Comparison of the axioms of local and universal choice*

by

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In the v. Neumann–Bernays–Gödel set theory (in short: **NBG**-set theory) one can give the axiom of choice in two versions: a local and a universal version. The local version asserts that for every set x of non-empty sets there exists a function f such that $f(y) \in y$ for every $y \in x$. The universal version asserts the existence of a function F which is defined on the class of all sets such that for every set y either $y = \emptyset$ or $F(y) \in y$. The local version of the axiom of choice, call it (AC), is therefore the axiom of choice of the Zermelo–Fraenkel set-theory (in short: **ZF**-set theory) and the universal version is the axiom (E) in Gödel [5]. Here we shall discuss the relative strength of these two versions and prove by means of Cohen's forcing method the following result: **NBG**+(E) *is a conservative extension of* **ZF**+(AC) *with respect to* **ZF**-*formulas*. This result will be generalized at the end of the paper.

§ 1. We are working in **NBG**-set theory as presented in Gödel [5]. Here we consider the **NBG**-set theory as a theory formulated in the two-sorted lower predicate calculus without equality whose unique non-logical constant is "ε". The language $\mathfrak{L}_{\mathbf{NBG}}$ is built up from the primitive expressions $x \varepsilon y$, $x \varepsilon Y$, $X \varepsilon y$ and $X \varepsilon Y$ as usual (using the sentential connectives $\neg$, $\vee$ (negation, disjunction), the quantifier $\bigvee$ (there exists) and brackets). Equality $=$ is introduced as a defined relation: e.g. $X = Y$ is defined by $\bigwedge_z (z \varepsilon X \Longleftrightarrow z \varepsilon Y)$; $x = Y$, $X = y$ and $x = y$ are defined similarly. Remark that the logical symbols $\wedge$, $\Rightarrow$, $\Longleftrightarrow$ (conjunction, implication, equivalence) and $\bigwedge$ (quantification "for all") are introduced by definition (by means of $\neg$, $\vee$, $\bigvee$) as usual. The axioms of the **NBG**-set theory are the axioms of groups A, B, C and D, but with the difference

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that we replace Gödel's axiom (A 3) by the following axiom:

$$\bigwedge_x \bigwedge_Y \bigwedge_z [(x \in Y \wedge x = z) \Rightarrow z \in Y].$$

This replacement is necessary, because Gödel [5] treats equality in its logical meaning as identity and we are treating equality as a mathematically defined relation. Axiom (A 1) is the following sentence: $\bigwedge_x \bigvee_X (x = X)$. Axiom (A 2) reads as follows: $\bigwedge_X \bigwedge_Y (X \in Y \Rightarrow \bigvee_x x = X)$, etc. The system of axioms of groups A, B, C, D is called Σ. Lower case roman letters are called "set variables" and capital roman letters "class variables".

Let **ZF** be the system of axioms of Zermelo–Fraenkel set theory (including the axioms of replacement and regularity but without choice). The axioms of **ZF** are formulated in a sublanguage $\mathfrak{L}_{ZF}$ of the language $\mathfrak{L}_{NBG}$, where $\mathfrak{L}_{ZF}$ is built up from the atomic formulae $x \in y$ using $\neg$, $\vee$, $\bigvee$ and brackets. It is well-known that Σ is a conservative extension of **ZF** with respect to **ZF**-formulas:

(#) $$(\forall \varphi \in \mathfrak{L}_{ZF})[\mathbf{ZF} \vdash \varphi \text{ iff } \Sigma \vdash \varphi]$$

(see e.g. Cohen [2], p. 77). A question which arises now is, whether this is also true for **ZF**+*its axiom of choice* and Σ+*its axiom of choice*. The aim of the present paper is to show that this is in fact the case.

THEOREM 1. *If φ is a sentence of the language $\mathfrak{L}_{ZF}$, then φ is provable in $\Sigma+(E)$ if and only if φ is provable in* **ZF**+(AC).

Theorem 1 together with (#) gives the following

COROLLARY. *If φ is a sentence of the language $\mathfrak{L}_{NBG}$ in which no class variables occur and if φ is a theorem of $\Sigma+(E)$, then φ is also a theorem of $\Sigma+(AC)$.*

The corollary gives the solution to a problem posed by A. Lévy in [7]. Professor R. Solovay informed me, that theorem 1 was also obtained independently by P. J. Cohen, R. B. Jensen, S. Kripke and R. Solovay himself.

Before giving details we indicate briefly how we will prove theorem 1. Suppose theorem 1 is false; then [1] for some **ZF**-sentence φ, $\Sigma+(E) \vdash \varphi$ but $\sim \mathbf{ZF}+(AC) \vdash \varphi$. Then by (#) also $\sim \Sigma+(AC) \vdash \varphi$. Hence $\Sigma+(AC)+\neg\varphi$ is consistent, has a model and by the Löwenheim–Skolem argument a countable model $\mathcal{M}$. We want to extend $\mathcal{M}$ by adding to it a universa choice function F. We cannot take F as the limit of an arbitrary increasing sequence of (local) choice functions $f_i \in \mathcal{M}$, $F = \bigcup_i f_i$ (union in the sense of the meta-theory), because such arbitrary F's need not to generate models of the theory Σ. By means of a certain forcing relation we will define the notion of a *complete sequence* of (local) choice functions. Then it will be shown that, if F is the limit of a complete sequence, then F is *generic* (id est: generates a model of Σ). Let $\mathcal{N}$ be the extension of $\mathcal{M}$ by F. $\mathcal{N}$ will have only new classes but no new sets. Hence, since $\neg\varphi$ holds in $\mathcal{M}$ and φ speaks only about sets, $\neg\varphi$ holds also in $\mathcal{N}$. If F is the limit of a complete sequence, then F is a universal choice function in the extension $\mathcal{N}$, hence [2] $\mathcal{N} \models (E)$. But $\mathcal{N} \models \Sigma+(E)+\neg\varphi$ is the desired contradiction to $\Sigma+(E) \vdash \varphi$.

We have stated the corollary for the **NBG**-set theory. Clearly, it is in an appropriate form also true for **ZF** (add Hilbert's ε-operator to **ZF**—see Lévy [7]). In Ackermann's set theory with an axiom of foundation added, the corollary becomes trivial, since, as Lévy [6] has shown, in this theory (AC) and (E) are equivalent. But (E) is not equivalent to (AC) in the **NBG**-set theory as W. B. Easton [3] has shown.

§ 2. Proof of theorem 1: the forcing relation. Suppose that there is a sentence φ of the language $\mathfrak{L}_{ZF}$ such that φ is a theorem of $\Sigma+(E)$ but not a theorem of $\Sigma+(AC)$. Then there is a countable model $\mathcal{M} \cong \langle M, R\rangle$ of $\Sigma+(AC)$ in which $\neg\varphi$ holds. R is the interpretation of the binary predicate "ε" in the model and need not to be the actual membership relation (we can not suppose that $\mathcal{M}$ is a standard model!). To make easier the reading of the paper we will write from now on $a \in^{\mathcal{M}} b$ instead of aRb and the reader will remember that $\in^{\mathcal{M}}$ is the relation R and not the actual membership relation $\in$ restricted to M! We wish to extend $\mathcal{M}$ to a model $\mathcal{N}$ in such a way that $\mathcal{N}$ has no new sets but only some new classes, in particular a universal choice function. This is done by Cohen's method of forcing which is used to construct a *generic* class $\boldsymbol{F}$ such that the resulting relational system $\mathcal{N}$ will be a model of $\Sigma+(E)$. Since $\mathcal{N}$ shall not contain new sets, we can choose an unramified language in order to describe the structure $\mathcal{N}$ (usually one needs ramified languages).

The language $\mathfrak{L}^F$: *Primitive symbols*:

(a) Set-variables: v_i for each i of $\mathcal{M}$ such that i is an integer in the sense of $\mathcal{M}$ ($u, v, w, x, y, z, \ldots$ stand for these variables).

(b) Constants $\underline{S}$ for each class S of $\mathcal{M}$.

(c) A one-place predicate symbol F and a two-place predicate symbol ε.

(d) Sentential connectives $\neg$, $\vee$, a quantifier $\bigvee$ and brackets.

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[1] The symbol $\vdash$ is used to denote the syntactical provability relation, hence $S \vdash \Phi$ says that Φ is derivable from S and $\sim S \vdash \Phi$ says that Φ is not derivable from S.

[2] The symbol $\models$ is used to denote the semantical satisfaction relation, hence $\mathcal{M} \models \Phi$ says that Φ holds in the structure $\mathcal{M}$.

The formulae of $\mathfrak{L}^F$:

(the inductive definition takes place in the meta-language, so that the length of a formula will be finite in the "usual" sense):

(i) If a and b are variables or set-constants then $F(a)$ and $a \epsilon b$ are formulas [3].

(ii) If a is a variable or a set-constant and if $\underline{S}$ is a constant, then $a \varepsilon \underline{S}$ is a formula.

(iii) If Φ and Ψ are formulas, x a variable, then $\neg(\Phi)$, $(\Phi)\vee(\Psi)$ and $\bigvee_x (\Phi)$ are formulas.

(iv) Only those sequences of symbols of $\mathfrak{L}^F$ which can be obtained from the rules (i), (ii), (iii) in finitely many steps are formulas.

We follow the usual conventions which allow us to omit brackets in some cases (see Hilbert–Ackermann [10], p. 74). We remark that in contrast to the symbols of the "object-language" $\mathfrak{L}^F$ the following symbols are used in the meta-language: $\sim$, &, $\dot{\vee}$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$, $\bar{\bar{\circ}}$, ϵ, $\subseteq$, $\{.;..\}$, etc (negation, conjunction, disjunction, implication, equivalence, for all, there exists, equality, membership, inclusion, comprehension). The following convention is useful: a notion, operation or relation with a superscript $\mathcal{M}$ shall always mean the notion, operation, relation resp. in the sense of $\mathcal{M}$. Thus e.g. "x is a set$^{\mathcal{M}}$" means "x is an object of $\mathcal{M}$ which is a set in the sense of $\mathcal{M}$" and "f is a function$^{\mathcal{M}}$" means that f is a set$^{\mathcal{M}}$ and a function in the sense of $\mathcal{M}$. $\subseteq^{\mathcal{M}}$ is the inclusion-relation$^{\mathcal{M}}$ (in $\mathcal{M}$ with respect to $\epsilon^{\mathcal{M}}$) and $\mathrm{pr}_i^{\mathcal{M}}(x)$ is the projection$^{\mathcal{M}}$ of the set$^{\mathcal{M}}$ x of n-tuples$^{\mathcal{M}}$ to the ith coordinate$^{\mathcal{M}}$. Notions operations, and relations without superscript $\mathcal{M}$ are understood to be the corresponding notions, operations, relations resp. of the underlying meta-language. E.g. if a is a set of ordered pairs, then $\mathrm{pr}_1(a)$ is the set $\{b;\ (\exists c)(\langle b, c\rangle \epsilon a)\}$ and $\mathrm{pr}_2(a) \bar{\bar{\circ}} \{c;\ (\exists b)(\langle b, c\rangle \epsilon a)\}$.

The forcing argument:

DEFINITION. A *condition* is a set$^{\mathcal{M}}$ p such that p is a (local) choice function$^{\mathcal{M}}$ (this means, that p is a set$^{\mathcal{M}}$ of ordered pairs$^{\mathcal{M}}$ such that $\mathrm{pr}_1^{\mathcal{M}}(p)$, the domain$^{\mathcal{M}}$ of p, consists$^{\mathcal{M}}$ of non-empty$^{\mathcal{M}}$ sets$^{\mathcal{M}}$ and p is a function$^{\mathcal{M}}$ and $p(x) \epsilon^{\mathcal{M}} x$ for all $x \epsilon^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(p)$) [4].

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(3) A set-constant is a constant $\underline{S}$ where S is a set in the sense of $\mathcal{M}$. If s is a set in the sense of $\mathcal{M}$, then by (A 1) there is a class S of $\mathcal{M}$ such that both are equal in the sense of $\mathcal{M}$ and we will use the symbols $\underline{s}$ and $\underline{S}$ interchangably. Further we make the convention that $s \epsilon \mathcal{M}$, with the lower case latin variable s, shall always mean that s is a "set" of $\mathcal{M}$, whereas $S \epsilon \mathcal{M}$ means that S is a "class" of $\mathcal{M}$.

(4) In contrast to Gödel [5] we let $\mathrm{pr}_1(G)$ be the domain and $\mathrm{pr}_2(G)$ be the range of the function G.

It is clear that the collection$^{\mathcal{M}}$ of all conditions forms a proper class in $\mathcal{M}$, which will be called **Cond**. Now we will give (by a simple induction on the length of Φ) the definition of the relation $p \Vdash \Phi$ (p *forces* Φ) between conditions p and sentences Φ of $\mathfrak{L}^F$. We will use $p, p', p'', \ldots, q, q', \ldots$ as variables for conditions.

(1) $p \Vdash \underline{t} \varepsilon \underline{S} \leftrightarrow t \epsilon^{\mathcal{M}} S$,

(2) $p \Vdash F(\underline{s}) \leftrightarrow s \epsilon^{\mathcal{M}} p$,

(3) $p \Vdash \neg\Phi \leftrightarrow (\forall q)(p \subseteq^{\mathcal{M}} q \rightarrow \sim q \Vdash \Phi)$,

(4) $p \Vdash \Phi \vee \Psi \leftrightarrow (p \Vdash \Phi \,\dot{\vee}\, p \Vdash \Psi)$,

(5) $p \Vdash \bigvee_x \Phi(x) \leftrightarrow (\exists s \epsilon \mathcal{M})(p \Vdash \Phi(\underline{s}))$.

In the following three lemmata let Φ be any sentence of $\mathfrak{L}^F$.

CONSISTENCY LEMMA. *For no p do we have $p \Vdash \Phi$ and $p \Vdash \neg\Phi$.*

FIRST EXTENSION LEMMA. *If $p \Vdash \Phi$ and $p \subseteq^{\mathcal{M}} q$, then $q \Vdash \Phi$.*

SECOND EXTENSION LEMMA. *Every condition p can be extended to a condition q such that either $q \Vdash \Phi$ or $q \Vdash \neg\Phi$.*

The definition of the (strong) forcing relation $\Vdash$ was carried out in the underlying meta-theory. It is well-known that for each specific sentence Φ of $\mathfrak{L}^F$ the forcing relation can be defined in $\mathcal{M}$ (see e.g. Cohen [2], p. 120–121, Easton [3], p. 20) because Φ is finite and the construction of the class C_Φ of p's forcing Φ can be done in finitely many steps. For each specific Φ the mechanism of constructing C_Φ can be implemented within $\mathcal{M}$ but the mechanism is not universally applicable for all sentences Φ of $\mathfrak{L}^F$, so that within $\mathcal{M}$ we do not have the whole relation $\Vdash$. (This is not too much surprising since the definition of forcing resembles very much the definition of truth.) However, as we have already mentioned, we can define forcing for a single given sentence Φ or for some particular family of sentences within $\mathcal{M}$:

LEMMA 1. *Let $\Phi(x_1, \ldots, x_n)$ be a formula of $\mathfrak{L}^F$. There is a class C_Φ in $\mathcal{M}$ whose elements$^{\mathcal{M}}$ are the $n+1$-tuples$^{\mathcal{M}}$ $\langle p, s_1, \ldots, s_n\rangle^{\mathcal{M}}$ such that $p \Vdash \Phi(\underline{s}_1, \ldots, \underline{s}_n)$.*

Proof by induction on the length of Φ (see e.g. Easton [3], p. 20).

Remark that forcing does not obey some simple rules of the propositional calculus. E.g. p may force $\neg\neg\Phi$ but not Φ. Furthermore, the forcing relation $\Vdash$ has by definition, clauses (4) and (5), a homomorphism property with respect to disjunction ($\vee$, $\dot{\vee}$) and existential quantification ($\bigvee$, $\exists$). The relation $\Vdash$ does not have the homomorphism property for conjunction ($\wedge$, &) and universal quantification ($\bigwedge$, $\forall$). For example only the following holds:

(6) $p \Vdash \Phi \wedge \Psi \rightarrow (\exists q_1 \supseteq^{\mathcal{M}} p)(\exists q_2 \supseteq^{\mathcal{M}} p)(q_1 \Vdash \Phi \ \&\ q_2 \Vdash \Psi)$.

We will introduce a relation $\Vdash^*$ (called *weak forcing*), which has the property that $p \Vdash^* \Phi \leftrightarrow p \Vdash^* \neg\neg\Phi$ and the homomorphism property for conjunction and universal quantification $\Vdash^*$ does not have the homomorphism property for disjunction and existential quantification and is, as we may say, dual to the strong forcing relation $\Vdash$.

§ 3. Properties of the strong and weak forcing relation.

DEFINITION. $p \Vdash^* \Phi \leftrightarrow p \Vdash \neg\neg\Phi$ (*p weakly forces* Φ),
$p \| \Phi \leftrightarrow (p \Vdash \Phi \,\dot\vee\, p \Vdash \neg\Phi)$ (*p decides* Φ),
$p \|^* \Phi \leftrightarrow (p \Vdash^* \Phi \,\dot\vee\, p \Vdash^* \neg\Phi)$ (*p weakly decides* Φ),
$\Vdash \Phi \leftrightarrow (\forall p)(p \Vdash \Phi)$.
p and q are *compatible* $\leftrightarrow p \cup^{\mathcal{M}} q$ is a condition.

LEMMA 2. *The weak forcing relation has the following properties:*

(i) $p \Vdash^* \Phi \leftrightarrow \sim(\exists q)(p \subseteq^{\mathcal{M}} q \ \& \ q \Vdash \neg\Phi)$,

(ii) $p \Vdash \Phi \to p \Vdash^* \Phi$,

(iii) $p \Vdash^* \neg\Phi \leftrightarrow p \Vdash \neg\Phi$,

(iv) *If* Φ *is of the form* $\Psi_1 \wedge \Psi_2$, $\Psi_1 \Leftrightarrow \Psi_2$, $\bigwedge_x \Psi$ *or* $\underline{s} = \underline{t}$, *then* $p \Vdash \Phi \leftrightarrow p \Vdash^* \Phi$,

(v) $p \Vdash^* \Phi \wedge \Psi \leftrightarrow (p \Vdash^* \Phi \ \& \ p \Vdash^* \Psi)$,

(vi) $p \Vdash^* \bigwedge_x \Phi(x) \leftrightarrow (\forall s \in \mathcal{M})(p \Vdash^* \Phi(\underline{s}))$,

(vii) $p \Vdash^* \Phi \Leftrightarrow \Psi \to (p \Vdash^* \Phi \leftrightarrow p \Vdash^* \Psi)$,

(viii) $(\forall q \supseteq^{\mathcal{M}} p)(q \Vdash^* \Phi \leftrightarrow q \Vdash^* \Psi) \to p \Vdash^* \Phi \Leftrightarrow \Psi$.

Proof. (i), (ii) and (iii) are immediate consequences of the forcing definition. Notice that all sentences in (iv) are of the form $\neg\Gamma$, hence (iv) follows from (iii).

Ad (v). $p \Vdash^* \Phi \wedge \Psi$ is by (iv) equivalent to $p \Vdash \neg(\neg\Phi \vee \neg\Phi)$, which in turn is (by clauses (3) and (4) of the forcing definition) equivalent to $(\forall q \supseteq^{\mathcal{M}} p)[\sim q \Vdash \neg\Phi \ \& \sim q \Vdash \neg\Psi]$. Use now again (3) to get the equivalent form $p \Vdash \neg\neg\Phi \ \& \ p \Vdash \neg\neg\Psi$. The proof of (vi) is very similar to the proof of (v).

Ad (vii). Assume $p \Vdash^* \Phi \Leftrightarrow \Psi$ and $p \Vdash^* \Phi$ but $\sim p \Vdash^* \Psi$. Then by (i) $q \Vdash \neg\Psi$ for some q extending p. Hence by the first extension lemma $q \Vdash \neg\neg\Phi \ \& \ q \Vdash \neg\Psi$. Thus by (3) and (4) of the forcing definition $\sim(\exists q' \supseteq^{\mathcal{M}} q)(q' \Vdash \neg\Phi \vee \Psi)$. Use (3) in order to see that $q \Vdash \neg(\Phi \Rightarrow \Psi)$. This is by (v) and the first extension lemma in contradiction with $p \Vdash^* \Phi \Leftrightarrow \Psi$.

Ad (viii). Suppose that the conclusion does not hold. If $\sim p \Vdash^* \Phi \Rightarrow \Psi$, then by (i) and the forcing definition (3), (4):

(+) $$\sim(\exists q' \supseteq^{\mathcal{M}} q)(q' \Vdash \neg\Phi \,\dot\vee\, q' \Vdash \Psi),$$

for some q extending p. By the second extension lemma $\overline{q} \| \Phi$ for some $\overline{q}$ extending p. But both possibilities, $\overline{q} \Vdash \Phi$ or $\overline{q} \Vdash \neg\Phi$, are in contradiction with (+). Hence $p \Vdash^* \Phi \Rightarrow \Psi$ must hold. Similarly it follows that also $p \Vdash^* \Psi \Rightarrow \Phi$ holds. Thus $p \Vdash^* \Phi \Leftrightarrow \Psi$ holds by (v).

LEMMA 3. (a) *If* p *and* q *are compatible and* $p \Vdash \Phi$ *and* $q \Vdash \Psi$, *then* $p \cup^{\mathcal{M}} q \Vdash \Phi \wedge \Psi$.

(b) *If* $p \| \Phi$ *and* $p \| \Psi$, *then* $p \| \Phi \vee \Psi$, $p \| \Phi \Rightarrow \Psi$, $p \| \Phi \wedge \Psi$ *and* $p \| \Phi \Leftrightarrow \Psi$.

(c) *If* $p \| \Phi$ *and* $p \| \Psi$, *then* $p \Vdash \Phi \wedge \Psi \leftrightarrow (p \Vdash \Phi \ \& \ p \Vdash \Psi)$.

(d) *If* $p \| \Phi$ *and* $p \| \Psi$, *then* $p \Vdash \Phi \Leftrightarrow \Psi \leftrightarrow (p \Vdash \Phi \leftrightarrow p \Vdash \Psi)$.

(e) *Let* S *be a set*$^{\mathcal{M}}$ *or a class*$^{\mathcal{M}}$ *of* $\mathcal{M}$. *If* $p \Vdash \Phi(\underline{u}_1, \ldots, \underline{u}_n)$ *for all* $u_1 \in^{\mathcal{M}} S, \ldots, u_n \in^{\mathcal{M}} S$, *then* $p \Vdash \bigwedge_{x_1 \varepsilon \underline{S}} \cdots \bigwedge_{x_n \varepsilon \underline{S}} \Phi(x_1, \ldots, x_n)$.

Proof by direct computation (use lemma 2 and the consistency lemma!).

LEMMA 4. (a) *If* $p \|^* \Phi$ *and* $p \|^* \Psi$, *then* $p \|^* \neg\Phi$, $p \|^* \Phi \wedge \Psi$, $p \|^* \Phi \vee \Psi$, $p \|^* \Phi \Rightarrow \Psi$ *and* $p \|^* \Phi \Leftrightarrow \Psi$.

(b) *If* $p \|^* \Phi$ *and* $p \|^* \Psi$ *then* $p \Vdash^* \Phi \vee \Psi \leftrightarrow (p \Vdash^* \Phi \,\dot\vee\, p \Vdash^* \Psi)$.

(c) *If* $p \|^* \Phi$ *and* $p \|^* \Psi$ *then* $p \Vdash^* \Phi \Leftrightarrow \Psi \leftrightarrow (p \Vdash^* \Phi \leftrightarrow p \Vdash^* \Psi)$.

Proof by direct computation (use lemma 2 (vii) and (viii)).

Notice that formulas Φ of $\mathfrak{L}^F$, in which the symbol F does not occur, have a natural interpretation in $\mathcal{M}$ (the interpretation of ε is $\epsilon^{\mathcal{M}}$ and the interpretation of a constant $\underline{S}$ is the class$^{\mathcal{M}}$ S).

LEMMA 5. *If* Φ *is a sentence of the language* $\mathfrak{L}^F$ *in which the symbol* F *does not occur, then* $\Vdash \Phi \leftrightarrow \mathcal{M} \models \Phi \leftrightarrow (\exists p)(p \Vdash \Phi)$.

Proof by induction on the length of Φ.

§ 4. Definition of the extension.

DEFINITION. A sequence $\mathfrak{C} \overset{\circ}{=} (p^{(0)}, p^{(1)}, \ldots)$ of conditions is *complete* iff the following two requirements are satisfied:

(I) $\mathfrak{C}$ is well-ordered by $\subseteq^{\mathcal{M}}$ and of order type ω,

(II) If C is a class$^{\mathcal{M}}$ of conditions such that every condition has an extension$^{\mathcal{M}}$ in$^{\mathcal{M}}$ C, then $p^{(k)} \in^{\mathcal{M}} C$ for some $p^{(k)} \in \mathfrak{C}$.

Remarks. A complete sequence $\mathfrak{C}$ of conditions is a set in the meta-language and need not to be a class in the sense of $\mathcal{M}$. Our definition of completeness seems to be more complicated than the usual definitions. In fact, the usual definitions are given for standard models (that is, for those models whose membership relation is the actual membership relation ϵ) but our ground-model $\mathcal{M}$ can be non-standard. So we had to distinguish carefully between "in the sense of $\mathcal{M}$" and "in the sense of the meta-language". Thus, e.g., $\omega^{\mathcal{M}}$ is the set$^{\mathcal{M}}$ of finite$^{\mathcal{M}}$ ordinals$^{\mathcal{M}}$ but ω is the meta-linguistical collection of all (actually) finite ordinals. A complete sequence $\mathfrak{C}$ of conditions is by (I) a sequence (in the sense of the meta-language) $p^{(0)}, p^{(1)}, p^{(2)}, \ldots$ such that $p^{(0)} \subseteq^{\mathcal{M}} p^{(1)} \subseteq^{\mathcal{M}} p^{(2)} \subseteq^{\mathcal{M}} \ldots$ can be seen from the outside of $\mathcal{M}$. Call a class$^{\mathcal{M}}$ C of conditions *dense* in **Cond** iff the following holds $(\forall p)[p \in^{\mathcal{M}} \mathbf{Cond} \to (\exists q \in^{\mathcal{M}} \mathbf{Cond})(q \in^{\mathcal{M}} C \;\&\; p \subseteq^{\mathcal{M}} q)]$. Thus (II) requires that $\mathbb{C}$ "intersects" every dense subclass$^{\mathcal{M}}$ of **Cond**.

We remind the reader to the following important lemmata.

Lemma 6. *If $\mathbb{C}$ is a complete sequence of conditions, then every sentence Φ of $\mathfrak{L}^F$ is decided by some $p^{(k)} \in \mathbb{C}$.*

Proof. Let Φ be given. By lemma 1 there is a class$^{\mathcal{M}}$ C whose elements$^{\mathcal{M}}$ are precisely those conditions p for which $p \| \Phi$ holds. By the second extension lemma C is a dense subclass$^{\mathcal{M}}$ of **Cond**.

Lemma 7. *There exists a complete sequence of conditions. Moreover, for every condition p there is a complete sequence $\mathbb{C}$ in which p occurs as first element.*

Proof. Since $\mathcal{M}$ is countable there is an enumeration of all classes$^{\mathcal{M}}$ of $\mathcal{M}$: $C_0, C_1, C_2, \ldots$ Let p be any condition and put $p \circeq p^{(0)}$. If $p^{(n)}$ is defined let $p^{(n+1)}$ be any element$^{\mathcal{M}}$ of C_n which extends$^{\mathcal{M}}$ $p^{(n)}$ if such an element$^{\mathcal{M}}$ exists, otherwise put $p^{(n+1)} \circeq p^{(n)}$. The so-defined sequence satisfies (I) of the completeness definition. To see that also (II) is satisfied, let C be any dense class$^{\mathcal{M}}$ of Conditions. In the enumeration C is a certain C_n $(n \in \omega)$. By definition we have $p^{(n+1)} \in^{\mathcal{M}} C_n \circeq C$. Hence (II) is satisfied.

Definition. Let $\mathbb{C}$ be any collection of conditions and Φ a sentence of the language $\mathfrak{L}^F$. Define

$$\mathbb{C} \Vdash \Phi \leftrightarrow (\exists p \in \mathbb{C})(p \Vdash \Phi) \quad \text{and} \quad \mathbb{C} \Vdash^* \Phi \leftrightarrow (\exists p \in \mathbb{C})(p \Vdash^* \Phi)\,.$$

Notice that if $\mathbb{C}$ is a complete sequence then $\mathbb{C} \Vdash \Phi$ is equivalent to $\mathbb{C} \Vdash^* \Phi$.

Lemma 8. *Let $\mathbb{C}$ be a complete sequence. If $\Phi(x_1, \ldots, x_n) \in \mathfrak{L}^F$ and*

$$\mathbb{C} \Vdash \underline{s}_1 = \underline{t}_1, \ldots, \mathbb{C} \Vdash \underline{s}_n = \underline{t}_n,$$

then

$$\mathbb{C} \Vdash \Phi(\underline{s}_1, \ldots, \underline{s}_n) \leftrightarrow \mathbb{C} \Vdash \Phi(\underline{t}_1, \ldots, \underline{t}_n)\,.$$

Proof by induction on the length of Φ. Remark first that by lemma 5 for every condition p the following holds: $p \Vdash \underline{S} = \underline{S}$, $p \Vdash \underline{S} = \underline{T} \to p \Vdash \underline{T} = \underline{S}$, $(p \Vdash \underline{S}_1 = \underline{S}_2 \;\&\; p \Vdash \underline{S}_2 = \underline{S}_3) \to p \Vdash \underline{S}_1 = \underline{S}_3$, $(p \Vdash \underline{S}_1 \varepsilon \underline{T} \;\&\; p \Vdash \underline{S}_1 = \underline{S}_2) \to p \Vdash \underline{S}_2 \varepsilon \underline{T}$ and $(p \Vdash \underline{S} \varepsilon \underline{T}_1 \;\&\; p \Vdash \underline{T}_1 = \underline{T}_2) \to p \Vdash \underline{S} \varepsilon \underline{T}_2$. Furthermore it is easily seen that for any condition p, $p \Vdash F(\underline{s}) \;\&\; p \Vdash \underline{s} = \underline{t}$ implies that $p \Vdash F(\underline{t})$. Hence the lemma is true for atomic formulas Φ. For arbitrary formulas Φ the lemma follows by a simple induction.

Definition of the relational system $\mathcal{N}[\mathbb{C}]$ with respect to a complete sequence $\mathbb{C}$:

(i) Let $\Phi(x)$ be a formula of $\mathfrak{L}^F$. The collection of all set-constants $\underline{s}$ such that $p^{(k)} \Vdash \Phi(\underline{s})$ for some $p^{(k)} \in \mathbb{C}$ will be a *class* of $\mathcal{N}[\mathbb{C}]$. This class will be denoted by $\mathrm{K}x\Phi(x)$: $\mathrm{K}x\Phi(x) \circeq \{\underline{s};\ \mathbb{C} \Vdash \Phi(\underline{s})\}$.

(ii) *Sets* of $\mathcal{N}[\mathbb{C}]$ will be classes of the form $\mathrm{K}x(x \varepsilon \underline{s})$.

(iii) The *membership* relation $\epsilon^{\mathcal{N}[\mathbb{C}]}$ (or shortly $\epsilon^{\mathcal{N}}$) of $\mathcal{N}[\mathbb{C}]$ is defined as follows: $\mathrm{K}x\Phi(x) \epsilon^{\mathcal{N}} \mathrm{K}y\Psi(y)$ will hold iff there is a set-constant $\underline{s}$ such that $\underline{s} \in \mathrm{K}y\Psi(y)$ and $\mathrm{K}x\Phi(x) \circeq \mathrm{K}x(x \varepsilon \underline{s})$.

(iv) $F(\mathrm{K}x\Phi(x))$ will hold iff $\mathrm{K}x\Phi(x) \epsilon^{\mathcal{N}} \mathrm{K}x(F(x))$.

(v) The interpretation of a constant $\underline{S}$ is the class $\mathrm{K}x(x \varepsilon \underline{S})$ of $\mathcal{N}[\mathbb{C}]$.

The following convention is useful: a notion, operation or relation with superscript $\mathcal{N}[\mathbb{C}]$, or simply $\mathcal{N}$, shall always mean the corresponding notion, operation, relation resp. in the sense of $\mathcal{N}[\mathbb{C}]$.

Next we have to show that for every complete sequence $\mathbb{C}$, $\mathcal{N}[\mathbb{C}]$ is a model of the **NBG**-set theory. This will follow from the following two facts:

(1) $\mathcal{M}$ is contained in $\mathcal{N}[\mathbb{C}]$ as a complete submodel, and

(2) a sentence Φ of $\mathfrak{L}^F$ is true in $\mathcal{N}[\mathbb{C}]$ iff Φ is forced by some condition $p \in \mathbb{C}$.

Hence by (2) questions about the extension $\mathcal{N}[\mathbb{C}]$ can be reduced to questions which can be answered in the groundmodel $\mathcal{M}$. It is therefore allowed to say that: $\mathcal{N}[\mathbb{C}]$ *does not differ too much* from $\mathcal{M}$.

Henceforth we fix a complete sequence $\mathbb{C}$.

Lemma 9. $\mathrm{K}x\Phi(x) \circeq \mathrm{K}y\Psi(y) \leftrightarrow \mathbb{C} \Vdash \bigwedge_x [\Phi(x) \Leftrightarrow \Psi(x)]$.

Proof. By definition $\mathrm{K}x\Phi(x) \circeq \mathrm{K}y\Psi(y)$ is equivalent to

$$(\forall w \in \mathcal{M})[(\exists p \in \mathbb{C})(p \Vdash \Phi(\underline{w})) \leftrightarrow (\exists q \in \mathbb{C})(q \Vdash \Psi(\underline{w}))]\,.$$

Since $\mathbb{C}$ is totally ordered by $\subseteq^{\mathcal{M}}$ we get by lemma 6 and 3 (d):

$$(+) \qquad (\forall w \in \mathcal{M})(\exists p \in \mathbb{C})[p \Vdash \Phi(\underline{w}) \Leftrightarrow \Psi(\underline{w})]\,.$$

The sentence $\bigvee_x \neg(\Phi(x) \Leftrightarrow \Psi(x))$ is by lemma 6 decided by some $p^{(k)} \in \mathbb{C}$. If $p^{(k)} \Vdash \bigvee_x \neg(\Phi(x) \Leftrightarrow \Psi(x))$ would hold, we would get by clause (5) of the forcing definition a contradiction to (+). Hence $p^{(k)} \Vdash \neg \bigvee_x \neg(\Phi(x) \Leftrightarrow \Psi(x))$ must hold and we have proved the part "$\to$" of the lemma.

In order to prove the part "$\leftarrow$" assume that $p^{(k)} \Vdash \bigwedge_x [\Phi(x) \Leftrightarrow \Psi(x)]$ holds. By lemma 2 (iv) and (vi):

$$(++) \qquad (\forall s \in \mathcal{M})[p^{(k)} \Vdash^* \Phi(\underline{s}) \Leftrightarrow \Psi(\underline{s})]\,.$$

In order to prove $\mathrm{K}x\Phi(x) \subseteq \mathrm{K}y\Psi(y)$ assume $\underline{w} \in \mathrm{K}x\Phi(x)$, id est: $p^{(j)} \Vdash \Phi(\underline{w})$ for some $p^{(j)} \in \mathfrak{C}$. Define $i \overset{\circ}{=} \mathrm{Max}\{k, j\}$; hence by $(++)$ and the first extension lemma:

$$p^{(i)} \Vdash^* \Phi(\underline{w}) \Leftrightarrow \Psi(\underline{w}) \;\&\; p^{(i)} \Vdash \Phi(\underline{w}).$$

Thus by lemma 2 (ii) and (vii): $p^{(i)} \Vdash^* \Psi(\underline{w})$ and therefore by lemma 6 $p^{(n)} \Vdash \Psi(\underline{w})$ for some $p^{(n)} \in \mathfrak{C}$. This proves that $\underline{w} \in \mathrm{K}y\Psi(y)$. In the same way one shows that $\mathrm{K}y\Psi(y) \subseteq \mathrm{K}x\Phi(x)$ also holds.

Corollary 9a. *If U and W are class- or set-constants then*

$$\mathrm{K}x(x \varepsilon U) \overset{\circ}{=} \mathrm{K}y(y \varepsilon W) \leftrightarrow \mathfrak{C} \Vdash U = W.$$

Lemma 10. *A sentence Φ of $\mathfrak{L}^F$ holds in $\mathcal{N}[\mathfrak{C}]$ iff $\mathfrak{C} \Vdash \Phi$.*

Proof by a simple induction on the length of Φ.

Lemma 11. *$p \Vdash^* \Phi$ iff Φ is true in all models $\mathcal{N}[\mathfrak{C}]$ such that p occurs in $\mathfrak{C}$.* (Proof as in Easton [3], p. 32).

Lemma 12. *The mapping δ from $\mathcal{M}$ into $\mathcal{N}[\mathfrak{C}]$ given by $\delta(S) \overset{\circ}{=} \mathrm{K}x(x \varepsilon \underline{S})$ is an isomorphism with respect to the membership relations: $\delta(S_1) \epsilon^{\mathcal{N}} \delta(S_2) \leftrightarrow S_1 \epsilon^{\mathcal{M}} S_2$.*

Proof. This follows directly from Corollary 9a and the definition of the structure $\mathcal{N}[\mathfrak{C}]$.

Sometimes it will be convenient to identify $\mathcal{M}$ with its isomorphic image $\delta(\mathcal{M})$.

Remark. If we would have enriched the forcing language $\mathfrak{L}^F$ by a further one-place predicate $G(x)$, with the intended interpretation: *x is an object of the groundmodel $\mathcal{M}$* and the forcing definition by a further clause, saying that every condition forces $G(\underline{S})$, then lemma 12 could be strengthened by adding the assertion that $\delta(\mathcal{M})$ is a *complete inner submodel* of $\mathcal{N}[\mathfrak{C}]$ in the sense of Sherperdson [11], p. 170.

§ 5. Proof of the axioms in the relational system $\mathcal{N}[\mathfrak{C}]$.

Lemma 13. *If Φ is a sentence of the language $\mathfrak{L}^F$ in which the symbol F does not occur, then Φ holds in $\mathcal{N}[\mathfrak{C}]$ iff Φ holds in $\mathcal{M}$.*

Proof. This follows immediately from lemma 5 and lemma 10.

From lemma 13 it follows that the axioms (A 4), (C 1), (C 2) and $(\overline{\text{C 3}})$ hold in the extension $\mathcal{N}[\mathfrak{C}]$. Furthermore it follows that the sentence $\neg\varphi$ holds in $\mathcal{N}[\mathfrak{C}]$ and that the set-form of the axiom of regularity (*Fundierungsaxiom*) is true in $\mathcal{N}[\mathfrak{C}]$. Gödel has pointed out that on the basis of the axioms of groups A, B, C the set-form and the class-form (i.e. axiom (D)) are equivalent—see P. Bernays [1], p. 68.

The axioms (A 1) and (A 2) are obviously true in $\mathcal{N}[\mathfrak{C}]$. Ad (A 3): Let two sets$^{\mathcal{N}}$, $\mathrm{K}x(x \varepsilon \underline{s})$ and $\mathrm{K}y(y \varepsilon \underline{t})$, and a class$^{\mathcal{N}}$ $\mathrm{K}z\Phi(z)$ be given such that $\mathrm{K}x(x \varepsilon \underline{s}) \epsilon^{\mathcal{N}} \mathrm{K}z\Phi(z)$ and $\mathrm{K}x(x \varepsilon \underline{s}) \overset{\circ}{=} \mathrm{K}y(y \varepsilon \underline{t})$ holds. Hence by the definition of $\mathcal{N}[\mathfrak{C}]$ and lemma 9, $\mathfrak{C} \Vdash \underline{s} = \underline{t}$ and $\mathfrak{C} \Vdash \Phi(\underline{s})$. Therefore by lemma 8, $\mathfrak{C} \Vdash \Phi(\underline{t})$, hence $\mathrm{K}y(y \varepsilon \underline{t}) \epsilon^{\mathcal{N}} \mathrm{K}z\Phi(z)$. Thus we know that all the axioms of group A hold in $\mathcal{N}[\mathfrak{C}]$.

Lemma 14. *The axioms of group* B *hold in $\mathcal{N}[\mathfrak{C}]$.*

Proof (Easton [3]). The axioms of group B have the form

$$\bigwedge_{X_1} \dots \bigwedge_{X_k} \bigvee_{Y} \bigwedge_{z} [z \varepsilon Y \Leftrightarrow \Phi(X_1, \dots, X_k, z)]$$

with $k \overset{\circ}{=} 0, 1$ or 2, for $\Phi \in \mathfrak{L}^F$. Then, given Φ and classes$^{\mathcal{N}}$ $\mathrm{K}x_1\Psi_1(x_1), \dots, \mathrm{K}x_k\Psi_k(x_k)$ the required class$^{\mathcal{N}}$ is $\mathrm{K}z\Phi^+(\mathrm{K}x_1\Psi_1(x_1), \dots, \mathrm{K}x_k\Psi_k(x_k))$ where Φ^+ is the formula obtained from Φ by replacing each subformula $u \varepsilon X_i$ of Φ by $\Psi_i(u)$.

Lemma 15. *The universal axiom of choice* (E) *holds in $\mathcal{N}[\mathfrak{C}]$.*

Proof. First remark that $\boldsymbol{F} \overset{\circ}{=} \mathrm{K}x(\boldsymbol{F}(x))$ is a function$^{\mathcal{N}}$ and that [5] $\boldsymbol{F}(\mathrm{K}x(x \varepsilon \underline{s})) \epsilon^{\mathcal{N}} \mathrm{K}x(x \varepsilon \underline{s})$ for every non-empty$^{\mathcal{N}}$ set$^{\mathcal{N}}$ $\mathrm{K}x(x \varepsilon \underline{s})$. We have to show that $\boldsymbol{F}$ is defined on the whole universe$^{\mathcal{N}}$ of non-empty$^{\mathcal{N}}$ sets$^{\mathcal{N}}$. Let $\mathrm{K}x(x \varepsilon \underline{s})$ by any non-empty$^{\mathcal{N}}$ set$^{\mathcal{N}}$. Consider the following class$^{\mathcal{M}}$:

$$C \overset{\circ}{=} \{p;\ p \epsilon^{\mathcal{M}} \mathbf{Cond} \;\&\; (\exists t)(t \epsilon^{\mathcal{M}} s \;\&\; \langle s, t\rangle^{\mathcal{M}} \epsilon^{\mathcal{M}} p)\}^{\mathcal{M}}.$$

C is a dense subclass$^{\mathcal{M}}$ of **Cond**, hence by the completeness of $\mathfrak{C}$, $p^{(k)} \epsilon^{\mathcal{M}} C$ for some $p^{(k)} \in \mathfrak{C}$. Therefore $p^{(k)} \Vdash \boldsymbol{F}(\langle s, t\rangle^{\mathcal{M}})$. This gives us $\langle \mathrm{K}x(x \varepsilon \underline{s}), \mathrm{K}y(y \varepsilon \underline{t})\rangle^{\mathcal{N}} \epsilon^{\mathcal{N}} \boldsymbol{F}$ and $\mathrm{K}y(y \varepsilon \underline{t}) \epsilon^{\mathcal{N}} \mathrm{K}x(x \varepsilon \underline{s})$. Hence $\boldsymbol{F}$ is defined on the whole class$^{\mathcal{N}}$ of non-empty$^{\mathcal{N}}$ sets$^{\mathcal{N}}$ and is a choice-function$^{\mathcal{N}}$.

The proof that the axiom of replacement (C 4) holds in $\mathcal{N}[\mathfrak{C}]$ is organized as follows. In the first step (lemmata 16 and 17) we show that for every formula $\Phi(x_1, \dots, x_n)$ of $\mathfrak{L}^F$ and every set-constant $\underline{s}$ there exists a formula $\Psi(x_1, \dots, x_n)$ of $\mathfrak{L}^F$ in which all quantifiers are restricted to set-constants such that in $\mathcal{N}[\mathfrak{C}]$ both formulas are equivalent when the free variables are restricted to range over $\underline{s}$. In the second step (lemma 18) we show that the image$^{\mathcal{N}}$ of a set$^{\mathcal{N}}$ under the global choice function $\boldsymbol{F}$ is again a set$^{\mathcal{N}}$. This then allows us to prove (by induction) in the third step (lemma 19) that for every formula $\Psi(x_1, \dots, x_n)$ of $\mathfrak{L}^F$, in which all quantifiers are restricted to set-constants, there exists a formula $\Gamma(x_1, \dots, x_n)$ in which the symbol F does not occur such that both formulas are equivalent in $\mathcal{N}[\mathfrak{C}]$ when the free variables are restricted to range over $\underline{s}$. Combining these results we are able to prove in lemma 20 that the replacement axiom (C 4) holds in $\mathcal{N}[\mathfrak{C}]$.

Definition. Let $\Phi(x_0, x_1, \dots, x_n)$ be a formula of $\mathfrak{L}^F$ with no free variables other than $x_0, \dots, x_n$ and let $\underline{s}$ be a set-constant and y a variable

(5) $\boldsymbol{F}(u)$ is the image of u under $\boldsymbol{F}$ (Gödel [5] writes $\boldsymbol{F}'u$).

not occurring in Φ. Define $\mathrm{Res}(\Phi, \underline{s}, y)$ (*restriction of* Φ *to* $\underline{s}$ *and* y) to be the following formula:

$$\bigwedge_{x_1} \dots \bigwedge_{x_n} \Big[(x_1 \varepsilon \underline{s} \wedge \dots \wedge x_n \varepsilon \underline{s}) \Rightarrow \Big(\bigvee_{x_0} \Phi(x_0, \dots, x_n) \Longleftrightarrow \bigvee_{x_0 \varepsilon y} \Phi(x_0, \dots, x_n)\Big)\Big] .$$

We want to show that the class$^{\mathcal{M}}$ of conditions p for which $p \Vdash \bigvee_y \mathrm{Res}(\Phi, \underline{s}, y)$ holds is a dense subclass of **Cond**.

LEMMA 16. *Let p be a condition, s a set of $\mathcal{M}$ and $\Phi(x_0, \dots, x_n)$ a formula of $\mathfrak{L}^F$ with no free variables other than $x_0, \dots, x_n$. Then there exists an extension q of p such that*

$$q \Vdash \bigvee_y \mathrm{Res}(\Phi, \underline{s}, y) .$$

Let us first indicate how this lemma will be proved. Let s^n be the set$^{\mathcal{M}}$ of n-tuples$^{\mathcal{M}}$ of elements$^{\mathcal{M}}$ of s and let $u_0, u_1, \dots, u_\alpha, \dots, \alpha \epsilon^{\mathcal{M}} \lambda$, be a well-ordering of s^n. If u_α is the n-tuple $\langle z_1, \dots, z_n \rangle^{\mathcal{M}} \epsilon^{\mathcal{M}} s^n$ then let us simply write $\Phi(x_0, u_\alpha)$ instead of $\Phi(x_0, z_1, \dots, z_n)$. We define inductively a sequence $v_0, v_1, \dots, v_\alpha, \dots, \alpha \epsilon^{\mathcal{M}} \lambda$, of elements of $\mathcal{M}$, and a sequence of conditions $p \subseteq^{\mathcal{M}} p_0, p_1, \dots, p_\alpha, \dots, \alpha \epsilon^{\mathcal{M}} \lambda$, in the following way: if there is a set v and an extension p' of p such that $p' \Vdash \Phi(\underline{v}, \underline{u}_0)$ then pick such a pair $\langle p', v \rangle^{\mathcal{M}}$ and call it $\langle p_0, v_0 \rangle^{\mathcal{M}}$. If there is no such pair, then define p_0 to be p. If conditions p_β for all $\beta <^{\mathcal{M}} \alpha$ are obtained, then look at the "equation" $p' \Vdash \Phi(\underline{x}, \underline{u}_\alpha)$. If there are solutions with $\bigcup^{\mathcal{M}} \{p_\beta;\ \beta <^{\mathcal{M}} \alpha\}^{\mathcal{M}} \subseteq^{\mathcal{M}} p'$ then pick such a solution $\langle p', v \rangle^{\mathcal{M}}$ and call it $\langle p_\alpha, v_\alpha \rangle^{\mathcal{M}}$. If there are no such solutions, define p_α to be the union$^{\mathcal{M}}$ of the p_β for $\beta <^{\mathcal{M}} \alpha$ and let v_α be undefined. Finally let p^* be the union$^{\mathcal{M}}$ of all the p_α, $\alpha <^{\mathcal{M}} \lambda$, and let t be the set$^{\mathcal{M}}$ of those sets$^{\mathcal{M}}$ v_α which are defined. Then it follows that $p^* \Vdash \mathrm{Res}(\Phi, \underline{s}, \underline{t})$. Clearly, the axiom of choice (AC) was used in order to obtain a well-ordering of s^n. Furthermore we used the universal version of the axiom of choice (E) in order to pick out the pairs $\langle p_\alpha, v_\alpha \rangle^{\mathcal{M}}$. Since only (AC) and not (E) holds in $\mathcal{M}$ we must carefully avoid any use of (E) in the rigourous proof of lemma 16. This will be done by defining (by means of a certain tree) a subset of $\mathbf{Cond} \times^{\mathcal{M}} V^{\mathcal{M}}$ such that the solutions $\langle p', v \rangle^{\mathcal{M}}$ of $p' \Vdash \Phi(\underline{v}, \underline{u}_\alpha)$ are selected only from this set$^{\mathcal{M}}$ ($V^{\mathcal{M}}$ is the class$^{\mathcal{M}}$ whose elements$^{\mathcal{M}}$ are just the sets$^{\mathcal{M}}$).

Proof of lemma 16. Let p, s and $\Phi(x_0, \dots, x_n)$ be given. s^n is the set$^{\mathcal{M}}$ of all n-tuples$^{\mathcal{M}}$ of elements$^{\mathcal{M}}$ of s. By (AC) there is in $\mathcal{M}$ a well-ordering of s^n. Hence let u_α, $\alpha <^{\mathcal{M}} \lambda$, be the elements$^{\mathcal{M}}$ of s^n (λ is an ordinal$^{\mathcal{M}}$). $\varrho^{\mathcal{M}}$ is the rank-function [6] of $\mathcal{M}$ in the sense of $\mathcal{M}$. By lemma 1 there is in $\mathcal{M}$ a class D,

$$D \circeq \{\langle p', v, u_\alpha \rangle^{\mathcal{M}};\ p' \Vdash \Phi(\underline{v}, \underline{u}_\alpha)\ \&\ \alpha <^{\mathcal{M}} \lambda\ \&\ p \subseteq^{\mathcal{M}} p'\}^{\mathcal{M}} .$$

[6] $\varrho(x)$ is the least α such that x is an element of $V_\alpha = \bigcup_{\beta<\alpha} \mathfrak{P}(V_\beta)$, $\mathfrak{P}(x)$ is the power-set of x.

We define inductively sets g_α, $\alpha <^{\mathcal{M}} \lambda$,

$$g_0 \circeq \{\langle q, v \rangle^{\mathcal{M}};\ \langle q, v, u_0 \rangle^{\mathcal{M}} \epsilon^{\mathcal{M}} D\ \&$$
$$\&\ (\forall p' \epsilon^{\mathcal{M}} \mathbf{Cond})(\forall v' \epsilon \mathcal{M})[\langle p', v', u_0 \rangle^{\mathcal{M}} \epsilon^{\mathcal{M}} D \rightarrow$$
$$\rightarrow \varrho^{\mathcal{M}}(\langle q, v \rangle^{\mathcal{M}}) \leqslant^{\mathcal{M}} \varrho^{\mathcal{M}}(\langle p', v' \rangle^{\mathcal{M}})]\}^{\mathcal{M}} .$$

Suppose we have defined the sets g_β for all $\beta <^{\mathcal{M}} \alpha$. We want to define g_α. Call a subset$^{\mathcal{M}}$ d of **Cond** a *regular α-chain* iff d has p as least$^{\mathcal{M}}$ element$^{\mathcal{M}}$,

$$d \subseteq^{\mathcal{M}} \{p\}^{\mathcal{M}} \cup^{\mathcal{M}} \bigcup^{\mathcal{M}} \{\mathrm{pr}_1^{\mathcal{M}}(g_\beta);\ \beta <^{\mathcal{M}} \alpha\}^{\mathcal{M}} ,$$

d is totally ordered by $\subseteq^{\mathcal{M}}$ and if g_β is non-empty$^{\mathcal{M}}$ for $\beta <^{\mathcal{M}} \alpha$, then there is exactly one q in$^{\mathcal{M}}$ d such that $q \epsilon^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(g_\beta)$. Let Reg_α be the set$^{\mathcal{M}}$ of all regular α-chains. Let $\Delta(p', \alpha, c)$ be the following expression: *c is a regular α-chain and p' extends$^{\mathcal{M}}$ the union$^{\mathcal{M}}$ of c*. Let further $\Gamma(p, v, c, u_\alpha)$ be the following expression:

$$(\forall p' \epsilon^{\mathcal{M}} \mathbf{Cond})(\forall v' \epsilon \mathcal{M})\big[(\langle p', v', u_\alpha \rangle^{\mathcal{M}} \epsilon^{\mathcal{M}} D\ \&\ \Delta(p', \alpha, c)) \rightarrow$$
$$\rightarrow \varrho^{\mathcal{M}}(\langle p, v \rangle^{\mathcal{M}}) \leqslant^{\mathcal{M}} \varrho^{\mathcal{M}}(\langle p', v' \rangle^{\mathcal{M}})\big] .$$

For $c \epsilon^{\mathcal{M}} \mathrm{Reg}_\alpha$ define $g(\alpha, c)$ to be the set$^{\mathcal{M}}$:

$$\{\langle q, v \rangle^{\mathcal{M}};\ \langle q, v, u_\alpha \rangle^{\mathcal{M}} \epsilon^{\mathcal{M}} D\ \&\ \Delta(q, \alpha, c)\ \&\ \Gamma(q, v, c, u_\alpha)\}^{\mathcal{M}} .$$

Now, g_α is defined to be the union$^{\mathcal{M}}$ of all sets $g(\alpha, c)$ where c ranges over Reg_α:

$$g_\alpha \circeq \bigcup^{\mathcal{M}} \{g(\alpha, c);\ c \epsilon^{\mathcal{M}} \mathrm{Reg}_\alpha\}^{\mathcal{M}} .$$

(Let Reg_λ be the union$^{\mathcal{M}}$ of all sets$^{\mathcal{M}}$ Reg_α, $\alpha <^{\mathcal{M}} \lambda$, and define $c_1 \trianglelefteq c_2$ to express that c_1 is an initial segment$^{\mathcal{M}}$ of c_2. $\langle \mathrm{Reg}_\lambda, \trianglelefteq \rangle^{\mathcal{M}}$ is the tree of which we have spoken in the discussion above.) Define finally:

$$\eta \circeq \sup{}^{\mathcal{M}} \{\varrho^{\mathcal{M}}(x) + 1;\ x \epsilon^{\mathcal{M}} \bigcup^{\mathcal{M}} \{g_\alpha;\ \alpha <^{\mathcal{M}} \lambda\}^{\mathcal{M}}\}^{\mathcal{M}}$$

and let $V_\eta^{\mathcal{M}}$ be the set$^{\mathcal{M}}$ of sets$^{\mathcal{M}}$ of rank$^{\mathcal{M}}$ less$^{\mathcal{M}}$ than η.

By the axiom of choice (AC) in $\mathcal{M}$ the set $V_\eta^{\mathcal{M}}$ has a well-ordering w and therefore we can choose inductively from each set $\mathrm{pr}_1^{\mathcal{M}}(g_\alpha)$ an element$^{\mathcal{M}}$ p_α such that p_α extends$^{\mathcal{M}}$ $\bigcup^{\mathcal{M}} \{p_\beta;\ \beta <^{\mathcal{M}} \alpha\}^{\mathcal{M}}$ if such an element exists in $\mathrm{pr}_1^{\mathcal{M}}(g_\alpha)$; otherwise let p_α be the union$^{\mathcal{M}}$ of the p_β for $\beta <^{\mathcal{M}} \alpha$. For those conditions p_α for which $p_\alpha \epsilon^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(g_\alpha)$ holds we can choose a v_α such that $\langle p_\alpha, v_\alpha \rangle^{\mathcal{M}} \epsilon^{\mathcal{M}} g_\alpha$. By the axiom of replacement in $\mathcal{M}$ the sets$^{\mathcal{M}}$ v_α form a set$^{\mathcal{M}}$ t. Define $p^* \circeq \bigcup^{\mathcal{M}} \{p_\alpha;\ \alpha <^{\mathcal{M}} \lambda\}^{\mathcal{M}}$. p^* is a set$^{\mathcal{M}}$ and, since the p_α's are totally ordered by $\subseteq^{\mathcal{M}}$, p^* is a condition. We claim that $p^* \Vdash \mathrm{Res}(\Phi, \underline{s}, \underline{t})$ holds. The first step will be to show that

$$(+)\qquad (\forall \overline{p} \supseteq^{\mathcal{M}} p^*)(\forall u_\alpha \in^{\mathcal{M}} s^n)[\overline{p} \Vdash \bigvee_v \Phi(v, \underline{u_\alpha}) \rightarrow p^* \Vdash \bigvee_{v \varepsilon \underline{t}} \Phi(v, \underline{u_\alpha})] .$$

Assume that $(\exists v \in \mathcal{M})(\overline{p} \Vdash \Phi(\underline{v}, \underline{u_\alpha}))$. $\{p\}^{\mathcal{M}} \cup^{\mathcal{M}} \overline{\{p_\beta;\, \beta <^{\mathcal{M}} \alpha \;\&\; p_\beta \in^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(g_\beta)\}^{\mathcal{M}}}$ is a regular α-chain and $\overline{p}$ is an extension of its union$^{\mathcal{M}}$. Therefore g_α is non-empty$^{\mathcal{M}}$ and $p_\alpha \Vdash \Phi(\underline{v_\alpha}, \underline{u_\alpha})$ where $v_\alpha \in^{\mathcal{M}} t$. Hence by lemma 3 (a) and the definition of forcing: $p_\alpha \Vdash \bigvee_v (v \varepsilon \underline{t} \wedge \Phi(v, \underline{u_\alpha}))$. Since $p_\alpha \subseteq^{\mathcal{M}} p^*$, (+) follows by the first extension lemma.

The second step will be to show that

$$(++)\qquad (\forall \overline{p} \supseteq^{\mathcal{M}} p^*)(\forall u_\alpha \in^{\mathcal{M}} s^n)[\overline{p} \Vdash \bigvee_{v \varepsilon t} \Phi(v, \underline{u_\alpha}) \rightarrow p^* \Vdash \bigvee_{v \varepsilon \underline{t}} \Phi(v, \underline{u_\alpha})] .$$

Assume that $\overline{p} \Vdash \bigvee_v (v \varepsilon \underline{t} \wedge \Phi(v, \underline{u_\alpha}))$. Hence there is an extension $\hat{p}$ of $\overline{p}$ such that $\hat{p} \Vdash \Phi(\underline{v}, \underline{u_\alpha})$ for some $v \in^{\mathcal{M}} t$. Now continue as in the proof of (+). The third step:

$$(\circ)\qquad (\forall \overline{p} \supseteq^{\mathcal{M}} p^*)(\forall u_\alpha \in^{\mathcal{M}} s^n)[\overline{p} \Vdash^* \bigvee_v \Phi(v, \underline{u_\alpha}) \rightarrow \overline{p} \Vdash^* \bigvee_{v \varepsilon \underline{t}} \Phi(v, \underline{u_\alpha})] .$$

If $\overline{p} \Vdash^* \bigvee_v \Phi(v, \underline{u_\alpha})$ then $\hat{p} \Vdash \bigvee_v \Phi(v, \underline{u_\alpha})$ for some $\hat{p} \supseteq^{\mathcal{M}} \overline{p}$. Hence by (+): $p^* \Vdash \bigvee_v (v \varepsilon \underline{t} \wedge \Phi(v, \underline{u_\alpha}))$ and ($\circ$) follows from the first extension lemma The fourth step:

$$(\circ\circ)\qquad (\forall \overline{p} \supseteq^{\mathcal{M}} p^*)(\forall u_\alpha \in^{\mathcal{M}} s^n)[\overline{p} \Vdash^* \bigvee_{v \varepsilon \underline{t}} \Phi(v, \underline{u_\alpha}) \rightarrow \overline{p} \Vdash^* \bigvee_v \Phi(v, \underline{u_\alpha})] .$$

Let $\overline{p}$ be an extension$^{\mathcal{M}}$ of p^* and assume that $\overline{p} \Vdash^* \bigvee_{v \varepsilon \underline{t}} \Phi(v, \underline{u_\alpha})$. Since $\bigvee_v (v \varepsilon \underline{t} \wedge \Phi(v, \underline{u_\alpha})) \Rightarrow \bigvee_v \Phi(v, \underline{u_\alpha})$ is a tautology, we get that also $\overline{p} \Vdash^* \bigvee_v \Phi(v, \underline{u_\alpha})$ (see e.g. Lévy [8], lemma 34).

From ($\circ$) and ($\circ\circ$) we get:

$$(\forall \overline{p} \supseteq^{\mathcal{M}} p^*)(\forall u_\alpha \in^{\mathcal{M}} s^n)[\overline{p} \Vdash^* \bigvee_v \Phi(v, \underline{u_\alpha}) \leftrightarrow \overline{p} \Vdash^* \bigvee_{v \varepsilon \underline{t}} \Phi(v, \underline{u_\alpha})] .$$

Thus lemma 16 follows by lemma 2 (viii), (iv) and the forcing definition, clause (5).

Corollary 16 a. *Let $\Phi(x_0, \ldots, x_n)$ be a formula of $\mathfrak{L}^F$ with no free variables other than $x_0, \ldots, x_n$, and let $\underline{s}$ be a set-constant.*

Then there exists a set$^{\mathcal{M}}$ t such that $\mathcal{N}[\mathfrak{C}] \models \mathrm{Res}(\Phi, \underline{s}, \underline{t})$.

Proof. By lemma 1 there is a class$^{\mathcal{M}}$ C whose elements$^{\mathcal{M}}$ are precisely those conditions p which force $\bigvee_y \mathrm{Res}(\Phi, \underline{s}, y)$. By lemma 16 C is a dense subclass$^{\mathcal{M}}$ of **Cond.** Hence $p^{(k)} \Vdash \bigvee_y \mathrm{Res}(\Phi, \underline{s}, y)$ for some $p^{(k)} \in \mathfrak{C}$ and the corollary follows from clause (5) of the forcing definition and lemma 10.

Lemma 17. *Let $\Phi(x_1, \ldots, x_n)$ be a formula* (7) *of $\mathfrak{L}^F$ and $\underline{s}$ a set-constant. There exists a formula $\Phi^{\nabla}(x_1, \ldots, x_n)$ of $\mathfrak{L}^F$ in which all quantifiers are restricted to set-constants such that*

$$\mathcal{N}[\mathfrak{C}] \vdash \bigwedge_{v_1 \varepsilon \underline{s}} \ldots \bigwedge_{x_n \varepsilon \underline{s}} [\Phi(x_1, \ldots, x_n) \Leftrightarrow \Phi^{\nabla}(x_1, \ldots, x_n)] .$$

Proof by induction on the length of Φ: if Φ is atomic, then let Φ^{∇} be Φ. If Φ is of the form $\neg\Psi$ or $\Psi_1 \vee \Psi_2$ then let Φ^{∇} be $\neg(\Psi^{\nabla})$, $(\Psi_1^{\nabla}) \vee (\Psi_2^{\nabla})$ respectively. If Φ is of the form $\bigvee_y \Psi$, then by lemma 16 there is a set$^{\mathcal{M}}$ t such that

$$\mathcal{N}[\mathfrak{C}] \models \bigwedge_{v_1 \varepsilon s} \ldots \bigwedge_{x_n \varepsilon \underline{s}} [\bigvee_y \Psi \Leftrightarrow \bigvee_{y \varepsilon t} \Psi] ,$$

if $x_1, \ldots, x_n$ are precisely the free variables of Φ. Let u be the union$^{\mathcal{M}}$ of s and t. By the induction hypothesis there is a formula $\Psi^{\nabla}(y, x_1, \ldots, x_n)$ of $\mathfrak{L}^F$ containing no unrestricted quantifier such that

$$\bigwedge_{y \varepsilon \underline{u}} \bigwedge_{x_1 \varepsilon \underline{u}} \ldots \bigwedge_{x_n \varepsilon \underline{u}} [\Psi(y, x_1, \ldots, x_n) \Leftrightarrow \Psi^{\nabla}(y, x_1, \ldots, x_n)]$$

is true in $\mathcal{N}[\mathfrak{C}]$. Define Φ^{∇} to be $\bigvee_y (y \varepsilon \underline{t} \wedge \Psi^{\nabla}(y, x_1, \ldots, x_n))$.

Lemma 18. $(\forall s \in \mathcal{M})(\forall p)[\mathrm{pr}_1^{\mathcal{M}}(s) \subseteq^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(p) \rightarrow p \Vdash \bigwedge_{x \varepsilon \underline{s}} (F(x) \Leftrightarrow x \varepsilon \underline{p})]$.

Proof. From the definition of forcing we get that

$$(\forall x \in \mathcal{M})(\forall p)[p \Vdash F(\underline{x}) \leftrightarrow p \Vdash \underline{x} \varepsilon \underline{p}] .$$

Obviously $(\forall x \in \mathcal{M})(\forall p)[p \| \underline{x} \varepsilon \underline{p}]$ and if $\mathrm{pr}_1^{\mathcal{M}}(s) \subseteq^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(p)$ then $(\forall x \in^{\mathcal{M}} s)(p \| F(\underline{x}))$. Hence by lemma 3 (d):

$$(\forall x \in \mathcal{M})(\forall p)[\mathrm{pr}_1^{\mathcal{M}}(s) \subseteq^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(p) \rightarrow (x \in^{\mathcal{M}} s \rightarrow p \Vdash F(\underline{x}) \Leftrightarrow \underline{x} \varepsilon \underline{p})] .$$

Since $x \notin^{\mathcal{M}} s \leftrightarrow p \Vdash \neg \underline{x} \varepsilon \underline{s}$ we get using the definition of forcing:

$$(\forall x \in \mathcal{M})(\forall p)\big[\mathrm{pr}_1^{\mathcal{M}}(s) \subseteq^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(p) \rightarrow p \Vdash \big(\neg \underline{x} \varepsilon \underline{s} \vee (F(\underline{x}) \Leftrightarrow \underline{x} \varepsilon \underline{p})\big)\big] .$$

Hence the lemma follows by lemma 2 (ii), (vi) and (iv).

Lemma 19. *Let $\Phi(x_1, \ldots, x_n)$ be a formula of $\mathfrak{L}^F$ with no free variables other than $x_1, \ldots, x_n$ such that all quantifiers in Φ are restricted to set-*

(7) We have the following convention: in simplifying the symbolism we mostly do not mention the free variables of a formula. But if some variables are listed, then it is understood that these are all the free variables of the formula.

constants. For every set$^{\mathcal{M}}$ s *there is a formula* $\Phi^0(x_1, \ldots, x_n)$ *of* $\mathfrak{L}^F$ *with the same set of free variables and in which the symbol* F *does not occur such that*

$$\mathcal{N}[\mathfrak{C}] \models \bigwedge_{x_1 \varepsilon \underline{s}} \cdots \bigwedge_{x_n \varepsilon \underline{s}} [\Phi(x_1, \ldots, x_n) \Leftrightarrow \Phi^0(x_1, \ldots, x_n)].$$

Proof by induction on the length of Φ.

Case 1. If Φ is atomic and of the form $x \varepsilon y$, or $x \varepsilon \underline{S}$ (x, y variables or set-constants), then let Φ^0 be Φ. If Φ is of the form $F(x)$, then take a condition p such that $\mathrm{pr}_1^{\mathcal{M}}(s) \subseteq^{\mathcal{M}} \mathrm{pr}_1^{\mathcal{M}}(p)$ and define $\Phi^0(x)$ to be the formula $x \varepsilon \underline{p}$. By lemma 18 the class$^{\mathcal{M}}$ D of conditions q for which $q \Vdash \bigwedge_x \big(x \varepsilon \underline{s} \Rightarrow (F(x) \Leftrightarrow x \varepsilon \underline{p})\big)$ holds, is a dense subclass of **Cond.** Hence $\mathfrak{C}$ "intersects" D and by lemma 10 $\bigwedge_x \big[x \varepsilon \underline{s} \Rightarrow \big(\Phi(x) \Leftrightarrow \Phi^0(x)\big)\big]$ holds in $\mathcal{N}[\mathfrak{C}]$.

Case 2. If Φ is of the form $\neg\Psi$ or $\Psi_1 \vee \Psi_2$ then define Φ^0 to be $\neg(\Psi^0)$, $(\Psi_1^0) \vee (\Psi_2^0)$ respectively.

Case 3. Let $\Phi(x_1, \ldots, x_n)$ be of the form $\bigvee_{x_0 \varepsilon \underline{t}} \Psi(x_0, x_1, \ldots, x_n)$ and let u be the union$^{\mathcal{M}}$ of s and t. By the induction hypothesis there is a F-free formula $\Psi^0(x_0, \ldots, x_n)$ such that

$$\bigwedge_{x_0 \varepsilon \underline{u}} \cdots \bigwedge_{x_n \varepsilon \underline{u}} [\Psi(x_0, \ldots, x_n) \Leftrightarrow \Psi^0(x_0, \ldots, x_n)]$$

holds in $\mathcal{N}[\mathfrak{C}]$. Hence

$$\bigwedge_{x_1 \varepsilon \underline{s}} \cdots \bigwedge_{x_n \varepsilon \underline{s}} [\bigvee_{x_0 \varepsilon \underline{t}} \Psi(x_0, \ldots, x_n) \Leftrightarrow \bigvee_{x_0 \varepsilon \underline{t}} \Psi^0(x_0, \ldots, x_n)]$$

holds in $\mathcal{N}[\mathfrak{C}]$. Thus we can define $\Phi^0(x_1, \ldots, x_n)$ to be $\bigvee_{x_0} \big(x_0 \varepsilon \underline{t} \wedge \wedge \Psi^0(x_0, \ldots, x_n)\big)$.

LEMMA 20. *The axiom of replacement* (C 4) *holds in* $\mathcal{N}[\mathfrak{C}]$.

Proof. Let $\mathrm{K}x(x \varepsilon \underline{s})$ be a set$^{\mathcal{N}}$ and $\mathrm{K}y\Phi(y)$ be a class$^{\mathcal{N}}$ such that $\mathrm{K}y\Phi(y)$ is a function$^{\mathcal{N}}$. By lemma 16 there is a set$^{\mathcal{M}}$ u such that

(i) $$\mathcal{N}[\mathfrak{C}] \models \bigwedge_{x \varepsilon \underline{s}} [\bigvee_y \Phi(\langle x, y\rangle^{\mathcal{M}}) \Leftrightarrow \bigvee_{y \varepsilon \underline{u}} \Phi(\langle x, y\rangle^{\mathcal{M}})].$$

Let w be the union$^{\mathcal{M}}$ of s and u. By lemma 19 and 17 there is a formula Φ^0 of $\mathfrak{L}^F$ in which the symbol F does not occur such that

$$\mathcal{N}[\mathfrak{C}] \models \bigwedge_{x \varepsilon \underline{w}} \bigwedge_{y \varepsilon \underline{w}} [\Phi(\langle x, y\rangle^{\mathcal{M}}) \Leftrightarrow \Phi^0(\langle x, y\rangle^{\mathcal{M}})].$$

It follows that

(ii) $$\mathcal{N}[\mathfrak{C}] \models \bigwedge_{x \varepsilon \underline{s}} \bigwedge_{y \varepsilon \underline{u}} [\Phi(\langle x, y\rangle^{\mathcal{M}}) \Leftrightarrow \Phi^0(\langle x, y\rangle^{\mathcal{M}})].$$

Since the symbol F does not occur in Φ^0, the formula Φ^0 has an interpretation in $\mathcal{M}$. Hence there is a class$^{\mathcal{M}}$ G in $\mathcal{M}$ such that G contains precisely those sets$^{\mathcal{M}}$ $\langle x, y\rangle^{\mathcal{M}}$ such that $\Phi^0(\langle x, y\rangle^{\mathcal{M}})$. G is a function$^{\mathcal{M}}$ (by lemma 13 and (ii)) because $\mathrm{K}y\Phi(y)$ is a function$^{\mathcal{N}}$. By the axiom of replacement (C 4) in $\mathcal{M}$ there is a set$^{\mathcal{M}}$ t containing$^{\mathcal{M}}$ precisely those sets$^{\mathcal{M}}$ y such that for some $x \varepsilon^{\mathcal{M}} s$, G contains$^{\mathcal{M}}$ $\langle x, y\rangle^{\mathcal{M}}$. We claim that $\mathrm{K}z(z \varepsilon \underline{t})$ is the image$^{\mathcal{N}}$ of $\mathrm{K}x(x \varepsilon \underline{s})$ under $\mathrm{K}y\Phi(y)$ in $\mathcal{N}[\mathfrak{C}]$. Thus by lemma 9 and lemma 10 we have to prove that

(iii) $$\bigwedge_z [z \varepsilon \underline{t} \Leftrightarrow \bigvee_{x \varepsilon \underline{s}} \Phi(\langle x, z\rangle^{\mathcal{M}})]$$

holds in $\mathcal{N}[\mathfrak{C}]$. Since t is the image$^{\mathcal{M}}$ of s under G in $\mathcal{M}$:

(iv) $$\bigwedge_z [z \varepsilon \underline{t} \Leftrightarrow \bigvee_{x \varepsilon \underline{s}} \Phi^0(\langle x, z\rangle^{\mathcal{M}})]$$

holds in $\mathcal{M}$. Hence by lemma 13, (iv) holds also in $\mathcal{N}[\mathfrak{C}]$. From (i) and (iv) we get that $t \subseteq^{\mathcal{M}} u$. Therefore (iii) follows from (ii) and (iv). This finishes the proof of lemma 20.

We have shown that for every complete sequence $\mathfrak{C}$, $\mathcal{N}[\mathfrak{C}]$ is a model of the axioms of groups A, B, C, D, E and that $\neg\varphi$ holds in $\mathcal{N}[\mathfrak{C}]$. Thus we have obtained a contradiction. Therefore, if $\Sigma + (\mathrm{E}) \vdash \varphi$ for **ZF**-sentences φ, then $\Sigma + (\mathrm{AC}) \vdash \varphi$, and our theorem 1 is proved.

We remark that we have actually proved a more general result, namely that every countable model $\mathcal{M}$ of Σ + the local axiom of choice can be embedded into a model $\mathcal{N}$ of Σ + the global axiom of choice. Since $\mathcal{M}$ is countable, hence well-orderable in the meta-language, one can easily show that the extension $\mathcal{N}[\mathfrak{C}]$ is also countable, hence:

THEOREM 2. *Every countable model* $\mathcal{M}$ *of* $\Sigma + (\mathrm{AC})$ *can be extended to a countable model* $\mathcal{N}$ *of* $\Sigma + (\mathrm{E})$ *such that the "sets" of* $\mathcal{N}$ *are precisely the "sets" of* $\mathcal{M}$ *and* $\mathcal{M} \models \varphi \leftrightarrow \mathcal{N} \models \varphi$ *for all formulas* φ *not involving class-variables.*

§ 6. Generalizations and problems.

We restate our theorem 1 in terms of well-orderings:

If φ *is a formula of set-theory not involving class-variables and if* φ *follows (in* Σ*) from the assertion that there is a class* R *which well-orders the universe* V*, then* φ *already follows (in* Σ*) from the assertion that every set can be well-ordered* [8].

It is natural to ask, whether this statement remains true if one replaces *well-ordering* for example by *total-ordering* or by *order-extension of* $\subseteq$ *to a total-ordering*:

PROBLEM 1. *Let* φ *be a formula of set theory without class-variables and suppose that* φ *follows (in* Σ*) from the assertion that there is a binary relation* R *such that* R *is a total-ordering of* V *and* R *extends the inclusion*

[8] V is the class of all sets.

relation (id est: if $x \subseteq y$ then $\langle x, y\rangle \in R$). Does φ then follows (in Σ) from the (local) order-extension principle, which says that every partial-ordering can be extended to a linear-ordering?

Another problem is the following. Let (Loc-GCH) be the usual (local form of the) generalized continuum hypothesis: $\bigwedge_{\alpha} 2^{\aleph_\alpha} = \aleph_{\alpha+1}$ and let (Univ-GCH) be the universal form of the GCH:

*There is a function F from **On** into V such that for all ordinals α, $F(\alpha)$ is a one-one-mapping from $2^{\aleph_\alpha}$ onto $\aleph_{\alpha+1}$* (9).

PROBLEM 2. *Let φ be a formula of set theory without class-variables and suppose that $\Sigma+$(Univ-GCH) $\vdash \varphi$. Does it then follow that $\Sigma+$(Loc-GCH) $\vdash \varphi$ also holds?*

(Remark that from the results of Easton [3] it follows that (E) is independent from $\Sigma+$(Loc-GCH), hence (Univ-GCH) is as well independent from $\Sigma+$(Loc-GCH).)

Clearly, the list of problems can be continued *ad infinitum*. All these problems have a common form. In order to state this general form it is best to use the notions of "local form" and "universal form" (corresponding to a pair of formulas $\Phi(x_1, \ldots, x_n)$, $\Psi(x_1, \ldots, x_{n+1})$ introduced in Felgner [4], p. 230).

($P_{\Phi,\Psi}$) *Let φ be a formula of set theory without class-variables and suppose that $\Sigma+$(Univ-Φ, Ψ) $\vdash \varphi$ holds. Is it true that then $\Sigma+$(Loc-Φ, Ψ) $\vdash \varphi$ holds?*

The real problem is to find nice and simple conditions $\mathfrak{C}(\Phi, \Psi)$ such that ($P_{\Phi,\Psi}$) has a positive solution whenever $\mathfrak{C}(\Phi, \Psi)$ holds. The conditions we have obtained are very restrictive. With these conditions we obtain a solution of problem 2 but unfortunately not of problem 1. The reason for this is that in the proof of

$$(\forall \Phi \in \mathfrak{L}^F)(\forall s \in \mathcal{M})(\forall p)(\exists q \supseteq^{\mathcal{M}} p)(\exists t \in \mathcal{M})\big(q \Vdash \mathrm{Res}(\Phi, \underline{s}, \underline{t})\big)$$

(see lemma 16) we need that the local form of the axiom of choice (AC) holds in the groundmodel $\mathcal{M}$.

THEOREM 3. *Let φ be a formula of set theory without class-variables and let $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_{n+1})$ be formulas of the language $\mathfrak{L}_{\mathrm{NBG}}$ such that Φ and Ψ are ppf's in the sense of Gödel [5]. Suppose that $\Sigma+$(Loc-Φ, Ψ) $\vdash$ (AC) holds. Then φ is provable in $\Sigma+$(Univ-Φ, Ψ) if and only if φ is provable in $\Sigma+$(Loc-Φ, Ψ).*

The proof is as follows. Suppose that there is a formula φ of $\mathfrak{L}_{\mathrm{ZF}}$ such that $\Sigma+$(Univ-Φ, Ψ) $\vdash \varphi$ but $\sim(\Sigma+$(Loc-Φ, Ψ) $\vdash \varphi)$. Then $\Sigma+$(Loc-Φ, Ψ)$+\neg\varphi$ has a countable model $\mathcal{M}$. Since $\Sigma+$(Loc-Φ, Ψ) $\vdash$ (AC), (AC) holds in $\mathcal{M}$. By theorem 2 there is an extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N} \models \Sigma+(E)+$(Loc-$\Phi$, Ψ)$+\neg\varphi$. But $\Sigma+$(E)$+$(Loc-Φ, Ψ) $\vdash$ (Univ-Φ, Ψ), hence $\mathcal{N} \models$ (Univ-Φ, Ψ), which is a contradiction.

(9) **On** is the class of all ordinal numbers.

Our best partial solution of problem 1 is contained in the following theorem. Σ_A is the axiom system Σ but with the Aussonderungsaxiom instead of the replacement axiom (C 4).

THEOREM 4. *Let φ be a formula of $\mathfrak{L}_{\mathrm{NBG}}$ without class-variables and let $\Phi(x)$ and $\Psi(x, y)$ be formulas of $\mathfrak{L}_{\mathrm{ZF}}$. Suppose that* (10) *$\Psi(x, y)$ implies $\varrho(y) < \varrho(x)+\omega$. If φ is provable in Σ_A+(Univ-Φ, Ψ), then φ is provable in $\Sigma+$(Loc-Φ, Ψ).*

The proof can be taken almost verbatim from Lévy [7], p. 85. Assume that φ is a theorem of Σ_A+(Univ-Φ, Ψ) but (Loc-Φ, Ψ)$\Rightarrow\varphi$ is not a theorem of Σ. Hence $\Sigma+$(Loc-Φ, Ψ)$+\neg\varphi$ has a model $\mathfrak{M} \mathrel{\underline{\circ}} \langle M, \epsilon^{\mathfrak{M}}\rangle$. By a theorem of Montague–Lévy (see [9], theorem 6) — applied inside of $\mathfrak{M}$ — there is a limit-ordinal$^{\mathfrak{M}}$ α such that $\neg\big((\text{Loc-}\Phi, \Psi)\Rightarrow\varphi\big)$ holds in $\mathfrak{M}_\alpha \mathrel{\underline{\circ}} \langle V^{\mathfrak{M}}_{\alpha+1}, \epsilon^{\mathfrak{M}}\rangle$ ($V^{\mathfrak{M}}_{\alpha+1}$ is the set$^{\mathfrak{M}}$ of sets$^{\mathfrak{M}}$ of rank$^{\mathfrak{M}}$ less$^{\mathfrak{M}}$ than $\alpha+1$, sets of $\mathfrak{M}_\alpha$ are the elements$^{\mathfrak{M}}$ of $V^{\mathfrak{M}}_\alpha$, *classes* of $\mathfrak{M}_\alpha$ are the elements$^{\mathfrak{M}}$ of $V^{\mathfrak{M}}_{\alpha+1}$ and ε is interpreted by the *membership relation* of $\mathfrak{M}$). Since α is a limit-ordinal$^{\mathfrak{M}}$, $\mathfrak{M}_\alpha$ is a model of Σ_A. By the axiom (Loc-Φ, Ψ) in $\mathfrak{M}$ there is a function$^{\mathfrak{M}}$ f defined on the set $V^{\mathfrak{M}}_\alpha \cap^{\mathfrak{M}} \{x;\ \Phi(x)\}^{\mathfrak{M}}$ such that $\Psi\big(x, f(x)\big)$ holds. Since $\Psi(x, y)$ implies that if $\varrho^{\mathfrak{M}}(x) <^{\mathfrak{M}} \alpha$ then $\varrho^{\mathfrak{M}}(y) <^{\mathfrak{M}} \alpha$, we have that f is a subset$^{\mathfrak{M}}$ of $V^{\mathfrak{M}}_\alpha \times^{\mathfrak{M}} V^{\mathfrak{M}}_\alpha$, hence a subset$^{\mathfrak{M}}$ of $V^{\mathfrak{M}}_\alpha$ (because α is a limit-number$^{\mathfrak{M}}$) and therefore an element$^{\mathfrak{M}}$ of $V^{\mathfrak{M}}_{\alpha+1}$. Thus (Univ-$\Phi$, Ψ) holds in $\mathfrak{M}_\alpha$ and therefore φ must hold also in $\mathfrak{M}_\alpha$, a contradiction! Hence φ must be a theorem of $\Sigma+$(Loc-Φ, Ψ).

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(10) ϱ is the rank function—see footnote 6.

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Proximity approach to extension problems

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1. Introduction. Let X and Y be dense subspaces of topological spaces αX and αY respectively. An important class of problems in Topology deals with necessary and/or sufficient conditions under which a continuous function $f\colon X\to Y$ has a continuous extension $\bar{f}\colon \alpha X\to\alpha Y$ (or Y). Among several known results in this class, the following result, due to Taimanov [12], has many applications:

(1.1) *A necessary and sufficient condition that a continuous function $f\colon X\to Y$, where X is dense in a T_1-space αX and Y compact Hausdorff, has a continuous extension $\bar{f}\colon \alpha X\to Y$ is that for every pair of disjoint closed sets F_1, F_2* of Y,

$$\mathrm{Cl}_{\alpha X} f^{-1}(F_1)\cap \mathrm{Cl}_{\alpha X} f^{-1}(F_2)=\emptyset .$$

Lodato [7] has shown that a generalized proximity δ_0 (called LO-*proximity* in this paper) can be introduced in αX as follows: $A\,\delta_0 B$ iff $A^-\cap B^-\neq\emptyset$ (we use the bar to denote closure when no confusion is possible). It is well known that in the case of a compact Hausdorff space, δ_0, as defined above, is a unique compatible Efremovič proximity (called EF-*proximity* in this paper) (see Efremovič [3]). Taimanov's Theorem can now be interpreted as follows: If αX and Y are assigned the LO-proximity δ_0 and the EF-proximity δ_0 respectively, then f has a continuous extension if and only if f is proximally continuous. It is interesting to note that whereas X has the subspace LO-proximity induced by δ_0 on αX, Y has an EF-proximity.

This investigation began with an attempt to prove Taimanov's Theorem by the use of bunches and clusters (see Lodato [7] and Leader [6]). However, we found a general theorem which includes several results, including Taimanov's result mentioned above, as special cases.

The 2nd Section gives preliminary results needed to prove our theorems. For a survey of EF-proximity spaces see for example [10]. An uptodate account of LO-proximity is written by Mozzochi [9].