A)$ *is bounded or* $\mathfrak{A}_A \models \varphi$.

Proof. The equivalence of conditions (ii) and (iii) is immediate from Theorem 3.1 and the fact that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ if and only if $\mathfrak{A}_A \equiv \mathfrak{B}_A$. The equivalence of conditions (i) and (ii) follows immediately from Lemma 2.2 of [8]. QED

In [9], § 3, Węglorz asks whether every algebra having a closure has an atomic compactification. We ask a similar question in the present context. Namely, if the relational structure $\mathfrak{A}$ has a pure closure, then does $\mathfrak{A}$ have an atomic-compact pure extension?

6. A correction to [5]. We take this opportunity to correct the proof of the second part of Theorem 4.7 of [5], which asserts *the existence of an atomic-compact algebra which is not a retract of a compact topological algebra*. The algebra $\langle G^*, F\rangle$ defined in the proof given there does not seem to be atomic-compact. We now define another algebra $\mathfrak{A} = \langle A, f, g\rangle$, where f and g are unary operations, which satisfies our assertion.

Let $\langle G, R\rangle$ be the graph defined in § 4 of [5]. Let N be a countably infinite set disjoint from G, and let a and b be distinct elements not in $N \cup G$. Let $A = N \cup G \cup \{a, b\}$. Define the unary operations f and g as follows: $\langle f, g\rangle \restriction N$ is a bijection of N onto R, and for $x \notin N$, $f(x) = a$ and $g(x) = b$. Notice that any ultrapower $\mathfrak{A}'$ of $\mathfrak{A}$ is the reduct of a structure $\langle A', f', g', R'\rangle$, where $A' = N' \cup G' \cup \{a, b\}$, $\langle G', R'\rangle$ is an elementary extension of $\langle G, R\rangle$, $\langle f', g'\rangle \restriction N'$ is a bijection of N' onto R', and for $y \notin N'$, $f'(y) = a$ and $g'(y) = b$. Clearly a retraction of $\langle G', R'\rangle$ onto $\langle G, R\rangle$ extends to a retraction of $\mathfrak{A}'$ onto $\mathfrak{A}$. Thus $\mathfrak{A}$ is atomic-compact (by [8], Theorem 2.3).

Our proof that $\mathfrak{A}$ is not a retract of a compact topological relational structure uses a chromatic number of the kind defined in § 1. Consider the sentence

$$\varphi = \exists x \exists y [f(x) = y \wedge g(x) = y].$$

It is clear that $\mathfrak{A} \models \neg\varphi$. But suppose that $F\colon \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, where $\mathfrak{B}$ is a finite structure. Since the chromatic number of the graph $\langle G, R\rangle$ is infinite, there exists $\langle u, v\rangle \in R$ such that $F(u) = F(v) = y$. By definition of f and g, we may find $s \in N$ such that $u = f(s)$ and $v = g(s)$. Taking $x = F(s)$ and y as above, we see that $\mathfrak{B} \models \varphi$. Thus $\mathfrak{A}$ is not a retract of a compact topological algebra (or structure) since $\chi(\varphi, \mathfrak{A})$ is infinite.

The existence of an algebra with two unary operations which is atomic-compact but not a retract of a compact topological algebra is to be contrasted with the result of Wenzel [10] that there is no such algebra with only one unary operation. On the other hand, there exists such an algebra with only one binary operation, which we now define.

Let $A = N \cup G \cup \{a, b\}$ be as above, and again take f and g to be the two components of a bijection from N onto R. We define

$$F(x, y) = \begin{cases} f(x) & \text{if } x \in N \text{ and } y = a; \\ g(y) & \text{if } y \in N \text{ and } x = a; \\ a & \text{if } x \in G \text{ and } y \in G; \\ b & \text{otherwise.} \end{cases}$$

The proof that $\langle A, F\rangle$ is atomic-compact is very similar to the above proof that $\langle A, f, g\rangle$ is atomic-compact, and will therefore be omitted. To see that $\langle A, F\rangle$ is not a retract of a compact topological relational structure, it suffices to show that $\chi(\varphi, \langle A, F\rangle) = \aleph_0$, where

$$\varphi = \exists x \exists y [F(x, x) = a \wedge F(y, a) = x \wedge F(a, y) = x].$$

This is shown by an argument similar to the preceeding one.

Added in proof (February 19, 1971). I stated Problem 1.14 (for graphs), together with a related problem in pure graph theory, as Problem 43 in *Combinatorial Structures and their Applications*, Gordon and Breach, New York, 1970. My note, *Generalized chromatic numbers*, in the same volume, gives some further information on chromatic numbers.

For a syntactic condition equivalent to the conditions of Theorem 3.1 [or of Corollary 5.1], see my paper with G. Fuhrken, *Weakly atomic-compact relational structures*, to appear in J. Symbolic Logic.

I give a positive answer to the question asked here in the last sentence of § 5, in my paper, *Some constructions of compact algebras*, to appear in Annals of Mathematical Logic.

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Reçu par la Rédaction le 19. 2. 1969

Retracting fans onto finite fans

by

J. B. Fugate (Lexington, Ken.)

1. Introduction. By a *continuum* we mean a compact connected metric space. A continuum which is hereditarily unicoherent and arcwise connected is a *dendroid*. A dendroid which has only one ramification point (a point which is the common part of 3 arcs, and an end point of each) is called a *fan*. A locally connected dendroid is called a *tree* or *dendrite*. A dendroid is *finite* if the set of end points is finite. Clearly, each finite dendroid is a tree and finite fans are the union of a finite collection of arcs, whose common part is a single point. The cone over the Cantor set, on $[0, 1]$ is a planar fan.

It is easy to see that dendroids are hereditarily decomposable and thus one-dimensional. In this paper we will establish that fans have a very strong one-dimensional structure, namely, they can be approximated from within by finite fans. This is the content of Theorem 1, which states that each fan can be retracted onto a finite fan, by a map which does not move points very far. From this it follows that each fan is tree-chainable, indeed is an inverse limit of finite fans, and (in a joint work with C. A. Eberhart) that the product of any collection of fans has the fixed point property.

2. Preliminary results. A *chain*, in a metric space, is a collection $\mathcal{E} = \{E_1, \ldots, E_m\}$ of open sets such that $E_i \cap E_j \neq \emptyset$ iff $|i-j| \leqslant 1$. The elements of $\mathcal{E}$ are *links*; frequently we denote $\mathcal{E}$ by $E(1, m)$ and denote $\bigcup\{E_i\colon 1 \leqslant i \leqslant m\}$ by $E^*(1, m)$ or $\mathcal{E}^*$. If each link of $\mathcal{E}$ has diameter $< \varepsilon$, we call $\mathcal{E}$ an *ε-chain*. A *tree chain* is a finite collection of open sets, no three of which have a point in common and the collection contains no circular chains. We shall often use Z^n to denote the first n positive integers.

The ramification point of a fan is called the *top*. It is shown in [1] that each point of a fan, except the top, lies on a unique arc from the top to an end point. We wish to commence our proof of Theorem 1 by covering each such arc by a chain in which the arc is straight.

DEFINITION. If $[a, b]$ is a an arc and $\mathcal{E} = E(1, m)$ is a chain covering $[a, b]$ then $[a, b]$ is *straight* in $\mathcal{E}$ provided