Decompositions of E^3 which satisfy a uniform Lipschitz condition are factors of E^4

by

Ralph J. Bean * (Knoxville, Tenn.)

1. Introduction. Bing has developed a technique which has been modified by several authors (see [1], [2], [3], [4]) to show that many spaces, not homeomorphic to E^3, are such that their Cartesian product with E^1 is E^4. The spaces considered are decompositions of E^3. The strongest theorem to date is contained in Bailey's thesis (University of Tennessee) which states that if G is an upper semi-continuous decomposition of E^3 whose non-degenerate elements are a Cantor set times an interval, then $E^3/G \times E^1 = E^4$. In this paper we prove a much stronger result.

2. Notation. The notation is standard. A collection G of compact subsets of a space X is an *upper semi-continuous decomposition of* X if and only if

(1) G is a partition of the space X;

(2) if $g \in G$ and U is an open subset of X with $g \subset U$, then there is an open set V, $g \subset V \subset U$ so that if $g' \in G$ and $g' \cap V \neq \emptyset$, then $g' \subset U$.

The decomposition space, X/G, is the space whose points are the elements of G and for which a set W is open if and only if $\bigcup_{g \in W} g$ is open in X. G is monotone if and only if each $g \in G$ is connected. We use π to denote the natural map from X to X/G. Let H denote the collection of non-degenerate elements of a decomposition G and let H^* denote their union. G is *compact* 0-*dimensional* if $\pi(H)$ is compact and 0-dimensional in X/G.

We will consider E^3 to be the $\omega = 0$ level in

$$E^4 = \{(x, y, z, \omega) \mid x, y, z, \omega \in \text{Reals}\}.$$

Throughout this paper all decompositions of E^3 will be monotone, upper semi-continuous, and compact 0-dimensional. If G is such a decomposition there is a map $s\colon E^3 \to [0, 1]$ which takes each non-degenerate

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element of G to a distinct point of $[0,1]$. Then $s^*\colon E^4 \to E^4$ defined by $s^*(x,y,z,\omega) = (x,y,z,s(x,y,z)+\omega)$ is a homeomorphism and lifts each non-degenerate element of G to a different level in the ω-direction. We will use E^3_a to denote $\{(x,y,z,\omega)|\ x,y,z \in \text{Reals},\ \omega = a\}$. Let C be a 3-cell in E^3 which contains each element of H in its interior. Then C_a will denote $C \times \{a\}$. We will index H by $\{s(g)\}_{g\in H}$, i.e., $H = \{g_a|\ a \in s(H)\}\cdot \tilde{g}_a$ will denote $s^*(g_a)$.

A subset X of E^3 is *cellular* if and only if for any $\varepsilon > 0$ there is a 3-cell D in E^3 with $X \subset \operatorname{Int} D \subset D \subset S_\varepsilon(X)$. Suppose D is a cell which contains X in its interior in E^3. A map $r\colon D\times[0,1)\to D$ will be called a *cellularity map of D around X* if and only if the following conditions are satisfied:

(1) $r|D+\{0\} = \text{identity}$,

(2) $r|D\times\tau$ is a homeomorphism for $\tau \in [0,1)$,

(3) $r(\operatorname{Bd} D\times\tau) \cap r(\operatorname{Bd} D\times\tau') = \emptyset$ for $\tau \neq \tau'$,

(4) $X \subset \operatorname{Int} r(D\times\tau)$ for all $\tau \in [0,1)$, and

(5) $\forall \varepsilon > 0, \exists \delta > 0$ so that $r(D\times\tau) \subset S_\varepsilon(X)$ for $\tau \in [1-\delta, 1)$.

The maps are very much like the maps required for strong cellularity, however, here we don't have the map defined for the closed interval $[0,1]$.

LEMMA 1. *$X \subset E^3$ is cellular if and only if for any cell D with $X \subset \operatorname{Int} D$, there is a cellularity map of D around X, $r\colon D\times[0,1)\to D$.*

Proof. This lemma follows easily from Bing's Approximation Theorem and the Annulus Theorem.

Suppose G is an upper semi-continuous decomposition of E^3 which is compact, 0-dimensional, and cellular (each element is cellular). Let C be a 3-cell such that $H^* \subset \operatorname{Int} C$. Then it is clear that we can get a collection $\{r_a\colon C_a\times[0,1)\to C_a\}_{a\in s(H)}$ of cellularity maps of C_a around $\tilde{g}_a$, however, we need a stronger condition. The collection $\{r_a\}_{a\in s(H)}$ will be called a *uniform collection of cellularity maps* if and only if for each $\varepsilon > 0$ and each $a \in s(H)$ there is a $\delta > 0$ and an $\eta \in [0,1)$ so that for any a' with $|a'-a| < \delta$ we have for all $x \in C$, $p\big(r_a(s^*(x)\times\eta)\big) = p\big(r_{a'}(s^*(x)\times\eta)\big)$ (where p is the projection of E^4 onto E^3) and $r_a(C_a\times\eta) \subset S_\varepsilon(\tilde{g}_a)$.

LEMMA 2. *If G is an upper semi-continuous, compact, 0-dimensional, cellular decomposition of E^3 then there is a uniform collection of cellularity maps.*

Proof. Simply construct these maps.

We start in the obvious way. Take a sequence $\{\varepsilon_i\}$ so that $\lim \varepsilon_i = 0$. $\forall \tilde{g}_a$, there is a cellularity map f_a which pulls C_a into $S_{\varepsilon_1}(\tilde{g}_a)$. Once we are inside $S_{\varepsilon_1}(\tilde{g}_a)$ we will stop and call the cell $f_a(C_a)$.

We thicken each $f_a(C_a)$ less than ε_1 and so that the boundary of the resulting 4-cell misses every $\tilde{g}_a \in s^*(H)$. Their interiors form an open cover so finitely many will do. We can assume these are disjoint. Then each $\tilde{g}_a$ is in one such cell and we can define $r_a\colon C_a\times[0,\frac{1}{2}]$ to follow the function f which defined this 4-cell. We can iterate this process to define $r_a\colon C_a\times\times[0,1)\to C_a$. We now claim that this collection of maps $\{r_a\}$ is uniform. The proof of this claim is a simple application of upper semi-continuity.

We need one more definition before we can proceed with the statement of the theorem. We will say that the decomposition satisfies a *uniform Lipschitz condition* if and only if there is a uniform collection of cellularity maps $\{r_a\}_{a\in s(H)}$ and an integer M so that for each a and for each pair of points $x,y \in C_a$, then

$$|r_a(x,\tau) - r_a(y,\tau)| < M|x-y|$$

for each $\tau \in [0,1)$.

The condition on a decomposition that it be uniform Lipschitz is quite strong. It has essentially two effects. First, it is a restriction on the type of non-degenerate elements; a pseudo-arc is not permitted. Second, it restricts the way in which the non-degenerate elements fit together. For example, if there is a sequence of tame arcs in G which

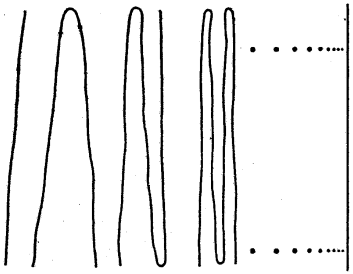

Fig. 1

converge to a tame arc (as shown in Figure 1), where the members of the converging sequence twist more and more as you move through the sequence and the twists do not shrink, then the decomposition is not uniform Lipschitz.

3. The main theorem.

THEOREM. *If G is an upper semi-continuous decomposition of E^3 which is compact 0-dimensional, cellular, and satisfies the uniform Lipschitz condition then $E^3/G\times E^1 = E^4$.*

A word about the proof. The method used is a combination of Bing's technique [2] and Bryant's modification of Bing's method [3].

We will use the uniform collection of cellularity maps and the uniform Lipschitz condition to find a nested finite sequence of finite collections of 4 cells which contain H in their intersection and are themselves within any preassigned neighborhood. We will then stack collections like this in E^4 as Bing does and shrink as Bryant does.

The unfortunate fact about this technique is that it still uses, in an essential way, the idea of Bing, that one must be able to shrink the non-degenerate elements in two ways, that there must be sort of two ends. Bryant's modification of Bing's technique still needs this idea, used three times, so there are six directions to move. A great contribution could be made to this entire study by someone who could develop an entirely new method, for I believe the above theorem is essentially at the limit of the ability of this technique.

4. Finding the cells. We will show that $E^3/G\times E^1$ is E^4 by showing the following lemma.

LEMMA 3. *There is a pseudo-isotopy F: $E^4\times[0,1]\to E^4$ satisfying*

(1) $f(x,0)=x$,

(2) *for each $t_0<1$, $f(x,t_0)$ is a homeomorphism of E^4 onto E^4.*

(3) *$f(x,1)$ takes E^4 onto itself and each element of G' onto a distinct point of E^4, where G' is the decomposition of E^4 defined by the decomposition G of E^3 taken at each level of E^4.*

The proof of this lemma parallels the proof of Bing's Theorem 3 of [2]. We need a sequence of lemmas.

LEMMA 4. *Let $\varepsilon>0$ be given. Then there is a finite collection K of mutually disjoint 4-cells such that*

(1) $H^*\subset \operatorname{Int}K$,

(2) *s^* takes each component of K onto a 3-cell times an interval of the form $\big(r_a(C_a\times\eta)\big)\times[a,b]$ for some a, η, a and b.*

(3) $K\subset S_\varepsilon(H^*)$.

Proof. This lemma follows from a simple application of the uniform Lipschitz condition and the uniform continuity of s^*. Any collection of 4-cells satisfying condition (2) will be called an allowable collection.

The collection $\{X_i\}_{i=1}^{\infty}$ is called a *defining sequence* for G iff each X_i is the union of finitely many mutually disjoint 3-manifolds with boundary, $X_{i+1}\subset\operatorname{Int}X_i$, and the non-degenerate elements of G are precisely the components of $\bigcap_{i=1}^{\infty}X_i$. It is well known that every monotone compact 0-dimensional decomposition of E^3 has a defining sequence.

LEMMA 5. *If $\{X_i\}_{i=1}^{\infty}$ is a defining sequence for G in E^3 and m is given then there is an integer $\omega>m$, an allowable collection of 4-cells K, and four numbers $p_{-2}<p_{-1}<0<p_1<p_2$ so that*

$$X_m\times[p_{-2},p_2]\supset K\supset\operatorname{Int}K\supset X_\omega\times[p_{-1},p_1].$$

LEMMA 6. *If $\{X_i\}_{i=1}^{\infty}$ is a defining sequence for G in E^3 and m and N are given then there is a sequence $m<i_1<i_2<\ldots<i_N$, a sequence of allowable collections of 4-cells $\{K_i\}_1^N$ and numbers $p_{-(N+1)}<p_{-N}<\ldots<p_{-1}<0<p_1<\ldots<p_{N+1}$ so that*

$$\begin{aligned}X_m\times[p_{-(N+1)},p_{N+1}]&\supset K_1\supset\operatorname{Int}K_1\supset X_{i_1}\times[p_{-N},p_N]\\&\supset K_2\supset\operatorname{Int}K_2\supset\ldots\supset K_N\\&\supset\operatorname{Int}K_N\supset X_{i_N}\times[p_{-1},p_1].\end{aligned}$$

5. The shrinking map. We will denote by X^* the image of a set X under s^*. Thus, if K is an allowable collection of 4-cells, then K^* is embedded in the nicest possible way; i.e., each member of K^* is a product, $r_a(C_a\times\eta)\times[a,b]$ for some a, η, a, and b.

If we had the ability to insure that all of the cells given to us by Lemma 6 were "long and skinny" we would be finished with the proof of the theorem, for then Bing's technique used on the Dogbone coupled with the existence of small chambers (given to us by the uniform Lipschitz condition) would show us how to perform the shrinking. However, since we don't have this ability we will have to resort to Bryant's method of shrinking. The thing which makes this method work is the following lemma.

LEMMA 7. *If $\{K_i\}_i^m$ is a sequence of nested allowable collections of 4-cells we may find another sequence $\{K_i'\}_1^m$ such that $K_m'=K_m$ and for each cell $T_j^i\in K_i'$, there is precisely one cell $T_l^{i+1}\in K_{i+1}'$ contained in it $(i=1,\ldots,m-1)$.*

Proof. We simply define K_i' by defining $K_i'^*$. Each cell in $K_i'^*$ is split, in the obvious way, into a finite number of cells, one for each member of K_{i+1}' contained in it.

We will now show how we can determine the length of the sequence of allowable collections of cells and indicate how we cut up the cells in this collection into chambers so that we can define the shrinking map.

Let U be a neighborhood of the non-degenerate elements and let $\varepsilon>0$ be given. Lift $E^3\times 0$ by s^*. Cover the lifted non-degenerate elements by a collection of 4-cells, each within $s^*(U)$ and each of the proper form. i.e., $r_a(C_a\times\eta)\times[a,b]$ for some a, η, a, b.

By uniform continuity we know that given any $\delta > 0$ there is a $\delta' > 0$ so that if $x, y \in r_a(C_a \times \eta)$ and $|x-y| < \delta'$ then there is an x' and a y' in C_a with $|x'-y'| < \delta$ and $x = r_a(x' \times \eta)$, $y = r_a(y' \times \eta)$. We can pick $\delta > 0$ so small that for any η' and any $x', y' \in C_a$ with $|x'-y'| < \delta$ we have $|r_a(x' \times \eta') - r_a(y' \times \eta')| < \varepsilon$.

Thus we see that if we subdivide $r_a(C_a \times \eta)$ into chambers of diameter less than δ', then at each later stage of the retraction of C_a, these chambers will have diameter less than ε.

Subdivide $r_a(C_a \times \eta)$ as though it were a cube being cut by planes parallel to the sides, an equal number of planes in each direction, and so that the resulting chambers are of diameter less than δ'. The number of planes required determines the length of the finite sequence of collections of 4-cells that we need to do our shrinking. This length we call N then use Lemma 6 to obtain the sequence. Lemma 7 allows us to make this sequence nice, in that the members nest properly. We will now define the final chambers we need. Let T_j^1 be a member of K_1. $T_j^1 = r_a(C_a \times \eta) \times \times [a, b]$ for some a, η, a, b. In T_j' there is one cell T_j^N of K_n,

$$T_j^N = r_a(C_a \times \eta') \times [a', b'] .$$

We define the chambers in $r_a(C_a \times \eta')$ by simply retracting those chambers we have in $r_a(C_a \times \eta)$. Some of these chambers intersect $\mathrm{Bd}\big(r_a(C_a \times \eta')\big)$ in a 2-cell and each of the chambers of this type will be enlarged by adding to it the space swept out by this 2-cell in reversing the retraction from η' to η.

The final chambers we end with in $r_a(C_a \times \eta)$ can be expanded in the ω-direction so we obtain chambers for the 4-cells.

The rest of the proof of this theorem is exactly like Bryant's theorem applied finitely often with one additional comment. We do have exactly his conditions save one. Our cells are broken into chambers just like his except for size. Recall that the size of the image of the non-degenerate elements is determined by the number of chambers it ends up in. We see that in our case all of the chambers are small except those which we caused to grow outside the central cell. The reason Bryant's proof works in our case is that at no time does any move cause points of the central cell to leave that cell.

References

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THE UNIVERSITY OF TENNESSEE

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