On a generalization of absolute neighborhood retracts*

by

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1. Introduction. In 1953 H. Noguchi [12] introduced a generalization of the notion of a retraction mapping. This led in a natural way to generalizations of the notions of Absolute Retracts and Absolute Neighborhood Retracts. He called his generalized retracts ε-retracts, and proved that several properties of compact Absolute Retracts and Absolute Neighborhood Retracts were also valid for his spaces. More recently A. Gmurczyk [4] and A. Granas [5] have extended this list of properties. They also introduced the terms Approximative Absolute Retracts (AAR) and Approximative Absolute Neighborhood Retracts (AANR) to replace Noguchi's less convenient ε-AR and ε-ANR.

In this paper we continue the investigation of Noguchi's generalization. In Section 2 we state the basic definitions, introducing a slight generalization in the definition of Approximative Absolute Neighborhood Retracts. The larger class of spaces thus obtained still possesses all of the properties of compact ANRs considered by Noguchi. In Section 3 we consider a metric on the set of non-empty compact subsets of a metric space, which leads to a sufficient condition for a space to be an AANR. In Section 4 we obtain a characterization of AANRs which shows that they are exactly the limits of polyhedra in the metric of continuity. Section 5 contains some results on the fixed point property, and in particular shows that Granas' result [5] extending the Hopf–Lefschetz Theorem to AANRs has a generalization to our more general setting. Section 6 is devoted to the hyperspaces of non-empty closed and non-empty closed connected subsets of a connected AANR, and Section 7 contains some miscellaneous results relating to AARs and AANRs.

All spaces considered in this paper are assumed to be separable metric, and d will denote the metric. By a compactum we mean a compact metric space, and by a continuum a connected compactum. A mapping will always mean a continuous function. Q will denote the Hilbert cube, and E^n, $n \geqslant 1$, Euclidean space of dimension n.

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2. Definitions and elementary properties.

DEFINITION 2.1. Let M be a metric space, X a subset of M and ε a positive number. A mapping $r_\varepsilon\colon M\to X$ is an *ε-retraction of M into X* if for each $x\in X$, $d(r_\varepsilon(x), x) < \varepsilon$.

DEFINITION 2.2. A subset X of a metric space M is an *approximative retract of M* if for each positive number ε there is an ε-retraction of M into X. The subset X of M is an *approximative neighborhood retract of M* if for each positive number ε there is a neighborhood $U = U(\varepsilon)$ of X in M and an ε-retraction of U into X.

Remark. In Noguchi's definition [12] of what we call here an approximative neighborhood retract, it is implied that the neighborhood U is independent of ε. By allowing the neighborhood to change, a wider class of spaces is obtained (Example 2.1 below is an approximative neighborhood retract by our definition but not by Noguchi's). Nevertheless it is easily verified that all of Noguchi's results remain valid with this definition.

DEFINITION 2.3. A compactum X is an *Approximative Absolute Retract* (AAR) provided that for every homeomorphism h mapping X onto a subset $h[X]$ of a metric space M, $h[X]$ is an approximative retract of M. X is an *Approximative Absolute Neighborhood Retract* (AANR) provided that for every homeomorphism h mapping X onto a subset of a metric space M, $h[X]$ is an approximative neighborhood retract of M.

Two useful characterizations of AARs and AANRs are the following:

THEOREM 2.1 ([12]). *A space X is an* AAR (*respectively* AANR) *if and only if X is homeomorphic to a closed approximative retract* (*respectively approximative neighborhood retract*) *of the Hilbert cube Q.*

It should be noted that when the dimension of X is finite, Q can be replaced in this theorem by a Euclidean space of sufficiently high dimension.

THEOREM 2.2 ([12]). *A space X is an* AAR (*respectively* AANR) *if and only if X is a compactum with the property that for any $\varepsilon > 0$, given a mapping f from a closed subset A of a metric space M into X, there is a mapping $f_\varepsilon\colon M\to X$* (*respectively a neighborhood $U = U(\varepsilon)$ of A in M and a mapping $f_\varepsilon\colon U\to X$*) *which satisfies $d(f_\varepsilon(a), f(a)) < \varepsilon$ for each $a\in A$.*

Theorem 2.2 makes it easy to prove the following sufficient condition for a space to be an AAR or an AANR. We have

THEOREM 2.3. *Let X be a compactum. Suppose that for each positive integer n, X contains a subset X_n such that X_n is an* AAR (*respectively* AANR) *and there is a mapping $r_n\colon X\to X_n$ such that for $x\in X$, $d(r_n(x), x) < 1/n$. Then X is an* AAR (*respectively* AANR).

It is clear that every compact metric ANR (AR) X is an AANR (AAR), for a retraction mapping r is an ε-retraction for every positive number ε.

From this and Theorem 2.3 it follows that Example 2.1 below is an AANR, and Example 2.2 is an AAR.

EXAMPLE 2.1. In E^2, let

$$A_n = \{(x, y)\mid (x-1/n)^2+y^2 = 1/n^2\}, \qquad n = 1, 2, \ldots,$$

$$X_m = \bigcup_{n=1}^{m} A_n, \qquad X = \bigcup_{n=1}^{\infty} A_n.$$

Then each X_m is an ANR, and the map

$$r_m(x, y) = \begin{cases} (x, y), & \text{if} \quad (x, y)\in X_m, \\ (0, 0), & \text{if} \quad (x, y)\in X - X_m \end{cases}$$

satisfies $d(r_m(x, y), (x, y)) < 1/m$, so X is an AANR. Note that X is not an ANR, for it fails to be locally contractible at $(0, 0)$.

In [4], Gmurczyk shows that, with his definition of AANR, almost all homology groups of an AANR are trivial, and each of the non-trivial groups is finitely generated. This example indicates that the result is no longer true in the context of our definition of AANR.

EXAMPLE 2.2. In E^2, let

$$A_n = \left\{(x, y)\mid y = \sin\frac{\pi}{2x},\ 1/(2n+1) \leqslant x \leqslant 1\right\}, \qquad n = 1, 2, \ldots,$$

$$A_0 = \{(0, y)\mid -1 \leqslant y \leqslant 1\},$$

$$X = \bigcup_{n=0}^{\infty} A_n.$$

Each A_n is a simple arc in the plane, hence an AR. The mappings $r_m\colon X\to X_m$ may be defined by collapsing $X - X_m$ onto that portion of X_m lying above the interval $[(2m+1)^{-1}, (2m-1)^{-1}]$. Theorem 2.3 implies that X is an AAR. Since X is neither contractible nor locally connected, it fails to be an AR or an ANR.

3. The metric of continuity.

In [1], Borsuk defined the following metric on the class of all nonempty compacta lying in a metric space.

DEFINITION 3.1. Let M be a metric space and A and B nonempty compact subsets of M. The distance $d_c(A, B)$ between A and B is defined as the infimum of the set of numbers t such that there exist maps $g\colon A\to B$, $h\colon B\to A$ with

$$d(g(a), a) \leqslant t \quad \text{for each } a\in A,$$

$$d(h(b), b) \leqslant t \quad \text{for each } b\in B.$$

The metric d_c is called the *metric of continuity*, and we denote by 2_c^M the metric space of nonempty compact subsets of M with the metric d_c.

The next theorem indicates the importance of this metric.

THEOREM 3.1. *Let M be a metric space, and (X_n) a sequence of compacta in M such that each X_n is an* AANR (*resp.* AAR). *Then if $X = \operatorname{Lim} X_n$ in the metric of continuity, X is an* AANR (*resp.* AAR).

Proof. We use the characterization of Theorem 2.2. Suppose ε is a positive number, A is a closed subset of a metric space Y, and f is a mapping from A to X. We may assume $d_c(X, X_n) < 1/n$. Choose n so that $1/n < \varepsilon/4$. By definition of d_c, there are maps $g\colon X \to X_n$, $h\colon X_n \to X$ which satisfy $d\big(g(x), x\big) < 1/n$ for $x \in X$, $d\big(h(y), y\big) < 1/n$ for $y \in X_n$. Thus for $x \in X$

$$d\big(hg(x), x\big) < 2/n \,. \tag{1}$$

Since X_n is compact, there is a $\delta > 0$ such that $y, z \in X_n$ and $d(y, z) < \delta$ imply

$$d\big(h(y), h(z)\big) < 2/n \,. \tag{2}$$

The map $gf\colon A \to X_n$ is a mapping into an AANR, so by Theorem 2.2 there is a neighborhood U of A in Y and a mapping $f_\delta\colon U \to X$ such that for $a \in A$,

$$d\big(f_\delta(a), gf(a)\big) < \delta \,. \tag{3}$$

Now define $f_\varepsilon\colon U \to X$ by $f_\varepsilon = hf_\delta$. If $a \in A$, (3) and (2) imply

$$d\big(hf_\delta(a), hgf(a)\big) < 2/n \,, \tag{4}$$

and this with (1) yields

$$d\big(f_\varepsilon(a), f(a)\big) < \varepsilon \,.$$

It follows that X is an AANR. The proof in case X is an AAR is similar.

This result shows that the collections of all AANRs and all AARs in a metric space M form closed subsets of 2_c^M.

We remark that in general the space 2_c^M is not separable. De Groot verified this in [6], and in the same paper showed that if M is a separable metric space, then the subspace of 2_c^M consisting of the compact ANRs $X \subset M$ is separable. Actually a larger subspace is separable. We have

THEOREM 3.2. *Let M be separable, and let H be the collection of* AANRs *$X \subset M$. Then H is a separable subspace of 2_c^M.*

The proof is a straightforward generalization of de Groot's, and is omitted.

4. LP-spaces. If X is a space and α is a finite open covering of X, there is an abstract simplicial complex associated with X and α, called the nerve of α, which is defined as follows: The vertices of the complex are the members of α, and the simplices are those subcollections of members of α with nonempty intersection. We denote this complex by $N(\alpha)$. By a realization of $N(\alpha)$ we mean a geometric complex K in Q or E^n with the property that there is a one-to-one correspondence between the vertices of $N(\alpha)$ and those of K subject to the condition that if the vertices $a_0, a_1, \ldots, a_n$ of $N(\alpha)$ correspond to the vertices $a_0', a_1', \ldots, a_n'$ of K, $a_0, a_1, \ldots, a_n$ span a simplex of $N(\alpha)$ if and only if $a_0', a_1', \ldots, a_n'$ span a simplex of K. We call the underlying point set of K a polyhedron, and denote it by $|K|$. Every abstract simplicial complex has a realization, and any two realizations of a given abstract complex have homeomorphic polyhedra. Because of this we shall abuse language somewhat and refer to a realization of $N(\alpha)$ as simply the nerve of α.

As Theorem 2.1 indicates, every AANR may be considered as a closed subset of the Hilbert cube. It will be useful to consider an AANR X and nerves of certain open coverings of X simultaneously in Q.

DEFINITION 4.1. Let X be a compact subset of Q. Then X is an LP-space if there is a sequence (P_n) of polyhedra in Q such that $X = \operatorname{Lim} P_n$ in the metric of continuity in 2_c^Q.

Remark. When the dimension of X is finite, Q can be replaced by a Euclidean space of sufficient dimension in the previous definition and the following lemmas.

If α is a finite open covering of a subset X of a metric space, we say the diameter of α is less than a positive number δ (written $\operatorname{diam}\alpha < \delta$) provided that for each $U \in \alpha$, $\operatorname{diam} U < \delta$. The following lemmas are simple to verify.

LEMMA 4.1. *Suppose X is compact, V is open in Q, $X \subset V$ and δ is a positive number such that $\delta = d(X, Q - V)$. Let α be a finite open covering of X of diameter less than $\delta/2$, and let $N(\alpha)$ have the property that if x is the vertex of $N(\alpha)$ associated with the set $U \in \alpha$, then $d(x, U) < \delta/2$. Then $|N(\alpha)| \subset V$.*

LEMMA 4.2. *Suppose M is a metric space, X is compact, V is open in M and $X \subset V$. For $\varepsilon > 0$, let $r\colon V \to X$ be an ε-retraction of V into X. Then there is an open set U in M with $X \subset U \subset V$ such that for all $x \in U$ $d\big(r(x), x\big) < \varepsilon$.*

The last lemmas of this section make use of the technique of mapping a compact metric space X into the nerve of a finite open covering of X. The proofs are modifications of those found in [11], and are omitted.

LEMMA 4.3. *Suppose X is compact, V is open in Q and $X \subset V$. Then for any $\varepsilon > 0$ there is a polyhedron $P \subset V$ and a mapping $g\colon X \to P$ such* *that for each $x \in X$, $d(g(x), x) < \varepsilon$. Moreover if X is connected, P is connected.*

It is possible to obtain a form of this lemma which is also useful. If X is a compact space of dimension $\leqslant n-1$, every open covering α of X has a finite open refinement β of order $\leqslant n$, so that $|N(\beta)|$ has dimension $\leqslant n-1$. Using this information, it is not difficult to alter the proof of the previous lemma to obtain the next result.

LEMMA 4.4. *Suppose X is a compact space of dimension $\leqslant n$, V is open in Q and $X \subset V$. Then for any $\varepsilon > 0$ there is a polyhedron $P \subset V$ of dimension $\leqslant n$ and a mapping $g: X \to P$ such that for each $x \in X$, $d(g(x), x) < \varepsilon$.*

We are now in a position to obtain a useful characterization of the class of AANRs. We have seen that if X is the limit in the metric of continuity of a sequence of AANRs, then X itself is an AANR. Actually a much stronger statement can be made, for it is possible to characterize the class of AANRs in terms of limits of sequences in the metric of continuity. Every AANR can be obtained as a limit of a sequence of polyhedra. In terms of LP-spaces, we may state this more precisely as

THEOREM 4.5. *A space X is an AANR if and only if X is an LP-space.*

Proof. Suppose X is an AANR. Without loss of generality we may assume that X is a subset of the Hilbert cube Q. It suffices to show that given a positive number ε, there is a polyhedron P such that $d_c(X, P) < \varepsilon$. Let $\varepsilon > 0$ be given. There is an open set $U \supset X$ in Q and an ε-retraction $r: U \to X$. By Lemma 4.2, we may assume that U has been chosen so that for $x \in U$, $d(r(x), x) < \varepsilon$. By Lemma 4.3 there is a polyhedron $P \subset U$ and a mapping $g: X \to P$ such that

(1) $$d(g(x), x) < \varepsilon \quad \text{for } x \in X.$$

Define $f: P \to X$ by $f = r|P$. Then for $y \in P$

(2) $$d(f(y), y) < \varepsilon,$$

since $P \subset U$. But then $d_c(X, P) < \varepsilon$ follows from the definitions of the metric of continuity and (1) and (2). Hence X is an LP-space.

Conversely, if X is an LP-space, then X is a compact subset of Q, and there is a sequence of polyhedra (P_n) which converges to X in the metric of continuity. Every polyhedron is an ANR, hence an AANR, so that by Theorem 3.4 X is an AANR.

Remark. Lemma 4.3 together with this proof imply that if X is a connected AANR, then $X = \operatorname{Lim} P_n$ where each P_n is a connected polyhedron.

Borsuk has shown [1] that the collection of compacta having dimension $\leqslant k$ in a metric space M forms a closed subset of 2^M_c. Hence if $X = \operatorname{Lim} P_n$ in the metric of continuity and $\dim P_n \leqslant k$ for each n, we may conclude that $\dim X \leqslant k$. Conversely if $\dim X \leqslant k$, by using Lemma 4.4 we may choose the polyhedra P in the proof of Theorem 4.5 so that $\dim P \leqslant k$. We summarize this in the following corollary:

COROLLARY. *A space X is an AANR of dimension $\leqslant k$ if and only if X is an LP-space obtained as a limit of polyhedra each of dimension $\leqslant k$.*

We may use Theorem 4.5 to verify that each of (a) the Cantor set, (b) the Sierpiński Universal Plane Curve, (c) the Menger Universal Curve is an AANR.

(a) Let the Cantor set C in $[0, 1]$ be written as the intersection of a countable sequence of closed sets A_n, where each A_n is a union of 2^n disjoint closed intervals. Then each A_n is a polyhedron, and $C = \operatorname{Lim} A_n$ in the metric of continuity.

(b) The Sierpiński Plane Curve S may be described as follows: Let

$$D_{nk} = \{t \in [0, 1] |\ 2k \cdot 3^{-n} \leqslant t \leqslant (2k+1) \cdot 3^{-n}\}$$

for $n = 1, 2, \ldots$, and $k = 0, 1, \ldots, (3^n-1)/2$, and

$$D_n = \bigcup \{D_{nk} |\ k = 0, 1, \ldots, (3^n-1)/2\},$$

$$H_n = [0, 1] - D_n, \qquad F_n = D_n \cap \overline{H}_n.$$

Then $S = [0, 1] \times [0, 1] - \bigcup_{n=1}^{\infty} (H_n \times H_n)$.

If we set $A_n = S \cap \big((F_n \times [0, 1]) \cup ([0, 1] \times F_n)\big)$, each A_n is a polyhedron, and $S = \operatorname{Lim} A_n$ in the metric of continuity. S is a continuum which fails to be locally contractible at each of its points, in marked contrast to the local contractibility at each point of an ANR.

(c) The Menger Curve is a one-dimensional locally connected continuum which contains homeomorphic images of all one-dimensional continua. It may be defined as the set of all points of the unit cube which project in each of the three directions (of the edges) into Sierpiński Universal Plane Curves, constructed as in (b), on the various faces of the cube. A construction analogous to that in (b) shows that the Menger Curve is an AANR.

For AARs the problem of using the metric of continuity to obtain a characterization seems more difficult. The following sufficient condition is an immediate application of Theorem 3.1.

THEOREM 4.6. *A compactum X is an AAR if there is a sequence (P_n) of polyhedra, each of which is an Absolute Retract, such that $X = \operatorname{Lim} P_n$ in the metric of continuity.*

5. The fixed point property. If f is a mapping of a space X into itself and for some $x \in X$, $f(x) = x$, the point x is called a fixed point of f. A topological space X is said to have the fixed point property (f.p.p.)

if every mapping of X into itself has a fixed point. We wish to consider next the relation of AANRs to this property. The class of ARs has the fixed point property, and Noguchi [12] showed that this property also holds for the larger class of AARs.

One cannot hope to show that each AANR has the f.p.p., since there are already ANRs which admit maps into themselves without fixed points. One result for AANRs was obtained by Noguchi. A mapping f on a metric space X is an ε-map if for each $x \in f[X]$, the diameter of $f^{-1}(x)$ is less than ε.

We then have

THEOREM 5.1 (Noguchi [12]). *Let X be an AANR. Suppose that for each $\varepsilon > 0$ there is a space Y having the fixed point property and an ε-map of X onto Y. Then X has the fixed point property.*

In connection with the f.p.p. and the metric of continuity, Borsuk [1] proved that if in a metric space M one considers the collection of all compacta $X \subset M$ having the fixed point property, then this collection forms a closed subset of 2^M_c. This gives the following result for AANRs.

THEOREM 5.2. *Suppose (X_n) is a sequence of compacta each having the fixed point property and X is an AANR such that $X = \operatorname{Lim} X_n$ in the metric of continuity. Then X has the fixed point property.*

The major result of this section is the generalization of the Lefschetz fixed point to a suitable class of AANRs. In this respect it should be noted that A. Granas [5] has proved this result for the type of AANR considered by Noguchi. However since our definition of AANR enlarges this class of spaces, our result would appear to have independent interest. Moreover, the techniques used by Granas in his work do not apply here, so our approach is quite different.

We make use of some recent material by Knill [9], in which he has developed a theory to give a unified treatment of the Lefschetz theorem on fixed points. We begin by introducing some of Knill's terminology.

DEFINITION 5.1. Let γ be a class of spaces. A compact subset K of a space X is approachable by γ if for every open covering α of K by open sets from X there is a space $Y \in \gamma$ and maps

$$f\colon K \to Y, \quad g\colon Y \to X$$

such that for each $x \in K$, $gf(x)$ and x are in a common set $U_x \in \gamma$.

Knill considers a class of spaces which he calls Q-simplicial (the Q has no relation to our notation for the Hilbert cube; it refers to a coefficient field). He observes that the quasi-complexes of Lefschetz are always Q-simplicial, so in particular every polyhedron is Q-simplicial. Knill obtains two results which are relevant to our discussion. We will state them in terms most useful to us, not in their most general form.

THEOREM 5.3 [9]. *A compact metric space is Q-simplicial if it is approachable by a class of polyhedra.*

The link between AANRs and Q-simplicial spaces is the following:

THEOREM 5.4. *Every AANR is Q-simplicial.*

Proof. Let X be an AANR, K a compact subset of X, and α a covering of K by open sets from X. Let $\alpha' = \alpha \cup \{X-K\}$. Since X is compact, there is a positive number δ such that if $x, y \in X$ and $d(x,y) < \delta$, then x and y lie in a common element of α'. Next let $\varepsilon < \delta/2$. Denote by γ_P the class of all polyhedra in the Hilbert cube. There is a $P \in \gamma_P$ such that $d_c(X, P) < \varepsilon$. Thus there are maps

$$f\colon X \to P, \quad g\colon P \to X$$

such that for $x \in X$, $q \in P$,

$$d\big(x, f(x)\big) < \varepsilon, \quad d\big(q, g(q)\big) < \varepsilon.$$

Denote also by f the restriction of f to K. Let $x \in K$. Evidently $f\colon K \to P$, $g\colon P \to K$ and

$$d\big(gf(x), x\big) \leqslant d\big(gf(x), f(x)\big) + d\big(f(x), x\big) < 2\varepsilon < \delta.$$

Thus x and $gf(x)$ lie in a common element of α'. Since $x \in K$, they must lie in a common element of α; therefore X is approachable by γ_P, and is Q-simplicial.

We consider Čech homology with rational coefficients, so the homology groups are vector spaces. We will say that a space X has finitely generated homology if each of its homology groups is finitely generated, and all but a finite number are trivial. For such a space, a mapping $f\colon X \to X$ induces for each n a vector space homomorphism f_{*n} for which the trace of f_{*n} is defined. The Lefschetz number of the map f is given by the formula

$$\lambda(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Trace}(f_{*n}).$$

We now state a second result of Knill's as

THEOREM 5.5 [9]. *Let X be a compact metric space with finitely generated homology, and suppose X is Q-simplicial. Then if $f\colon X \to X$ satisfies $\lambda(f) \neq 0$, f has a fixed point.*

This result together with Theorem 5.4 yields a generalization of the Lefschetz result to AANRs.

THEOREM 5.6. *Let X be an AANR with finitely generated homology, and $f\colon X \to X$ a mapping with $\lambda(f) \neq 0$. Then f has a fixed point.*

An important class of AANRs then always has the fixed point property.

Corollary. *Every acyclic* AANR *has the* f.p.p.

This corollary extends the class of acyclic compacta which are known to have the f.p.p. There are examples ([2], [8]) which show that one cannot extend fixed point theorems of this type to arbitrary compacta.

The last result of this section is a generalization of another theorem by Noguchi on fixed points and AANRs. He has proved [12] that every null-homotopic map of a finite dimensional AANR into itself has a fixed point. In fact, a stronger result is true.

Theorem 5.7. *Every nullhomotopic map of an* AANR X *into itself has a fixed point.*

Proof. Assume X to be contained in Q, let $f\colon X\to X$ be null-homotopic, and assume f has no fixed points. Then there is a $\delta>0$ such that for each $x\in X$, $d(x,f(x))\geqslant\delta$. Since X is an AANR, there is a polyhedron P such that $d_c(X,P)<\delta/4$. Hence there are maps $g\colon X\to P$, $h\colon P\to X$ such that for $x\in X$, $q\in P$, $d(x,g(x))<\delta/4$, $d(q,h(q))<\delta/4$. The mapping $gfh\colon P\to P$ is nullhomotopic, since f is. It follows from the Lefschetz theorem that any such map has a fixed point, say a. But

$$0<\delta\leqslant d\big(fh(a),h(a)\big)\leqslant d\big(fh(a),gfh(a)\big)+d\big(gfh(a),a\big)+d\big(a,h(a)\big)\leqslant\delta/2,$$

since $gfh(a)=a$. This is a contradiction, and it follows that f has a fixed point.

6. AANRs and the spaces $C(X)$ and 2^X. For this section we restrict our attention to compact metric continua. We denote by 2^X the collection of nonempty closed subsets of X.

Definition 6.1. Let $A,B\in 2^X$, and set

$$d_s(A,B)=\mathrm{MAX}\{\underset{a\in A}{\mathrm{MAX}}\,d(a,B),\underset{b\in B}{\mathrm{MAX}}\,d(b,A)\}.$$

d_s is a metric on 2^X and is called the *Hausdorff metric.*

We call the space 2^X with the Hausdorff metric the hyperspace of closed subsets of X. The subspace $C(X)$ of 2^X, consisting of all nonempty subcontinua of X we call the hyperspace of subcontinua of X.

X is locally connected if and only if 2^X (or $C(X)$) is an Absolute Retract [7]. Since every ANR X is locally connected, for these spaces 2^X and $C(X)$ are ARs. We next show that in the case X is an AANR, each of 2^X and $C(X)$ is an AAR. To do this we first prove the following lemma.

Lemma 6.1. *Suppose $X=\mathrm{Lim}\,X_n$ in the metric of continuity where $X,X_1,X_2,\ldots$, are subsets of a metric space M. Then in the metric of continuity, $2^X=\mathrm{Lim}\,2^{X_n}$, and $C(X)=\mathrm{Lim}\,C(X_n)$.*

Proof. We give the proof for $C(X)$; it is the same for 2^X. As usual we denote by d the metric on M and by d_c the metric of continuity on 2^M_c. The space $C(M)$ is metrized with the Hausdorff metric d_s. Denote the metric of continuity on the space $2_c^{C(M)}$ by D_c. The spaces $C(X)$, $C(X_1)$, $C(X_2)$, ..., are subsets of $C(M)$. We will show that $C(X)=\mathrm{Lim}\,C(X_n)$ in the metric of continuity on $2_c^{C(M)}$. Let $\varepsilon>0$ be given. Since $X=\mathrm{Lim}\,X_n$ in 2_c^M we can choose N such that for $n\geqslant N$, $d_c(X,X_n)<\varepsilon$. Fix $n>N$. There are maps $f\colon X\to X_n$, $g\colon X_n\to X$ such that for all $x\in X$, $y\in X_n$

$$\mathrm{MAX}\{d(x,f(x)),d(y,g(y))\}<\varepsilon. \tag{1}$$

The maps f and g induce maps $F\colon C(X)\to C(X_n)$ and $G\colon C(X_n)\to C(X)$ defined for $A\in C(X)$ and $B\in C(X_n)$ by

$$F(A)=f[A],\qquad G(B)=g[B].$$

It is not difficult to show from (1) that for $A\in C(X)$, $B\in C(X_n)$ we have

$$d_s(A,F(A))<\varepsilon,\qquad d_s(B,G(B))<\varepsilon. \tag{2}$$

Thus for this n we have by (2) and the definition of the metric of continuity that

$$D_c\big(C(X),C(X_n)\big)<\varepsilon. \tag{3}$$

But the choice of $n>N$ was arbitrary, so (3) holds for all $n>N$, and it follows that $\mathrm{Lim}\,C(X_n)=C(X)$ in the metric of continuity on $2_c^{C(M)}$.

We can now prove Theorem 6.2.

Theorem 6.2. *Let X be a connected* AANR. *Then each of 2^X and $C(X)$ is an* AAR.

Proof. Since X is a connected AANR, X is the limit in the metric of continuity of a sequence (P_n) of connected polyhedra. By Lemma 6.1 this means $\mathrm{Lim}\,C(P_n)=C(X)$ in the metric of continuity, and since each P_n is a locally connected continuum, $C(P_n)$ is an AR. By Theorem 3.1, $C(X)$ is an AAR. Similarly 2^{P_n} is an AR, so 2^X is an AAR.

Since AARs have the fixed point property, we have

Theorem 6.3. *If X is a connected* AANR, *both $C(X)$ and 2^X have the fixed point property.*

For any compact metric continuum X, each of 2^X and $C(X)$ is acyclic (see [7] for 2^X, [13] for $C(X)$). It then follows from the corollary to Theorem 5.6 that whenever either of 2^X or $C(X)$ is an AANR, it also has the fixed point property. It is not true that $C(X)$ is always an AANR. It can be shown that if X is the pseudo arc [10], $C(X)$ fails to be an AANR.

7. Some properties of AANRs. In this section, whenever it is convenient, we shall assume the AANRs to be subsets of the Hilbert cube Q.

A space X is an AR if and only if X is an ANR which is contractible in itself. Example 2.2 shows that an AAR need not be contractible. We can show, however, that the converse here is true: A contractible AANR is an AAR. This result follows from a generalization of a theorem of Borsuk [3].

Lemma 7.1. *Let X be a closed subset of X' and Y an AANR. Suppose F: $X \times [0,1] \to Y$ is continuous, and $F(\cdot, 0)$ admits a continuous extension to $F'(\cdot, 0)$: $X' \to Y$. Then for any $\varepsilon > 0$ there is a mapping $\overline{F}$: $X' \times [0,1] \to Y$ such that for $(x,t) \in X \times [0,1] \cup X' \times \{0\}$,*

$$d\big(\overline{F}(x,t), F'(x,t)\big) < \varepsilon .$$

The proof is omitted. It follows at once from Theorem 2.2 and the techniques of Borsuk's proof.

Theorem 7.2. *If X is a contractible AANR, then X is an AAR.*

Proof. X contractible implies that there exist $a \in X$ and F: $X \times [0,1] \to X$ such that $F(x,1) = x$, $F(x,0) = a$.

Then $C = X \times [0,1] \cup Q \times \{0\}$ is closed in $Q \times [0,1]$. Define f: $C \to X$ by

$$f(x,t) = \begin{cases} a, & \text{if } t = 0, \\ F(x,t), & \text{otherwise.} \end{cases}$$

For any $\varepsilon > 0$ there is by Lemma 7.1 a map $\overline{F}$: $Q \times [0,1] \to X$ such that for $(x,t) \in C$, $d\big(\overline{F}(x,t), f(x,t)\big) < \varepsilon$. Now define r_ε: $Q \to X$ by $r_\varepsilon(x) = \overline{F}(x,1)$. Then for any $x \in X$, since $f(x,1) = F(x,1) = x$, we have

$$d\big(r_\varepsilon(x), x\big) = d\big(\overline{F}(x,1), x\big) = d\big(\overline{F}(x,1), f(x,1)\big) < \varepsilon .$$

Thus r_ε is an ε-retraction of Q into X, and since X is compact, X is a closed ε-retraction of Q for each $\varepsilon > 0$. By Theorem 2.1, X is an AAR.

Every neighborhood retract of an ANR is itself an ANR. For AARs and AANRs we have two results of a similar nature.

Theorem 7.3. *Suppose A is a closed approximative retract of an AANR (resp. AAR) X. Then A is an AANR (resp. AAR).*

Proof. Let $\varepsilon > 0$ be given. There is a mapping r_1: $X \to A$ such that for $a \in A$, $d\big(r_1(a), a\big) < \varepsilon/2$. r_1 is uniformly continuous on X, so there is a $\delta > 0$ such that $a, b \in X$ with $d(a,b) < \delta$ implies $d\big(r_1(a), r_1(b)\big) < \varepsilon/2$.

Also there is a neighborhood U of X in Q and a mapping r_2: $U \to X$ which is a δ-retraction. Define r_ε: $U \to A$ to be $r_\varepsilon = r_1 r_2$. Then for $a \in A$, $d\big(r_2(a), a\big) < \delta$, so $d\big(r_\varepsilon(a), a\big) \leqslant d\big(r_1 r_2(a), r_1(a)\big) + d\big(r_1(a), a\big) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus r_ε is an ε-retraction of U into A. A is closed in X, so A is compact. By Theorem 2.1, A is an AANR.

If X is an AAR, the map r_2 may be taken with domain Q, from which it follows that r_ε is an ε-retraction of Q into A. Hence A is an AAR by Theorem 2.1, and this completes the proof.

Theorem 7.4. *Every closed approximative retract A of an ANR X is an AANR.*

Proof. Assume $A \subset X \subset Q$, so A is closed in Q. There is a neighborhood $U \supset X$ in Q and a retraction r: $U \to X$. For $\varepsilon > 0$ there is a neighborhood V of A in X and an ε-retraction r_ε: $V \to A$. Set $W = r^{-1}[V]$. W is a neighborhood of A in Q, and if $r_1 = r|W$, the map $f_\varepsilon = r_\varepsilon r_1$ is the desired ε-retraction of W into A.

Remark. Let us call an AANR of the type considered by Noguchi and Gmurczyk an AANR$'$. In [4], Gmurczyk shows that given an AANR$'$ A, there is an ANR X such that A is an approximative retract of X. From 7.4 we then have that A is an AANR$'$ if and only if A is a closed approximative retract of an ANR X.

Finally, we have the following result which says that a connected AANR has large locally connected subcontinua. More precisely, we have

Theorem 7.5. *Suppose X is a connected AANR. Then for each $\varepsilon > 0$ there is a locally connected subcontinuum K_ε of X such that for each $x \in X$, $d(x, K_\varepsilon) < \varepsilon$. Moreover, if $\dim X = n$, then for all sufficiently small ε, $\dim K_\varepsilon = n$.*

Proof. Since X is a connected AANR, we have $X = \operatorname{Lim} P_m$, where each P_m is a connected polyhedron, and the limit is taken in the metric of continuity on 2^Q_c. Let $\varepsilon > 0$ be given, and choose m so large that $d_c(X, P_m) < \varepsilon/2$. Then there are maps f: $X \to P_m$, g: $P_m \to X$ such that $d\big(x, f(x)\big) < \varepsilon/2$, $d\big(p, g(p)\big) < \varepsilon/2$. P_m is a locally connected continuum, hence $K_\varepsilon = g[P_m]$ is a locally connected continuum contained in X. Also if $x \in X$,

$$d(x, K_\varepsilon) \leqslant d\big(x, gf(x)\big) \leqslant d\big(x, f(x)\big) + d\big(f(x), gf(x)\big) < \varepsilon , \tag{1}$$

which proves the first part of Theorem 7.5. To prove the second, note that by (1), gf is an ε-displacement of X into X. If $\dim X = n$, it follows from a result in [11] that for all sufficiently small ε, $\dim gf[X] \geqslant n$. Since $gf[X] \subset K_\varepsilon \subset X$, we may conclude that $\dim K_\varepsilon = n$.

A continuum X is said to be indecomposable if it cannot be written as the union of two nondegenerate proper subcontinua, and is hereditarily indecomposable if each of its nondegenerate subcontinua is indecomposable. It is easy to show that an indecomposable continuum is not locally connected. This yields the following:

Corollary. *A connected AANR is not hereditarily indecomposable.*

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On the topology of curves II

by

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In the present paper we investigate two classes of curves which we call Suslinian and finitely Suslinian, respectively (see § 1). To motivate the terms which have been chosen, let us point out that the properties attached to them resemble a property of ordered sets introduced by M. Ja. Suslin and related to the famed Suslin problem. Our properties are intended to complete the well-known classification of curves (see [4], p. 96, and [5], p. 99) in which the notion of rational curves plays an essential role. Rational curves possess a decomposition property (see [2], p. 211), and an analogue for Suslinian curves is suggested here; it is, however, proved only in the case of hereditarily unicoherent curves (see § 2). We also prove the existence of Suslinian curves which are not rational (see § 3). A part of the material covered by this paper was mimeographed in [3].

§ 1. The concept of Suslinian curves. A curve X will be called *Suslinian* provided each collection of pairwise disjoint subcurves of X is countable (1). A curve X is called *hereditarily decomposable* provided each subcurve Y of X is decomposable, i.e. representable as the union of two proper subcurves of Y.

1.1. *Each Suslinian curve is hereditarily decomposable.*

Proof. This is because an indecomposable continuum has uncountably many pairwise disjoint composants and each of them is dense. Composants of a curve are countable unions of some subcurves (see [2], p. 147).

A space is called *ponctiform* provided each continuum contained in it is degenerate.

1.2. *If a curve X admits a decomposition $X = P \cup Q$ where P is ponctiform and Q is countable, then X is Suslinian.*

(1) We recall that, in our terminology, a continuum means a compact connected metric space, and a curve means a 1-dimensional continuum. Therefore the curves are non-degenerate sets. A subcurve means a curve which is contained in a curve under consideration.