From this corollary we get a complete characterization of the Lebesgue integral in terms of translations. This result was suggested to me by Professor Gödel.

COROLLARY 3.9. *Let f, g be non-negative finite integrable functions. Then $I(f) = I(g)$ iff there are non-negative finite integrable functions f_1, f_2, g_1, g_2 such that $f_i \cong g_i$ for $i = 1, 2$, $f = f_1 - f_2$ and $g = g_1 - g_2$.*

Proof. Suppose $I(f) = I(g)$, $f, g < \infty$. We can assume that $f, g \in \mathfrak{B}^+$. Then $f \approx g$. So, there are negligible f', g' such that $f + f' \cong g + g'$.

It is easy to show, since the set of finite Baire functions is a G.C.A., that we can take $f', g' < \infty$.

Let h be the characteristic function of a bounded open set. Then we have

$$f' + h \cong h \cong g' + h \quad \text{and} \quad f + f' + h \cong g + g' + h.$$

Thus, take

$$f_1 = f + f' + h, \qquad f_2 = f' + h,$$

$$g_1 = g + g' + h, \qquad g_2 = g' + h.$$

In a similar way we obtain for the Lebesgue measure, in the terminology of [1],

COROLLARY 3.10. *Let A, B be bounded measurable sets. Then $\lambda(A) = \lambda(B)$ iff there are bounded measurable sets A_1, A_2, B_1, B_2 such that $A_i \cong B_i$ for $i = 1, 2$, $A_2 \subseteq A_1$, $B_2 \subseteq B_1$, $A = A_1 - A_2$ and $B = B_1 - B_2$.*

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A representation theorem for linearly ordered cardinal algebras

by

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This paper gives a characterization of linearly ordered cardinal algebras. This result is a consequence of theorems proved in [1]. The idea for the theorems resulted from a conversation with Professor K. Gödel. I am also very grateful to Professor Gödel for several suggestions he made to improve the paper.

Cardinal algebras were introduced and extensively studied in [2]. Many results from this book will be needed. In what follows, I will use the terminology of [2]. In particular, $\overline{N}$ will be the set of non-negative real numbers including ∞, and $\overline{I}$ the set of non-negative integers including ∞. $\overline{\mathfrak{N}}$ and $\overline{\mathfrak{I}}$ will be the corresponding algebras. Let $\mathfrak{P}$ be $\langle \{0, \infty\}, +, \sum \rangle$ where $+$ and $\sum$ are ordinary addition and infinite addition.

From Theorem 15.12 of [2] it is clear that if $\mathfrak{A} = \langle A, \leqslant \rangle$ is a linearly ordered system with a first element and closed under least upper bounds (l.u.b.) of countable sets, then $\langle A, +, \sum \rangle$, where $a + b = a \cup b$ and $\sum_{i<\infty} a_i = \bigcup_{i<\infty} a_i$, is a linearly ordered idemmultiple cardinal algebra. On the other hand, the ordering in a linearly ordered idemmultiple cardinal algebra has a first element and is closed under l.u.b. of countable sets. So, I will identify linearly ordered systems with a first element and closed under l.u.b. of countable sets with linearly ordered idemmultiple cardinal algebras.

I will first prove an auxiliary theorem

THEOREM 1. *Let $\mathfrak{A} = \langle A, \leqslant \rangle$ be a linearly ordered system with first element 0 and closed under* l.u.b. *of countable sets. Let $\{\mathfrak{B}^x = \langle B^x, +^x, \sum^x \rangle : x \in A\}$ be a family of disjoint cardinal algebras with zero element 0_x such that:*

(i) *$\mathfrak{B}^0$ is the one element algebra,*

(ii) *If $x_i \in A$ for $i < \infty$ and $x = \bigcup_{i<\infty} x_i > x_j$ for every $j < \infty$, then $\mathfrak{B}^x$ is isomorphic to $\mathfrak{P}$. In this case, let $B^x = \{0_x, \infty_x\}$.*

Define $\mathfrak{B} = \langle B, +, \sum \rangle$ *as follows:*

(1) $B = \bigcup \{B^x \sim \{0_x\}: x \in A\} \cup B^0$.

(2) *Let* $a_i \in B^{x_i}$ *for* $i = 1, 2$. *Then*

$$a_1 + a_2 = \begin{cases} a_1 +^{x_1} a_2 & \text{if} \quad x_1 = x_2, \\ a_1 & \text{if} \quad x_2 < x_1, \\ a_2 & \text{if} \quad x_1 < x_2. \end{cases}$$

(3) *Let* $a_i \in B^{x_i}$ *for* $i < \infty$, $x = \bigcup_{i<\infty} x_i$. *Then*

(3.1) *If* $x = x_j$ *for a certain* $j < \infty$,

$$\sum_{i<\infty} a_i = \sum_{k<n}^{x} a_{i_k}$$

where a_{i_k} *is the sequence defined by* $i_k =$ *the least* $i < \infty$ *such that* $i \neq i_j$ *for every* $j < k$ *and* $a_i \in B^x$.

(3.2) *If* $x > x_j$ *for every* $j < \infty$,

$$\sum_{i<\infty} a_i = \infty_x.$$

Then $\mathfrak{B}$ *is a cardinal algebra.*

Furthermore, if for all $x \in A$, $\mathfrak{B}^x$ *is a simple cardinal algebra, then* $\mathfrak{A}$ *is isomorphic to the cardinal subalgebra of idemmultiple elements of* $\mathfrak{B}$.

Proof. It is easy to prove postulates I, II, III, IV and V of Definition 1.1 in [2].

(a) Proof of postulate VI. Let $a + b = \sum_{i<\infty} c_i$. Suppose that $a \in B^x$, $b \in B^y$, $c_i \in B^{z_i}$ for all $i < \infty$, $z = \bigcup_{i<\infty} z_i$. We have $x \cup y = z$, as $a + b \in B^{x \cup y}$ and $\sum_{i<\infty} c_i \in B^z$.

Case 1. $x = y$. Then $a + b = a +^x b$.

(i) $x = z_j = z$ for some $j < \infty$.

Then we have $\sum_{i<\infty} c_i = \sum_{k<n}^{x} c_{i_k}$, where the c_{i_k}'s are all the c_i's in B^x. Hence, $a +^x b = \sum_{k<n}^{x} c_{i_k}$.

Thus, since $\mathfrak{B}^x$ is a cardinal algebra, there are a_{i_k} and b_{i_k} such that

$$a = \sum_{k<n}^{x} a_{i_k}, \qquad b = \sum_{k<n}^{x} b_{i_k}$$

and

$$c_{i_k} = a_{i_k} +^x b_{i_k}.$$

Define

$$a = \begin{cases} a_{i_k} & \text{if} \quad i = i_k \text{ and } a_{i_k} \neq 0_x, \\ 0_0 & \text{if} \quad i = i_k \text{ and } a_{i_k} = 0_x, \\ c_i & \text{otherwise}; \end{cases}$$

$$b_i = \begin{cases} b_{i_k} & \text{if} \quad i = i_k \text{ and } b_{i_k} \neq 0_x, \\ 0_0 & \text{otherwise}. \end{cases}$$

Then

$$a = \sum_{i<\infty} a_i, \qquad b = \sum_{i<\infty} b_i$$

and

$$c_i = a_i + b_i.$$

(ii) Suppose $x = z > z_i$ for all $i < \infty$. Then

$$a + b = \infty_x = \sum_{i<\infty} c_i.$$

So,

$$a = \infty_x = b.$$

As $z > z_i$ for all $i < \infty$, there is a non-decreasing subsequence z_{k_n} for $n < \infty$ such that

$$\bigcup_{n<\infty} z_{k_{2n}} = \bigcup_{n<\infty} z_{k(2n+1)} = \bigcup_{n<\infty} z_{k_n} = z.$$

Let

$$a_i = \begin{cases} c_{k_{2n}} & \text{if} \quad i = k_{2n}, \\ 0_0 & \text{otherwise}; \end{cases}$$

$$b_i = \begin{cases} c_{k(2n+1)} & \text{if} \quad i = k_{(2n+1)}, \\ 0_0 & \text{if} \quad i = k_{2n}, \\ c_i & \text{otherwise}. \end{cases}$$

Then we have

$$a = \sum_{i<\infty} a_i, \quad b = \sum_{i<\infty} b_i \quad \text{and} \quad c_i = a_i + b_i \quad \text{for all } i < \infty.$$

Case 2. $x > y$. Then $a = \sum_{i<\infty} c_i$. As $x = z$, there is a $j < \infty$ such that

$$y < z_j.$$

Thus, define $a_i = c_i$

$$b_i = \begin{cases} 0_0 & \text{if} \quad i \neq j, \\ b & \text{if} \quad i = j. \end{cases}$$

Thus VI is proved.

(b) Proof of postulate VII. Let $a_n = b_n + a_{n+1}$.

Case 1. There is an $n_0 < \infty$ and an $x \in A$ such that $a_n \in B^x$ for all $n \geqslant n_0$ and $a_n \notin B^x$ for all $n < n_0$. Then we have, for $n \geqslant n_0$, $b_n \in B^y$ with $y \leqslant x$.

Let

$$d_n = \begin{cases} b_n & \text{if} \quad b_n \in B^x, \\ 0_x & \text{otherwise.} \end{cases}$$

Then $a_n = d_n +^x a_{n+1}$ and $a_n, d_n \in B^x$ for all $n \geqslant n_0$. As $\mathfrak{B}^x$ is a cardinal algebra, there is a $c' \in B^x$ such that

$$a_n = c' +^x \sum_{i<\infty}^{x} d_{n+1} \quad \text{for all } n \geqslant n_0 .$$

Let $c = c'$ if $c' \neq 0_x$ and $c = 0_0$ if $c' = 0_x$. Then $a_n = c + \sum_{i<\infty} b_{n+i}$ for all $n \geqslant n_0$.

On the other hand, we have,

$$a_{n_0-m} = \sum_{i<m} b_{n_0-m+i}$$

as $a_{n_0-m} \in B^y$ for a certain $y > x$.

So,

$$a_{n_0-m} = c + \sum_{i<\infty} b_{n_0-m+i} .$$

Case 2. Suppose that, for no $n_0 < \infty$ and no $x \in A$, we have $a_n \in B^x$ for all $n \geqslant n_0$.

Let $n < \infty$ be given. Suppose $a_n \in B^x$, $a_{n+m} \in B^x$ and $a_{n+m+1} \notin B^x$. Then $a_{n+m+1} \in B^y$ for a certain $y < x$. So

$$a_{n+m} = b_{n+m}$$

and

$$a_n = \sum_{i<m}^{x} b_{n+i} = \sum_{i<\infty} b_{n+i} .$$

Then taking $c = 0_0$, we obtain VII.

(c) Suppose now that $\mathfrak{B}^x$ is a simple cardinal algebra for every $x \in A$. Let $a \in B^x \sim \{0_x\}$. Then a is idemmultiple in $\mathfrak{B}$ if and only if a is the infinite element in $\mathfrak{B}^x$. Let $f(x)$ be the infinite element of $\mathfrak{B}^x$ for $x \neq 0$, and $f(0) = 0_0$. Then f is a one-one function from A onto the set of idemmultiple elements of $\mathfrak{B}$. We have $x \leqslant y$ if and only if $f(x) + f(y) = f(y)$ if and only if $f(x) \leqslant f(y)$. So f is the required isomorphism.

THEOREM 2 (Representation theorem). *An algebra $\mathfrak{C} = \langle C, +, \sum \rangle$ is a linearly ordered cardinal algebra if and only if it is of the form of the algebra $\mathfrak{B}$ of Theorem 1 with $\mathfrak{B}^x$ isomorphic to $\overline{\mathfrak{N}}$, $\overline{\mathfrak{J}}$ or $\mathfrak{P}$ for every $x \neq 0$.*

Furthermore, in any such representation $\mathfrak{A}$ is isomorphic to the cardinal algebra of idemmultiple elements of $\mathfrak{C}$.

Proof. It is clear from Theorem 1 that if $\mathfrak{C}$ is of the form indicated in the theorem it is a linearly ordered cardinal algebra as $\overline{\mathfrak{N}}$, $\overline{\mathfrak{J}}$ and $\mathfrak{P}$ are cardinal algebras (cf. 14.2, 14.4, 14.5 and 14.6 of [2]).

So, suppose $\mathfrak{C}$ is a linearly ordered cardinal algebra.

Define the equivalence relation R over C by aRb if and only if $\infty \cdot a = \infty \cdot b$ for $a, b \in C$. Then, by Theorem 8.5 of [2], R is an equivalence relation and $\mathfrak{C}/R$ is isomorphic to the idemmultiple cardinal subalgebra $\mathfrak{A} = \langle A, +, \sum \rangle$ of the idemmultiple elements of C.

For $x \in A$, let $\mathfrak{B}^x = \langle B^x, +^x, \sum^x \rangle$ be defined by

(1) $B^x = x/R \cup \{0_x\}$ for $x \neq 0$, where $0_x \notin C$, $B^0 = 0/R$,

(2) 0_x acts as a zero element for $\mathfrak{B}^x$,

(3) $+^x$, $\sum^x$ are the operation $+, \sum$ restricted to x/R for elements of x/R.

We have to prove now that all the conditions of Theorem 1 are satisfied

(a) $\mathfrak{B}^x$ is isomorphic to one of the algebras

$$\overline{\mathfrak{N}}, \overline{\mathfrak{J}} \text{ or } \mathfrak{P} \text{ if } x \neq 0 .$$

It is clear that if $a \in x/R$ with $x \in A$ and $r \in \overline{N}$ with $r \cdot a$ defined in C, then $r \cdot a \in x/R$, as $\infty \cdot r \cdot a = r \cdot \infty \cdot a = r \cdot x = x$ (cf. 1.6, 1.9 of [1]).

There are two cases to consider. Let $x \in A$.

Case 1. $x/R = \{x\}$. Then $B^x = \{0_x, x\}$ and

$$\mathfrak{B}^x \cong \mathfrak{P} .$$

Case 2. There is an $a \neq x$ such that $a \in x/R$. Then, for any $b \in x/R$ with $b \neq x$, we have $a + b \neq b$ and $a + b \neq a$. For suppose that $a + b = b$; then $x = \infty \cdot a \leqslant b \leqslant \infty \cdot b = x$ (by 1.29 of [2]). Similarly if $a + b = a$, then $a = x$. It is also true that $b \leqslant x = \infty \cdot a$.

So, by Theorem 1.18 of [1], we have, for any $b \in x/R$, $b = r \cdot a$ where $r \in \overline{N}$.

As $a + a \neq a$, we infer by Theorem 1.19 of [1] that if $r_1 \neq r_2$, then $r_1 \cdot a \neq r_2 \cdot a$.

Hence, if we define $f(r \cdot a) = r$ for all $r \cdot a \in B^x$, f is an isomorphism of $\mathfrak{B}^x$ into $\overline{\mathfrak{N}}$ (Theorems 1.9 and 1.10 of [1]).

We now have two subcases:

(i) There is a largest integer $n < \infty$ such that $a = n \cdot b$ for a certain b. Then by 1.7 of [1]

$$\mathfrak{B}^x \cong \langle P, +, \textstyle\sum \rangle \quad \text{where} \quad P = \left\{ \frac{m}{n} : m \in \overline{I} \right\}$$

and by 14.4 of [2], $\langle P, +, \Sigma \rangle \cong \overline{\mathfrak{J}}$.

(ii) There is no largest integer $n < \infty$ such that $a = n \cdot b$ for a certain b. Then by 1.8 of [1]

$$\mathfrak{B}^x \cong \overline{\mathfrak{N}}.$$

(b) Suppose $x_i \in A$ for $i < \infty$ and $x_j < \bigcup_{i<\infty} x_i = x$ for all $j < \infty$. Let $a \in C$, $a \in x/R$. Then $\infty \cdot a = x$ and $a \leqslant x$.

Suppose now that $a < x$. Then, as $\mathfrak{C}$ is linearly ordered, $a \leqslant x_i$ for a certain $i < \infty$. Hence $\infty \cdot a \leqslant \infty \cdot x_i = x_i < x$, as x_i is idemmultiple. But this contradicts $\infty \cdot a = x$.

So, $B^x = \{0_x, x\}$ as required by Theorem 1 (ii).

(c) $B^0 = 0/R = \{0\}$ as required by Theorem 1 (i).

(d) As R is an equivalence relation, the B^x's are disjoint and

$$C = \bigcup\{x/R:\ x \in A\} = \bigcup\{B^x \sim \{0_x\}:\ x \in A\} \cup \{0\}.$$

So (1) of Theorem 1 is satisfied.

(e) Suppose that $a \in x/R$, $b \in y/R$ with $y, x \in A$ and $y < x$.

We have, as C is linearly ordered,

$$y \leqslant a \quad \text{or} \quad a \leqslant y.$$

But if $a \leqslant y$, we would have

$$x = \infty \cdot a \leqslant \infty \cdot y = y.$$

So $y \leqslant a$.

Also $b \leqslant y$ and so,

$$\infty \cdot b \leqslant y \leqslant a.$$

Hence $a + b = a$ by 1.29 of [2].

From this, (2) and (3.1) of Theorem 1 are easy to prove.

(f) Suppose now that $a_i \in x_i/R$, $x_i \in A$ and $x = \bigcup_{j<\infty} x_j > x_i$ for all $i < \infty$. Then $x \in A$.

We have $\sum_{i<\infty} x_i = x$. So

$$\infty \sum_{i<\infty} a_i = \sum_{i<\infty} \infty a_i = \sum_{i<\infty} x_i = x,$$

i.e., $\sum_{i<\infty} a_i \in x/R$. But by (b), $x/R = \{x\}$.

So $\sum_{i<\infty} a_i = x$ and 3.2 is verified.

The last conclusion of Theorem 2 is an immediate consequence of Theorem 1 as it is clear that $\overline{\mathfrak{N}}$, $\overline{\mathfrak{I}}$ and $\mathfrak{P}$ are simple cardinal algebras (cf. 14.2, 14.4, 14.5 and 14.6 of [2]).

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