Internally standard set theories

by

George Wilmers (Manchester)

In this paper we prove results which show that the standard model property is extremely incompact. In particular we show that if there is an inaccessible cardinal then there is a complete extension T of first-order Zermelo–Fraenkel set theory such that every subtheory of T definable in T has a standard model but T has no standard model. Furthermore if we make a stronger assumption (the existence of a Ramsey cardinal suffices) then we can find a complete extension T such that every constructible subtheory of T has a standard model, but T has no standard model. These results are developed in Section 3. The method of proof uses a technique of model construction involving the compactness and completeness theorems of [1] which was first used by Barwise in [3].

Various results concerning pointwise definable models for set theory are also required in Section 3 and these are developed in Section 2. (A model is called pointwise definable if every member of the domain of the model is first-order definable in the model.) In particular we show that if T is a complete extension of first order Zermelo–Fraenkel set theory satisfying the axiom that every set is ordinal definable, then T has no standard model if and only if T has a pointwise definable model which is not well-founded. This provides a criterion which is useful in constructing set theories having no standard model.

In Section 2 we also introduce the idea of a *rank extension* of a model for set theory. This is a natural strengthening of the concept of an end extension obtained by adding the condition that every new element shall have a rank which is not an ordinal of the original model. This definition is clearly related to the idea of a natural model. In Section 3 we prove that every countable standard model for set theory has a proper rank extension and, further, if every set is ordinal definable in the original model then the rank extension can be chosen to be pointwise definable.

Section 1

Preliminaries. When we refer to formulae of set theory we shall mean formulae of the first order predicate calculus with $=$ and ϵ as the only non-logical constants. The following abbreviations for sentences and sets of sentences will be used throughout this paper:

ZF	The set of axioms of Zermelo–Fraenkel set theory.
AC	The axiom of choice.
$V = L$	Every set is constructible.
$V = OD$	Every set is ordinal definable (see e.g. [7] for definition).
In	There exists a strongly inaccessible cardinal greater than ω.

As our metatheory we shall use an informal theory similar to ZF+AC. Sentences will be identified with their Gödel numbers wherever this is possible without confusion.

$F'\alpha$ will denote the αth constructible set in Gödel's hierarchy and $F''\alpha = \{F'\beta \mid \beta < \alpha\}$. The *rank* of a set x is defined as usual and will be denoted by $\mathrm{rk}(x)$.

A set B is *transitive* if $x \in B$ implies $x \subseteq B$. A model for ZF of the form $\langle A, \epsilon\rangle$ where A is a transitive set will be called a *standard* model. (Note that this is a strengthening of the usual definition.) A model for ZF of the form $\langle B, E\rangle$ is *well-founded* if there is no infinite sequence $\langle b_i\rangle_{i<\omega}$ of elments of B such that $b_{n+1}\ E b_n$ for each $n < \omega$. By an oft quoted result of Mostowski every well-founded model of the axiom of extensionality is isomorphic to a unique standard model. By a theorem of Gödel each standard model of ZF+(V = L) is of the form $\langle F''\alpha, \epsilon\rangle$ where α is the supremum of the ordinals in the model. If $\langle B, E\rangle$ is a model for ZF an ordinal μ of the model is called *standard* if the set $\{\nu \mid \nu \in B \text{ and } \nu E \mu\}$ is well-ordered by the relation E. The model $\langle B, E\rangle$ is called ω_1-*standard* if every countable ordinal (in the sense of the model) is standard, and ω-*standard* if every finite ordinal of the model is standard.

If $\mathcal{M}$ is any structure $\mathfrak{D}(\mathcal{M})$ is defined to be the substructure of $\mathcal{M}$ whose domain is the set of elements of the domain of $\mathcal{M}$ which are first order definable in $\mathcal{M}$. [$a \in \mathrm{dom}(\mathcal{M})$ is said to be first-order definable in $\mathcal{M}$ if there is a $\varphi(x)$ with one free variable x in the ordinary first order language for $\mathcal{M}$ such that

$$\mathcal{M} \models \varphi[a] \quad \text{and} \quad \mathcal{M} \models \exists!\, x\varphi(x)]\,.$$

If $\mathfrak{D}(\mathcal{M}) = \mathcal{M}$ then $\mathcal{M}$ is said to be *pointwise definable*. Pointwise definable models for set theory have been studied in [6], [9] and [3]. The terminology is due to Barwise [3].

If $\langle A, E\rangle$ and $\langle A', E'\rangle$ are models for ZF, we say that $\langle A', E'\rangle$ is an *end extension* and write $\langle A, E\rangle \subseteq_e \langle A', E'\rangle$ if $\langle A', E'\rangle$ is an extension of $\langle A, E\rangle$ and whenever $aE'b$ and $b \in A$ then $a \in A$.

If $\mathcal{M}$ and $\mathcal{N}$ are relational systems of the same type, $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M} \prec \mathcal{N}$ will denote "$\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$" and "$\mathcal{M}$ is an elementary subsystem of $\mathcal{N}$" respectively. For definitions of these terms see e.g. [4]. For any system $\mathcal{M}$, the domain of that system will be denoted by $\mathrm{dom}(\mathcal{M})$.

In Section 3 we shall make use of the infinitary language $\mathfrak{L}_A$ of [1] where $\langle A, \ni\rangle$ is a countable standard model for ZF. Formulae of $\mathfrak{L}_A$ are sets in A and conjunctions and disjunctions over sets of $\mathfrak{L}_A$-formulae which are elements of A are permitted. A set of $\mathfrak{L}_A$-formulae which is a member of A is called A-*finite*. (For further details see [1] or [2]). We remark that for each $a \in A$ there is a constant c_a of the language $\mathfrak{L}_A$. Now we shall need to associate with each $a \in A$ a constant of $\mathfrak{L}_A$ in such a way that not all the constants of $\mathfrak{L}_A$ are used up. Accordingly for each $a \in A$ we denote by $\bar{a}$ the constant $c_{\langle a,0\rangle}$. A set Σ of $\mathfrak{L}_A$-formulae is said to be *definable* if there is a formula of set theory $\varphi(x)$ with one free variable x such that for any $a \in A$ $a \in \Sigma$ if and only if $\langle A, \epsilon\rangle \models \varphi[a]$. The notion of a derivation D from a set Σ of $\mathfrak{L}_A$-sentences is defined as in [1]. $\mathfrak{Der}(\Sigma)$ is the class of derivations from Σ and $\mathfrak{Der}_A(\Sigma) = A \cap \mathfrak{Der}(\Sigma)$.

Throughout this paper all theories to which the term complete is applied will be assumed to be consistent.

Section 2

We shall make use of the following consequence of theorems of Myhill and Scott [7], and Montague and Vaught [5]:

Result 2.1. *If $\mathcal{M}$ is a model for* ZF *then the following are equivalent*

(i) $\mathcal{M} \models (V = OD)$,

(ii) $\mathfrak{D}(\mathcal{M}) \prec \mathcal{M}$,

(iii) *there is a pointwise definable model $\mathcal{N}$ such that $\mathcal{M} \equiv \mathcal{N}$.*

It is easy to show that the model $\mathcal{N}$ of (iii) above is unique up to isomorphism. That the following more general theorem holds was observed by Jane Bridge.

Theorem 2.2. *If $\mathcal{M}$ and $\mathcal{N}$ are pointwise definable structures of the same type and $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. Let $\mathfrak{L}$ be the first-order language with equality corresponding to the type of $\mathcal{M}$ and $\mathcal{N}$. Let Σ be the set of formulae $\varphi(x)$ of $\mathfrak{L}$ with one free variable x such that

$$\mathcal{M} \models \exists!\, x\varphi(x).$$

We define an equivalence relation "$\approx$" on the members of Σ as follows:

$$\varphi(x)\approx\psi(x) \quad \text{if and only if} \quad \mathcal{M}\models\forall x\big(\varphi(x)\leftrightarrow\psi(x)\big).$$

We denote the equivalence class of $\varphi(x)$ by $[\![\varphi(x)]\!]$.

Now let $C=\{[\![\varphi(x)]\!] \mid \varphi(x) \text{ is in } \Sigma\}$ and let $\mathfrak{C}=\langle C, R_i\rangle$ be the structure of the same type as $\mathcal{M}$ with the n_i-ary relations R_i on C given by

$$\langle a_1,\ldots,a_{n_i}\rangle\in R_i \quad \text{if and only if}$$
$$\mathcal{M}\models\forall x_1\ldots\forall x_{n_i}[\varphi_1(x_1)\wedge\ldots\wedge\varphi_{n_i}(x_{n_i})\to P_i(x_1,\ldots,x_{n_i})]$$

where P_i is the predicate of $\mathfrak{L}$ corresponding to R_i and $\varphi_1(x),\ldots,\varphi_{n_i}(x)$ are members of Σ such that $[\![\varphi_r(x)]\!]=a_r$ for each r such that $1\leqslant r\leqslant n_i$.

It is easy to see that $\mathfrak{C}\cong\mathcal{M}$. We can define an isomorphism $f\colon \mathcal{M}\xrightarrow{\text{onto}}\mathfrak{C}$ as follows: for any m in the domain of $\mathcal{M}$, let $\varphi(x)$ be a formula defining m in $\mathcal{M}$ and let $f(m)=[\![\varphi(x)]\!]$. It is clear that this definition is independent of the particular choice of $\varphi(x)$, and that f defines an isomorphism. But the construction of $\mathfrak{C}$ depends only on the theory of $\mathcal{M}$ and so since $\mathcal{M}\equiv\mathcal{N}$ we have also $\mathfrak{C}\cong\mathcal{N}$. Hence $\mathcal{M}\cong\mathcal{N}$ and the theorem is proved.

The theorem above has a useful consequence which gives a criterion for certain set theories to have no standard model;

COROLLARY 2.3. *If T is a complete extension of* $\mathrm{ZF}+(\mathrm{V}=\mathrm{OD})$, *then T has no standard model if and only if T has a pointwise definable model which is not well-founded.*

Proof. Let $\mathcal{M}$ be a pointwise definable model for T, guaranteed by 2.1. If T has no standard model then $\mathcal{M}$ is not well-founded. Conversely if $\mathcal{M}$ is not well-founded suppose that there exists a standard model $\mathcal{N}$ for T. Then $\mathfrak{D}(\mathcal{N})$ is a well-founded pointwise definable model for T. But by theorem 2.2 we must have that $\mathcal{M}\cong\mathfrak{D}(\mathcal{N})$ which is clearly impossible.

The next two theorems concern end extensions of pointwise definable models.

THEOREM 2.4. *If $\mathcal{M}$ is a standard pointwise definable model for* $\mathrm{ZF}+(\mathrm{V}=\mathrm{L})$, *$\mathcal{N}$ is a model for* $\mathrm{ZF}+(\mathrm{V}=\mathrm{L})$ *which is a proper end extension of $\mathcal{M}$, and $\omega_1^{\mathcal{N}}$ is in $\mathcal{M}$ then*

(i) *$\mathcal{N}$ is not well-founded,*

(ii) *the standard ordinals of $\mathcal{N}$ are exactly the ordinals in $\mathcal{M}$.*

Proof. Suppose that there exists a supremum θ in $\mathcal{N}$ of the ordinals in $\mathcal{M}$. Let $\Phi(x,y)$ be the usual formula of set theory defining Gödel's F-function, so that $\Phi(x,y)$ reads "x is an ordinal and $F'x=y$". Now $\Phi(x,y)$ is equivalent to a Σ-formula and so is preserved under end extensions. It follows that the formula $\Psi(z,\alpha)$ given by

$$\Psi(z,\alpha)\leftrightarrow(\alpha \text{ is an ordinal})\wedge\forall x\big[x\in z\leftrightarrow\exists\beta\big(\beta<\alpha\wedge\Phi(\beta,x)\big)\big]$$

actually defines $F''\theta$ (i.e. the domain of $\mathcal{M}$) when we put α equal to θ in $\mathcal{N}$. Hence $\mathcal{M}\in\mathcal{N}$. But it is a theorem of ZF that the supremum of the ordinals in a standard pointwise definable model for ZF is less than ω_1. Since $\mathcal{N}\models\mathrm{ZF}$ this statement holds in $\mathcal{N}$ and so we must have that $\theta<\omega_1^{\mathcal{N}}$. But this contradicts the hypothesis that $\omega_1^{\mathcal{N}}$ is in $\mathcal{M}$. Thus the ordinals in $\mathcal{N}$ which are not in $\mathcal{M}$ have no infimum and so are non-standard. Since $\mathcal{N}\models(\mathrm{V}=\mathrm{L})$ and $\mathcal{N}\neq\mathcal{M}$ there must be such ordinals in $\mathcal{N}$ which are not in $\mathcal{M}$ and so the theorem is proved.

The following corollary to Theorem 2.4 is an analogue for the constructible hierarchy of Theorem 5.4 of [4]. It is interesting to observe that the original proof of [4] cannot be used in this case.

COROLLARY 2.5. *If $\langle F''\alpha,\epsilon\rangle$ is a model for* ZF, *$\alpha'<\alpha$ and if $\langle F''\alpha',\epsilon\rangle\prec\langle F''\alpha,\epsilon\rangle$, then there is an $\alpha^*<\alpha'$ such that*

$$\langle F''\alpha^*,\epsilon\rangle\equiv\langle F''\alpha,\epsilon\rangle .$$

Proof. This follows from the theorem above together with the fact that if $\mathcal{M}$ is a standard model for $\mathrm{ZF}+(\mathrm{V}=\mathrm{L})$, $\mathcal{M}$ is pointwise definable if and only if $\mathcal{M}$ contains no element which is a structure elementarily equivalent to $\mathcal{M}$ (see Corollary 2 of Theorem 1 of [9]).

We now introduce a strengthening of the concept of an end extension which we shall use in the next section. We first note that the notion of rank is absolute with respect to end extensions in the sense that if $\mathcal{M}$ and $\mathcal{N}$ are models for ZF, $\mathcal{M}\subseteq_e\mathcal{N}$, a is in $\mathcal{M}$ and $\mathrm{rk}(a)=\alpha$ within $\mathcal{M}$, then $\mathrm{rk}(a)=\alpha$ within $\mathcal{N}$.

DEFINITION 2.6. If $\mathcal{M}$ and $\mathcal{N}$ are models for ZF, $\mathcal{N}$ is said to be a *rank extension* of $\mathcal{M}$, and we denote this by $\mathcal{M}\subseteq_r\mathcal{N}$, if $\mathcal{N}$ is an end extension of $\mathcal{M}$ and for any a in $\mathcal{N}$, if $\mathrm{rk}(a)$ is in $\mathcal{M}$, then a is in $\mathcal{M}$.

Intuitively, if $\mathcal{N}$ is a rank extension of $\mathcal{M}$ then $\mathcal{M}$ is an "initial segment" of $\mathcal{N}$. If $\mathcal{N}$ is standard, then $\mathcal{M}$ is a natural model for ZF within $\mathcal{N}$.

We observe the following simple facts about rank extensions. Let $\mathcal{M}$ and $\mathcal{N}$ be models for ZF and $\mathcal{M}\subseteq_r\mathcal{N}$. Then if $a\in\mathrm{dom}(\mathcal{M})$, $b\in\mathrm{dom}(\mathcal{N})$, and $b\subseteq a$, it follows that $b\in\mathrm{dom}(\mathcal{M})$. A consequence of this is that if $\alpha\in\mathrm{dom}(\mathcal{M})$, then $R(\alpha)$, the set of all elements of rank less than α, is the same set when constructed in $\mathcal{N}$ as when constructed in $\mathcal{M}$. It follows in turn from this that all cardinals are preserved by rank extensions; thus if $\alpha\in\mathrm{dom}(\mathcal{M})$ and α is a cardinal of $\mathcal{M}$, then α is a cardinal of $\mathcal{N}$.

The following theorem is an analogue of Theorem 2.4.

THEOREM 2.7. *If $\mathcal{M}$ is a standard pointwise definable model for* ZF *and $\mathcal{N}$ is a model for* ZF *which is a proper rank extension of $\mathcal{M}$, then* (i) *and* (ii) *of Theorem* 2.4 *hold.*

Proof. The proof is similar to that of Theorem 2.4. For if the supremum of the ordinals of $\mathcal{M}$ were an ordinal θ of $\mathcal{N}$, then $R(\theta)$ in $\mathcal{N}$ would define dom$(\mathcal{M})$. But since $\theta > \omega$, $R(\theta)$ cannot be countable in $\mathcal{N}$ and hence $\langle R(\theta), \epsilon\rangle$ cannot be pointwise definable in $\mathcal{N}$. Since the notion of pointwise definability is absolute for ω-standard models for ZF it follows that $\langle R(\theta), \epsilon\rangle$ is not pointwise definable, which contradicts the assumption about $\mathcal{M}$. Hence there is no supremum in $\mathcal{N}$ of the ordinals in $\mathcal{M}$ and the theorem follows.

Section 3

Definition 3.1. If T is a complete extension of ZF, a set of integers s is said to be *T-definable* if there is a formula of set theory $\varphi(x)$ with one free variable x such that for each integer n, $n \in s$ if and only if $\vdash_T \varphi(\check{n})$ (here $\varphi(\check{n})$ is an abbreviation for $\forall x(\psi_n(x) \to \varphi(x))$ where $\psi_n(x)$ is the formula defining the numeral n). A theory is said to be T-definable if its set of Gödel numbers is T-definable.

In this section we answer the following question: is there a complete extension of ZF such that every T-definable subtheory of T has a well-founded model, but T has no well-founded model? The answer to this question is yes, if there exists an inaccessible cardinal. This shows that the property of having a standard model is highly incompact. In order to derive this result we introduce the following concept;

Definition 3.2. An extension T of ZF is said to be *internally standard* (abbreviated to i.s.) if T is complete and whenever S is a T-definable subtheory of T the sentence SM(S), which asserts that S has a standard model, is a theorem of T.

A model $\mathcal{M}$ for ZF is said to be *internally standard* if its theory is internally standard.

The condition of being internally standard can be regarded as a global reflection principle on T. The following lemma shows that quite weak model theoretic conditions on standard models are sufficient to generate i.s. theories.

Lemma 3.3. *If $\langle A, \epsilon\rangle$ is a standard model of* ZF *and there is an $x \in A$ such that $\langle x, \epsilon\rangle \equiv \langle A, \epsilon\rangle$ then the theory of $\langle A, \epsilon\rangle$ is* i.s.

The proof is clear.

The next lemma guarantees the existence of models with the properties required for our construction.

Lemma 3.4. *In* ZF+In *one can prove that there exists a standard, pointwise definable* i.s. *model for* ZF.

Proof. Let θ be the first inaccessible. Then θ is an inaccessible in the constructible universe and so $\langle F''\theta, \epsilon\rangle$ is a model for ZF. Also by Theorem 6.8 of [4] there is an $\alpha < \theta$ such that $\langle F''\alpha, \epsilon\rangle \equiv \langle F''\theta, \epsilon\rangle$. Now it is clear that $F''\alpha \in F''\theta$ and so by Lemma 3.3 the theory of $\langle F''\theta, \epsilon\rangle$ is i.s. Now let $\mathcal{M}$ be the standard model which is isomorphic to $\mathfrak{D}(\langle F''\theta, \epsilon\rangle)$. Then by 2.1, $\mathcal{M}$ has the required properties.

We shall need the following version of the completeness and compactness theorems of [1]. Both these results follow immediately from the generalized version of Theorem 2.12 of [1] if one observes that if $\langle A, \epsilon\rangle$ is a standard model for ZF then for any predicate P definable in the language of ZF, A is P-admissible. This was noted by Barwise in [2].

Result 3.5 (Barwise). *Let $\langle A, \epsilon\rangle$ be a countable standard model for set theory, and let Σ be a definable set of sentences of $\mathfrak{L}_A$. Then*

(a) *if φ is a sentence of $\mathfrak{L}_A$ which is a logical consequence of Σ, then there is a derivation $D \in \mathfrak{Der}_A(\Sigma)$ of $\vdash \varphi$,*

(b) *if every A-finite subset of Σ has a model, then Σ has a model.*

We now come to the main part of the construction which makes use of a technique of model construction first used in [3]. We shall call a theory Q *arithmetic* if the set of Gödel numbers of its sentences is definable by a first order formula with one free variable over the structure $\langle \omega, <, +, \cdot\rangle$

Lemma 3.6. *If Q is an arithmetic extension of* ZF+(V = OD) *which has a countable, standard,* i.s. *model $\mathcal{M}$, then Q has a pointwise definable,* i.s. *model $\mathcal{N}$ which is a proper rank extension of $\mathcal{M}$.*

Proof. Let $\mathcal{M}$ be $\langle A, \epsilon\rangle$. We shall assume first that $\mathcal{M}$ is pointwise definable. Let Gd be the set of Gödel numbers of sentences in the language of set theory and for each $n \in$ Gd let φ_n be the sentence with Gödel number n. Let SM(x) be the formula asserting that $x \subseteq$ Gd and x has a standard model.

Let Σ be the following set of sentences of $\mathfrak{L}_A$:

(1) $\bigwedge Q$,

(2) $(c_0 \text{ is an ordinal}) \wedge c_0 > \bar{\alpha}$ for each ordinal $\alpha \in A$,

(3) $\forall x[x \in \bar{a} \leftrightarrow \bigvee_{b \in a} x = \bar{b}]$ for each $a \in A$,

(4) $\forall x[rk(x) < \bar{\alpha} \leftrightarrow \bigvee_{b|\mathrm{rk}(b)<\alpha} x = \bar{b}]$ for each ordinal $\alpha \in A$,

(5) $\forall x \bigvee_{\psi(y)} [\forall y(\psi(y) \leftrightarrow y = x)]$,

(6) $\bigwedge_{\psi(y)}\Big[\bigwedge_{n \in \mathrm{Gd}} (\psi(\bar{n}) \to \varphi_n) \to \forall x[\forall y(y \in x \leftrightarrow y \in \mathrm{Gd} \wedge \psi(y)) \to \mathrm{SM}(x)]\Big]$.

The infinite disjunction in (4) is taken over all b whose rank in $\mathcal{M}$ is less than α. The disjunction in (5) and first conjunction in (6) are taken over all formulae of set theory with one free variable, y. Conditions (3), (4), (5) and (6) correspond to the property of being an end extension, a rank extension, pointwise definable and internally standard respectively. (1), (5) and (6) are single sentences of $\mathfrak{L}_A$ whereas (2), (3) and (4) are infinite schemas, but it is easily seen that the whole collection is definable over $\mathcal{M}$ by a formula of set theory, $H(x)$.

Now if Σ_0 is an A-finite subset of Σ we can make $\mathcal{M}$ a model for Σ_0 by taking the realization of each $\bar{a}$ occurring in Σ_0 to be a, and the realization of c_0 to be the supremum of the set $\{a|\ a$ is an ordinal in A and $\bar{a}$ occurs in $\Sigma_0\}$. (Since Σ_0 is A-finite we know that this supremum is an ordinal of A.) Thus each such subset Σ_0 has a model. Applying Result 3.5 we obtain a model $\mathcal{N}$ for Σ. By (3) we can take $\mathcal{N}$ to be an end extension of $\mathcal{M}$. Conditions (4) and (2) ensure that $\mathcal{N}$ is a proper rank extension of $\mathcal{M}$, conditions (5) and (6) that $\mathcal{N}$ is pointwise definable and internally standard respectively and condition (1) that $\mathcal{N}$ is a model for Q. This completes the proof for the case when $\mathcal{M}$ is pointwise definable.

Now suppose that $\mathcal{M}$ is not pointwise definable. By 2.1 there is an $\mathcal{M}_0 \equiv \mathcal{M}$ such that $\mathcal{M}_0$ is pointwise definable and standard. Now it is clear that the predicate $H(x)$ mentioned above can be chosen to be absolute in the sense that for *any* standard model $\langle A, \epsilon\rangle$ for ZF, the set of $\mathfrak{L}_A$-sentences (1)–(6) is given by $\{a|\ a \in A$ and $\langle A, \epsilon\rangle \models H[a]\}$. Hence by the above proof the following sentence of set theory holds in $\mathcal{M}_0$:

"For every set Φ of infinitary sentences such that $\forall x(x \in \Phi \to H(x))$, there is no $D \in \mathfrak{Der}(\Phi)$ which is a derivation of the empty sequent."

Since $\mathcal{M} \equiv \mathcal{M}_0$ the above sentence also holds in $\mathcal{M}$, and so by the completeness theorem (3.5 (a)) the set of $\mathfrak{L}_A$-sentences Σ has a model. This completes the proof of the lemma.

Although Lemma 3.6 has no special significance except for our construction, it is perhaps worth noting that the same proof establishes the following more general result:

THEOREM 3.7. *If Q is an arithmetic extension of* ZF *which has a countable standard model $\mathcal{M}$, then Q has a model $\mathcal{N}$ which is a proper rank extension of $\mathcal{M}$. Moreover if $\mathcal{M} \models \mathrm{V} = \mathrm{OD}$ then $\mathcal{N}$ can be chosen to be pointwise definable.*

We are now ready to stablish the principal results of this paper.

THEOREM 3.8. *If Q is an arithmetic extension of* $\mathrm{ZF}+(\mathrm{V}=\mathrm{OD})$ *which has a standard* i.s. *model $\mathcal{M}$, then Q has a complete extension T such that every T-definable subtheory of T has a well-founded model but T has no well-founded model.*

Proof. By 2.1 we can choose $\mathcal{M}$ to be pointwise definable. Let $\mathcal{N}$ be the rank extension of $\mathcal{M}$ constructed in Lemma 3.6 and let T be the theory of $\mathcal{N}$. By Theorem 2.7 $\mathcal{N}$ is not well-founded and so by Corollary 2.3, T has no well-founded model. The crucial step in proving that every T-definable subtheory of T has a well-founded model is to observe that $\mathcal{N}$ is ω_1-standard. (This follows from the fact that $\mathcal{N}$ is a rank extension of $\mathcal{M}$ so that $\omega_1^{\mathcal{N}} = \omega_1^{\mathcal{M}}$.) Now let S be a T-definable subtheory of T. Then $\mathrm{SM}(S)$ can be written as a sentence of set theory and $\mathcal{N} \models \mathrm{SM}(S)$ since $\mathcal{N}$ is i.s. Applying the downward Löwenheim–Skolem theorem and the Mostowski collapsing theorem in $\mathcal{N}$ we have

$\mathcal{N} \models$ "There is a standard model for S the supremum of whose ordinals is less than ω_1".

Since $\mathcal{N}$ is ω_1-standard such a model for S is actually well-founded and so the proof is complete.

COROLLARY 3.9. *In* ZF + In *one can prove that there exists a complete extension T of* ZF *such that every T-definable subtheory has a well-founded model, but T has no well-founded model.*

Proof. Apply Lemma 3.4 to Theorem 3.8.

If we make slightly stronger assumptions we can prove the existence of even stranger theories. Following our previous terminology we call a theory constructible if its set of Gödel numbers is a constructible set.

THEOREM 3.10. *In* ZF + *"There exists a Ramsey cardinal" one can prove that there is a complete extension T of* ZF *such that every constructible subtheory of T has a well-founded model, but T has no well-founded model.*

Proof. By a well-known result of Rowbottom (see e.g. [8]) if there is a Ramsey cardinal, then there is a countable ordinal a such that $\langle F''a, \epsilon\rangle \prec \langle L, \epsilon\rangle$. By Lemma 3.3 the theory of $\langle F''a, \epsilon\rangle$ is i.s. Also since $\omega_1^L < a$ every constructible set of integers is a member of $F''a$. By Lemma 3.6 there is a proper rank extension $\mathcal{N}$ of $\langle F''a, \epsilon\rangle$ which is a pointwise definable i.s. model for $\mathrm{ZF}+(\mathrm{V}=\mathrm{L})$. Let T be the theory of $\mathcal{N}$. $\mathcal{N}$ cannot be well-founded since the supremum of the ordinals in a standard pointwise definable model for $\mathrm{ZF}+(\mathrm{V}=\mathrm{L})$ is less than ω_1^L. Hence by Corollary 2.3, T has no well-founded models. On the other hand since $\mathcal{N}$ is a rank extension of $\langle F''a, \epsilon\rangle$, $\mathcal{N}$ is ω_1-standard. Hence since $\mathcal{N}$ is i.s., every T-definable subtheory of T has a well-founded model. But since $\mathcal{N}$ is pointwise definable and contains every constructible set of integers as an element, it follows that each constructible set of integers is T-definable. Thus every constructible subtheory of T has a well-founded model and the theorem is proved.

Note. The assumptions necessary for the above theorem can be weakened to ZF + In + "The set of constructible sets of integers is countable".

As a final corollary of Lemma 3.6 we give a theorem which enables one to pass from non-well-founded to well-founded models. However, it is doubtful whether this theorem has any application. Let us call a model for ZF+(V = L) *peculiar* if it is pointwise definable, ω_1-standard, i.s. and not well-founded.

THEOREM 3.11. *In* ZF+In *one can prove that if S is an arithmetic set of sentences of set theory each of which holds in every peculiar model for* ZF+(V = L), *then* ZF+(V = L)+S *has a standard model.*

Proof. Put Q = the set of logical consequences of ZF+(V = L) in 3.6, let $\mathcal{M}$ be the model constructed in 3.4, and observe that $\mathcal{N} \models \mathrm{SM}(\mathrm{ZF}+(\mathrm{V}=\mathrm{L})+S)$. Then by the argument of 3.8 it follows that ZF+(V = L)+S has a well-founded model.

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