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Cardinal algebras of functions and integration

by

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Introduction. The purpose of this work is to apply the methods of [1] to cardinal algebras of functions instead of algebras of sets. I believe that the results become more elegant in this way and somewhat stronger, because it is possible to obtain the integral directly.

In the first part it is proved that the non-negative Baire functions are a cardinal algebra, which is an interesting result in its own right. Finally, a complete characterization of the Lebesgue integral (and Lebesgue measure) is obtained in terms of translations.

As in the previous paper, [1], I shall quote the theorems in Tarski's book [2] by their number and a T.

I. Cardinal algebras of functions. I am first going to prove that the class of non-negative Baire functions is a cardinal algebra. I will adopt the following notation: $\mathcal{R}^+$ is the set of non-negative real numbers; $\wedge, \vee$ the lattice operations on the class of functions; ${}^{B}A$ the class of functions from B into A.

THEOREM 1.1. *Let $F \subseteq {}^{E}\mathcal{R}^+$ such that*

(i) *If $f, g \in F$ then $f+g$, $(f-g)\vee 0$, $f \wedge g \in F$,*

(ii) *If for every $n < \infty$, $f_n \in F$, $f_n \leqslant f_{n+1}$ and $\lim f_n = f < \infty$, then $f \in F$,*

(iii) *If for every $n < \infty$, $f_n \in F$ and $f_{n+1} \leqslant f_n$, then $\lim f_n \in F$.*

Then $\langle F, +, \sum \rangle$ is a finitely closed generalized cardinal algebra where $\sum_{i<\infty} f_i$ is defined only if $\sum_{i<\infty} f_i < \infty$.

Proof. We note that

(1) If $\sum_{i<\infty} f_i < \infty$ with $f_i \in F$, $\sum_{i<\infty} f_i \in F$.

So

(2) If $\sum_{i<\infty} f_i \in F$, $\sum_{i<\infty} f_{i+n} \in F$ for all $n < \infty$.

(3) If $\sum_{i<\infty} f_i \in F$, $g_i \leqslant f_i$ and $g_i \in F$ for all $i < \infty$, then $\sum_{i<\infty} g_i \in F$.

(4) $F \subseteq {}^E\mathcal{R}^+$ and $\langle {}^E\mathcal{R}^+, +, \sum\rangle$ is a generalized cardinal algebra (6.12 T).

(a) From (2) and (4) we get 5.1. I T.

(b) From (3) and (4) we get 5.1. II T.

(c) 5.1. III T we get from (1).

(d) Proof of 5.1. IV T. Suppose

$$f+g = \sum_{i<\infty} h_i \quad \text{with} \quad f, g, h_i, f+g \in F.$$

Define

$$f_n = \Big(h_n - \Big(\Big(g - \sum_{i<n} h_i\Big) \vee 0\Big)\Big) \vee 0\,,$$

$$g_n = h_n \wedge \Big(\Big(g - \sum_{i<n} h_i\Big) \vee 0\Big).$$

Then we have: $f_n, g_n \in F$ for every n

$$h_n = f_n + g_n \quad \text{for every } n\,,$$

$$f = \sum_{n<\infty} f_n\,, \quad g = \sum_{n<\infty} g_n\,.$$

(e) Proof of Axiom 5.1. V T. Suppose

(*) $$f_n = g_n + f_{n+1}\,, \quad f_n, g_n \in F \text{ for every } n < \infty\,.$$

We have $f_{n+1} \leqslant f_n$ for every n.

Let $h = \lim_{n\to\infty} f_n \in F$. From (*) we also get

$$f_n = \sum_{i<m} g_{n+i} + f_{m+n} \quad \text{for every } n, m < \infty\,.$$

Hence,

$$f_n = \lim_{m\to\infty} f_n = \sum_{i<\infty} g_{n+i} + h \quad \text{for every } n < \infty\,.$$

Now $\sum_{i<\infty} g_{n+i} \leqslant f_n < \infty$. So

$$\sum_{i<\infty} g_{n+i} \in F.$$

THEOREM 1.2. *Let $\mathfrak{F} = \langle F, +, \sum\rangle$ be as in Theorem 1.1. Let $\bar{\mathfrak{F}} = \langle \bar{F}, +, \sum\rangle$ be the cardinal algebra which is the closure of $\mathfrak{F}$* (cf. 7.1 T, 7.8 T).

Then $\bar{F}$ is the smallest set B such that

(i) $F \subseteq B$.

(ii) *If $f_n \leqslant f_{n+1}, f_n \in B$ for every $n < \infty$, then $\lim_{n\to\infty} f_n \in B$.*

Furthermore we have

(a) $\bar{F} \subseteq {}^E(\mathcal{R}^+ \cup \{\infty\})$,

(b) *If $f \in \bar{F}$ and $f < \infty$, then $f \in F$.*

Proof. A. (i) It is clear that $F \subseteq \bar{F}$.

(ii) Suppose $f_n \leqslant f_{n+1}, f_n \in F$ for every $n < \infty$. That is

$$f_{n+1} = f_n + g_{n+1} = \sum_{i<n+2} g_i \quad \text{where} \quad g_0 = f_0 \text{ and } g_n \in F.$$

Then

$$f = \bigvee_{n<\infty} f_n = \sum_{i<\infty} g_i = \lim_{n\to\infty} f_n\,.$$

So $\lim_{n\to\infty} f_n \in \bar{F}$.

Suppose now $f_n \leqslant f_{n+1}$ with $f_n \in \bar{F} - F$ for $n < \infty$. Then

$$f_n = \sum_{i<\infty} f_{ni}\,, \quad f_{ni} \in F \text{ for } i, n < \infty\,.$$

Define

$$g_0 = f_{00}\,, \quad g_{n+i} = \sum_{i<m_{n+1}} f_{n+1,i}$$

such that m_{n+1} is the least p with $\sum_{i<p} f_{n+1,i} \geqslant g_n, p > m_n$.

Then $g_n \in F$ for every $n < \infty$ and $g_n \leqslant g_{n+1}$.

Then $g = \lim_{n\to\infty} g_n \in \bar{F}$.

But $g = \lim_{n\to\infty} f_n$.

B. Suppose that B is such that

(i) $F \subseteq B$,

(ii) If $f_n \leqslant f_{n+1}$, $f_n \in B$, then $\lim f_n \in B$.

Let $f \in \bar{F}$. Then $f = \sum_{i<\infty} f_i$ with $f_i \in F$ (7.1 T). Then $\sum_{i<n} f_i \in F \subseteq B$ for $n < \infty$. So $f = \sum_{i<\infty} f_i \in B$ and $\bar{F} \subseteq B$.

C. Let $f \in \bar{F}$, $f = \sum_{i<\infty} f_i$ with $f_i \in F$. Then $f(x) \in \mathcal{R}^+ \cup \{\infty\}$ for every $x \in E$.

Suppose $f < \infty$. So $\sum_{i<\infty} f_i < \infty$ with $f_i \in F$. That is $f \in F$.

From these two theorems it is clear that the non-negative Baire functions ($\mathcal{B}^+$) on any space form a cardinal algebra. Thus, there are many interesting results about the Baire functions that we can obtain immediately, applying the theory of cardinal algebras. For instance:

(From 2.1 T) *If $n \leqslant \infty$ and $p \leqslant \infty$ and $\sum_{i<n} f_i = \sum_{i<p} g_j$ where $f_i, g_i \in \mathfrak{B}^+$, then there is a double sequence $h_{ij} \in \mathfrak{B}^+$ for $i < n, j < p$ such that*

$$f_i = \sum_{j<p} h_{ij} \text{ for } i < n \quad \text{and} \quad g_j = \sum_{i<n} h_{ij} \text{ for } j < p .$$

(From 2.28 T) *If $n \leqslant \infty$, $p \leqslant \infty$ and $f_i \leqslant g_j$ for every $i < n$, $j < p$, $f_i, g_j \in \mathfrak{B}^+$, then there is an $h \in \mathfrak{B}^+$ such that*

$$f_i \leqslant h \leqslant g_j \quad \text{for every } i < n,\ j < p .$$

(From 2.37 T) *If $m < \infty$ and $n < \infty$ are relatively prime and $m \cdot f = n \cdot g$ with $f, g \in \mathfrak{B}^+$, then there is an $h \in \mathfrak{B}^+$ such that*

$$f = n \cdot h \quad \text{and} \quad g = m \cdot h .$$

II. Equivalence relations determined by a group of functions. Theorem 6.10T asserts that if $\mathfrak{A}$ is a C.A. or G.C.A. and R is a countably additive and finitely refining equivalence relation, then $\mathfrak{A}/R$ is also a C.A. or a G.C.A. So it is important to see under what conditions this type of relations arise. The next theorem gives some sufficient conditions.

THEOREM 2.1. *Suppose:*

(i) $\langle A, +, \sum \rangle$ *is a* G.C.A.

(ii) $F \subseteq {}^E A$, $\langle F, +, \sum \rangle$ *a finitely closed* G.C.A. *such that $f \wedge g \in F$ for $f, g \in F$.*

(iii) *G a group of functions, $G \subseteq {}^E E$.*

Define

(a) *if $\sigma \in G$, $f \in F$*

$$\sigma f(x) = f(\sigma x) \quad \text{for all } x \in E ,$$

(b) *if $f, g \in F$, then $f \cong g$ iff there are $\sigma_i \in G$ for $i < \infty$, $f_i \in F$ for $i < \infty$ such that*

$$f = \sum_{i<\infty} f_i \quad \text{and} \quad g = \sum_{i<\infty} \sigma_i f_i .$$

Then $\cong$ is a countably additive, finitely refining equivalence relation over F.

Proof. (i) $\cong$ is clearly reflexive and symmetric.

(ii) Let $f \cong g \cong h$. Then

$$f = \sum_{i<\infty} f_i , \quad g = \sum_{i<\infty} g_i , \quad f_i = \sigma_i g_i ,$$

$$g = \sum_{i<\infty} g_i', \quad h = \sum_{i<\infty} h_i, \quad g_i' = \tau_i h_i .$$

Define

$$f_{ij} = \sigma_i(g_i \wedge g_j') , \quad h_{ij} = \tau_j^{-1}(g_i \wedge g_j') .$$

So

$$f_{ij} = \sigma_i \tau_j h_{ij} ,$$

$$f_i = \sigma_i g_i = \sigma_i \Big(g_i \wedge \sum_{j<\infty} g_j'\Big) \quad \text{as } g_i \leqslant g$$

$$= \sigma_i \Big(\sum_{j<\infty} g_i \wedge g_j' \Big) \quad \text{by 3.33T}$$

$$= \sum_{j<\infty} \sigma_i(g_i \wedge g_j')$$

$$= \sum_{j<\infty} f_{ij} .$$

Similarly $h_j = \sum_{i<\infty} h_{ij}$. So $f \cong h$ and $\cong$ is transitive.

(iii) $\cong$ is clearly countably additive.

(iv) Let $f \cong g$ and $f = h_1 + h_2$. We have $f = \sum_{i<\infty} f_i$, $g = \sum_{i<\infty} g_i$ and $g_i = \sigma_i f_i$. Also $h_1 + h_2 = \sum_{i<\infty} f_i$. As F is a G.C.A., by 5.1. IV T

$$f_i = f_{i1} + f_{i2} , \quad h_1 = \sum_{i<\infty} f_{i1} , \quad h_2 = \sum_{i<\infty} f_{i2} .$$

Take

$$g_{i1} = \sigma_i f_{i1} , \quad g_{i2} = \sigma_i f_{i2} , \quad g_1 = \sum_{i<\infty} g_{i1} , \quad g_2 = \sum_{i<\infty} g_{i2} .$$

We have

$$g_1 \cong h_1 , \quad g_2 \cong h_2 .$$

Also

$$g_1 + g_2 = \sum_{i<\infty} (g_{i1} + g_{i2}) \quad \text{by 5.1. II T}$$

$$= \sum_{i<\infty} \sigma_i(f_{i1} + f_{i2})$$

$$= g .$$

III. Applications to integration theory. In this section $\mathfrak{F} = \langle F, +, \sum \rangle$ will be a fixed cardinal algebra of functions, $\cong$ a fixed countably additive and finitely refining relation between elements of F, and h a fixed element of F.

DEFINITIONS 3.1. (i) $\mathfrak{A} = \langle A, +, \sum \rangle = \mathfrak{F}/\cong$, $\tau(f) = f/\cong$ (= the type of f),

(ii) $f \in F$ (or $\tau(f)$) is negligible iff $\infty\tau(f) \leqslant \tau(h)$,

(iii) Φ is the ideal of negligible elements of $\mathfrak{A}$,

(iv) $\mathfrak{B} = \mathfrak{A}/\Phi$, $\varrho(f) = \tau(f)/\Phi$,

(v) $f \approx g$ iff $\varrho(f) = \varrho(g)$.

COROLLARY 3.2. (i) *$f \approx g$ iff there is an $f' \in F$ such that f' is negligible and $f+f' \cong g+f'$,*

(ii) *$\tau(f) \leqslant \tau(g)$ iff there is an f' such that*

$$f' \cong f, \quad f' \leqslant g,$$

(iii) *$\varrho(f) \leqslant \varrho(g)$ iff there is an f' such that*

$$f' \approx f, \quad f' \leqslant g,$$

(iv) *f is negligible iff there are f_i for $i < \infty$ such that $f_i \cong f$ for every $i < \infty$ and $\sum_{i<\infty} f_i \leqslant h$.*

3.2 (i) is true in this case, because $\mathfrak{F}$ is a cardinal algebra and not only a generalized cardinal algebra. When we are dealing with sets instead of functions as in [1], this is not quite correct. In [1] there is a small error in this respect, which fortunately does not effect the main results. The only change necessary to correct this error is in Definition 2.5 (iii), which should read:

Let $A, B \in \mathfrak{K}$. Then $A \approx B$ iff there are $A', B', C', C'' \in \mathfrak{K}$ such that C', C'' are negligible, $A \cong A'$, $B \cong B'$, $A' \cap C' = 0 = B' \cap C''$ and $A' \cup C' \cong B' \cup C''$.

The simplicity of 3.2 (i) compared to this shows one of the advantages of working with functions.

In a completely similar manner as Theorem 2.10 of [1] we get the following:

THEOREM 3.3. *Let $\langle F, +, \sum\rangle$ be a cardinal algebra of functions, $\cong$ a countably additive, finitely refining equivalence relation between elements of F. If:*

(i) *h is not negligible.*

(ii) *For all $f, g \in F$ with $f, g \leqslant h$ there is an $f' \in F$ such that $f \approx f' \leqslant g$ or $g \approx f' \leqslant f$.*

(iii) *For all $f \in F$ there are $h_i \in F$ for $i < \infty$ such that $h \approx h_i$ and $f \leqslant \sum_{i<\infty} h_i$.*

Then there is a unique countable additive numerical integral I defined on F such that

(a) $I(h) = 1$,

(b) *$I(f) = I(g)$ iff $f \approx g$.*

I am now going to apply this theorem to the following case.

Let $\langle F, +, \sum\rangle$ be the class of non-negative Baire functions over a Euclidean space $\mathcal{R}^n$. Let G be the group of translations on $\mathcal{R}^n$ and let $\cong$ be defined from G as in II. Then we know that $\cong$ is a countably additive, finitely refining equivalence relation. Let h be the characteristic function of the unit cube.

We know that (i) and (iii) of Theorem 3.3 are true ((i) because there is an integral). Thus it is only necessary to prove (ii).

As $\mathfrak{A}$ is a homomorphic image of $\mathfrak{F}$ and $\mathfrak{B}$ of $\mathfrak{A}$, we have if $f_n \leqslant f_{n+1}$ for every $n < \infty$

$$\bigcup_{n<\infty} \tau(f_n) = \tau\Big(\lim_{n\to\infty} f_n\Big), \quad \bigcup_{n<\infty} \varrho(f_n) = \varrho\Big(\lim_{n\to\infty} f_n\Big)$$

because

$$\lim_{n\to\infty} f_n = \bigvee_{n<\infty} f_n .$$

On the other hand, we have the following lemma.

LEMMA 3.4. *Let h be not negligible, $f_{n+1} \leqslant f_n \leqslant h$ for every $n < \infty$. Then $\bigcap_{n<\infty} \varrho(f_n)$ exists and*

$$\bigcap_{n<\infty} \varrho(f_n) = \varrho\Big(\lim_{n\to\infty} f_n\Big).$$

The proof is entirely similar to 3.1 of [1].

From this lemma in the same way as in 3.2 of [1] we get

LEMMA 3.5. *Let F be the monotone class generated by L, $h \in L$, h not negligible. If for all $f, g \in L$ with $\varrho(f), \varrho(g) \leqslant \varrho(h)$ we have $\varrho(f) \leqslant \varrho(g)$ or $\varrho(g) \leqslant \varrho(f)$. Then for all $f, g \in F$ with $\varrho(f), \varrho(g) \leqslant \varrho(h)$ we have $\varrho(f) \leqslant \varrho(g)$ or $\varrho(g) \leqslant \varrho(f)$.*

From this lemma and the fact that F is the monotone class generated by simple functions with rational values, we get (ii) of 3.3 and hence infer that the Lebesgue integral I is such that

$$I(f) = I(g) \quad \text{iff} \quad f \approx g .$$

In the same way as in [1] we get the following corollaries.

COROLLARY 3.6. *$I(f) = 0$ iff f is negligible, that is, there are $f_i \in \mathcal{B}^+$ for $i < \infty$ such that $f \cong f_i$ for all $i < \infty$ and $\sum_{i<\infty} f_i \leqslant h$.*

COROLLARY 3.7. *$I(f) = 0$ iff for every characteristic function of an open set g, there is an $f' \in \mathcal{B}^+$ such that $f \cong f' \leqslant g$.*

COROLLARY 3.8. *Let $f, g \in \mathcal{B}^+$. Suppose that there are f', g' characteristic functions of open sets such that $1/nf' \leqslant f$ and $1/ng' \leqslant g$ for some $n < \infty$. Then $I(f) = I(g)$ iff $f \cong g$ (that is: there are integrable f_i and translations σ_i for $i < \infty$ such that $f = \sum_{i<\infty} f_i$ and $g = \sum_{i<\infty} \sigma_i f_i$).*

From this corollary we get a complete characterization of the Lebesgue integral in terms of translations. This result was suggested to me by Professor Gödel.

COROLLARY 3.9. *Let f, g be non-negative finite integrable functions. Then $I(f) = I(g)$ iff there are non-negative finite integrable functions f_1, f_2, g_1, g_2 such that $f_i \cong g_i$ for $i = 1, 2$, $f = f_1 - f_2$ and $g = g_1 - g_2$.*

Proof. Suppose $I(f) = I(g)$, $f, g < \infty$. We can assume that $f, g \in \mathfrak{B}^+$. Then $f \approx g$. So, there are negligible f', g' such that $f+f' \cong g+g'$.

It is easy to show, since the set of finite Baire functions is a G.C.A., that we can take $f', g' < \infty$.

Let h be the characteristic function of a bounded open set. Then we have

$$f'+h \cong h \cong g'+h \quad \text{and} \quad f+f'+h \cong g+g'+h .$$

Thus, take

$$f_1 = f+f'+h , \qquad f_2 = f'+h ,$$

$$g_1 = g+g'+h , \qquad g_2 = g'+h .$$

In a similar way we obtain for the Lebesgue measure, in the terminology of [1],

COROLLARY 3.10. *Let A, B be bounded measurable sets. Then $\lambda(A) = \lambda(B)$ iff there are bounded measurable sets A_1, A_2, B_1, B_2 such that $A_i \cong B_i$ for $i = 1, 2$, $A_2 \subseteq A_1$, $B_2 \subseteq B_1$, $A = A_1 - A_2$ and $B = B_1 - B_2$.*

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A representation theorem for linearly ordered cardinal algebras

by

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This paper gives a characterization of linearly ordered cardinal algebras. This result is a consequence of theorems proved in [1]. The idea for the theorems resulted from a conversation with Professor K. Gödel. I am also very grateful to Professor Gödel for several suggestions he made to improve the paper.

Cardinal algebras were introduced and extensively studied in [2]. Many results from this book will be needed. In what follows, I will use the terminology of [2]. In particular, $\overline{N}$ will be the set of non-negative real numbers including ∞, and $\overline{I}$ the set of non-negative integers including ∞. $\overline{\mathfrak{N}}$ and $\overline{\mathfrak{I}}$ will be the corresponding algebras. Let $\mathfrak{P}$ be $\langle \{0, \infty\}, +, \sum \rangle$ where $+$ and $\sum$ are ordinary addition and infinite addition.

From Theorem 15.12 of [2] it is clear that if $\mathfrak{A} = \langle A, \leqslant \rangle$ is a linearly ordered system with a first element and closed under least upper bounds (l.u.b.) of countable sets, then $\langle A, +, \sum \rangle$, where $a+b = a \cup b$ and $\sum_{i<\infty} a_i = \bigcup_{i<\infty} a_i$, is a linearly ordered idemmultiple cardinal algebra. On the other hand, the ordering in a linearly ordered idemmultiple cardinal algebra has a first element and is closed under l.u.b. of countable sets. So, I will identify linearly ordered systems with a first element and closed under l.u.b. of countable sets with linearly ordered idemmultiple cardinal algebras.

I will first prove an auxiliary theorem

THEOREM 1. *Let $\mathfrak{A} = \langle A, \leqslant \rangle$ be a linearly ordered system with first element 0 and closed under* l.u.b. *of countable sets. Let $\{\mathfrak{B}^x = \langle B^x, +^x, \sum^x \rangle : x \in A\}$ be a family of disjoint cardinal algebras with zero element 0_x such that:*

(i) $\mathfrak{B}^0$ is the one element algebra,

(ii) If $x_i \in A$ for $i < \infty$ and $x = \bigcup_{i<\infty} x_i > x_j$ for every $j < \infty$, then $\mathfrak{B}^x$ is isomorphic to $\mathfrak{P}$. In this case, let $B^x = \{0_x, \infty_x\}$.