Closed mappings on complete metric spaces

by

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In this note we prove the following [1]

THEOREM 1. *For every mapping $f\colon X \to Y$ of a space X metrizable in a complete manner into a T_2-space Y satisfying the first axiom of countability, the set of all points y at which f is closed is a G -set in Y.*

As a consequence of the above theorem we obtain two results of Vaĭnšteĭn's (announced in [5] and proved in [6]):

THEOREM 2. *For every mapping $f\colon X \to Y$ of a space X metrizable in a complete manner into a T_2-space Y satisfying the first axiom of countability and any set $A \subset X$ such that $f|A\colon A \to f(A)$ is closed, there exists a G_δ-set $B \subset X$ such that $A \subset B$ and $f|B\colon B \to f(B)$ is closed.*

THEOREM 3. *If there exists a closed mapping $f\colon X \to Y$ of a space X metrizable in a complete manner onto a metrizable space Y, then Y is metrizable in a complete manner*

Let us notice that the proof of Theorem 2 given in [6] was somewhat involved and that our proof of this result is shorter and simpler (although some ideas of the original proof are applied in it).

For any mapping $f\colon X \to Y$ we denote by $C(f)$ the subset of Y consisting of all points y at which f is closed, i.e. of such points $y \in Y$ that for every open $W \subset X$ containing $f^{-1}(y)$ there exists a neighbourhood $V \subset Y$ of y satisfying $f^{-1}(V) \subset W$. It is well-known (see for example [3], p. 117), that f is closed if and only if $C(f) = Y$.

Let $f\colon X \to Y$ be a mapping of a metrizable space X into a T_2-space Y satisfying the first axiom of countability and let ϱ be a metric in the space X. We shall prove three lemmas about f; Lemma 1, as well as Lemma 4 below, were proved in [6], we give here the proofs for the sake of completeness.

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(1) We adopt the terminology of [1], in particular *mapping* means a continuous function.

LEMMA 1. *For every $y \in C(f)$ the set $F = \operatorname{Fr} f^{-1}(y)$ is compact.*

Proof. Let $\{V_n\}_{n=1}^{\infty}$ be a base at y. Take an $A = \{x_1, x_2, \ldots\} \subset F$ such that $\overline{\overline{A}} = \aleph_0$, for every n pick a point $x_n' \in f^{-1}(V_n) \setminus f^{-1}(y)$ such that $\varrho(x_n, x_n') < \frac{1}{n}$ and denote by A' the set $\{x_1', x_2', \ldots\}$. As $y \in C(f)$, the set $W = X \setminus A' \supset f^{-1}(y)$ is not open, and $A'^{d} \neq 0$. It is easily seen that $0 \neq A'^{d} \subset A^{d}$, hence the set F is compact.

Let $W_i(f)$ denote for $i = 1, 2, \ldots$ the subset of Y consisting of all points $y \in Y$ which have such a neighbourhood $V \subset Y$ that every set $K \subset f^{-1}(V)$ satisfying the conditions

$$(1) \qquad \varrho(x, x') \geqslant \frac{1}{i} \quad \text{and} \quad f(x) \neq f(x') \quad \text{for } x, x' \in K,\ x \neq x'$$

is finite. Obviously, the sets $W_i(f)$ are open.

LEMMA 2. *$C(f) \subset W_i(f)$ for $i = 1, 2, \ldots$*

Proof. Suppose that there exists a point $y \in C(f) \setminus W_i(f)$ and let $\{V_n\}_{n=1}^{\infty}$ be a base at y. Choose for $n = 1, 2, \ldots$ an infinite subset $K = K_n$ of $f^{-1}(V_n)$ satisfying conditions (1). From Lemma 1 it follows that there exists a finite set of points $a_1, a_2, \ldots, a_k \in F = \operatorname{Fr} f^{-1}(y)$ such that $F \subset \bigcup_{j=1}^{k} B\left(a_j, \frac{1}{4i}\right)$.(²) It is easily seen that $B\left(F, \frac{1}{4i}\right) \subset \bigcup_{j=1}^{k} B\left(a_j, \frac{1}{2i}\right)$. Since by (1) every term of the last union contains at most one point from K_n for $n = 1, 2, \ldots$, it follows that for $n = 1, 2, \ldots$ there exists a point $x_n \in K_n \subset f^{-1}(V_n)$ such that

$$(2) \qquad f(x_n) \neq y \quad \text{and} \quad x_n \notin B\left(F, \frac{1}{4i}\right).$$

The first part of (2) implies that the set $A = \overline{\{x_1, x_2, \ldots\}}$ is disjoint to $\operatorname{Int} f^{-1}(y)$, and the second part implies that A is disjoint to $F = \operatorname{Fr} f^{-1}(y)$. Then $W = X \setminus A$ is open and contains $f^{-1}(y)$. As $y \in C(f)$, we have $f^{-1}(V_n) \subset W$ for some n. But this is impossible, because $x_n \in A \cap f^{-1}(V_n)$. The contradiction proves that $C(f) \subset W_i(f)$.

LEMMA 3. *If the metric ϱ is complete, then $\bigcap_{i=1}^{\infty} W_i(f) \subset C(f)$.*

Proof. Let y be an arbitrary point of $\bigcap_{i=1}^{\infty} W_i(f)$ and let $\{V_i\}_{i=1}^{\infty}$ be a base at y such that $V_{i+1} \subset V_i$ for $i = 1, 2, \ldots$ Without loss of generality we can suppose that

$$(3) \qquad \text{every set } K \subset f^{-1}(V_i) \text{ satisfying (1) is finite.}$$

Suppose that $y \notin C(f)$; there exists then an open set $W \subset X$ containing $f^{-1}(y)$ such that for $i = 1, 2, \ldots$ one can find points $y_i \in V_i$ and $x_i \in X \setminus W$, where $x_i \in f^{-1}(y_i)$. As $y \neq y_i$ for $i = 1, 2, \ldots$ and $\{V_i\}_{i=1}^{\infty}$ is a base at y, there exists an infinite sequence $i_1, i_2, \ldots$ such that $f(x_{i_j}) \neq f(x_{i_k})$ if $j \neq k$. For every k almost all elements of the set $A = \{x_{i_1}, x_{i_2}, \ldots\}$ belong to the set $f^{-1}(V_{i_k})$ and (3) implies that A contains a finite subset A_k such that every point in A is within a distance less than $\frac{1}{i_k} \leqslant \frac{1}{k}$ from a point of A_k. It follows that A is totally bounded, and that $\bar{A}$ is compact (see [1], Theorem 14, p. 191). Then $f(\bar{A})$ is also compact and closed in Y, as $f(A) \cap V_i \neq 0$ for $i = 1, 2, \ldots$, we have $y \in f(\bar{A})$. But $A \subset X \setminus W$, hence $\bar{A} \subset X \setminus W$ and $\bar{A} \cap f^{-1}(y) = 0$. This contradiction proves that $y \in C(f)$, i.e. that $\bigcap_{i=1}^{\infty} W_i(f) \subset C(f)$.

Theorem 1 is an immediate consequence of Lemmas 2 and 3.

LEMMA 4. *For every mapping $f\colon X \to Y$ of a normal space X into a T_1-space Y and a dense subset A of X we have $C(f|A) \subset C(f)$, where $f|A\colon A \to f(A)$ is the restriction of f.*

Proof. Let $y \in C(f|A)$ and W be an open subset of X such that $f^{-1}(y) \subset W$. By virtue of normality of X, there exists an open set $W_0 \subset X$ such that $f^{-1}(y) \subset W_0 \subset \overline{W}_0 \subset W$. For a neighbourhood $V \subset Y$ of y we have

$$(f|A)^{-1}\big(V \cap f(A)\big) = f^{-1}(V) \cap A \subset W_0 \cap A.$$

Since $f^{-1}(V)$ is open in X and A is dense

$$f^{-1}(V) \subset \overline{f^{-1}(V)} = \overline{f^{-1}(V) \cap A} \subset \overline{W_0 \cap A} = \overline{W}_0 \subset W,$$

i.e. $y \in C(f)$.

The definition of the set $C(f)$ immediately implies the following two lemmas.

LEMMA 5. *For every mapping $f\colon X \to Y$, any set $C \subset Y$ and the restriction $f_C\colon f^{-1}(C) \to C$ of f we have $C \cap C(f) \subset C(f_C)$.*

LEMMA 6. *For every mapping $f\colon X \to Y$ we have $C(f) \cap \big(\overline{f(X)} \setminus f(X)\big) = 0$.*

Proof of Theorem 2. Without loss of generality we can suppose that $\bar{A} = X$. From Lemma 4 it follows that $f(A) = C(f|A) \subset C(f)$. By virtue of Theorem 1, $C(f)$ is a G_δ-set in Y. Hence $B = f^{-1}\big(C(f)\big)$ is

(²) $B(a, r) = \{x \in X\colon \varrho(a, x) < r\}$ and $B(A, r) = \bigcup_{a \in A} B(a, r)$ for $A \subset X$.

a G_δ-set in X, moreover $A \subset f^{-1}f(A) \subset B$. Putting $C = C(f)$ in Lemma 5 we see that $f(B) \subset C(f) \subset C(f_{C(f)})$, then $f_{C(f)}: f^{-1}(C(f)) \to C(f)$, and also $f|B: B \to f(B)$ are closed.

Proof of Theorem 3. We can suppose (see [1], Theorem 10, p. 190), that Y is a dense subspace of a complete metric space Z; let $g: X \to Z$ denote the composition of f and of the embedding of Y into Z. By Theorem 1, $C(g)$ is a G_δ-set in Z. From Lemma 6 it follows that $C(g) \subset Y$, and from Lemma 4 that $Y = C(f) = C(g|X) \subset C(g)$. Hence $Y = C(g)$ and Y is metrizable in a complete manner (see [1], Theorem 9, p. 189).

Let us finish with a few remarks.

1. For the embedding $f: Q \to R$ of the set of rationals Q into the reals R we have $C(f) = Q$, hence in Theorem 1 the assumption of completness of X is essential.

2. In Theorem 2 one can replace "closed" by "perfect". In fact, if $f|A$ is perfect, then $f|C: C \to f(C)$ is perfect for $C = \bar{A} \cap B$.

3. The assumption of metrizability of Y in Theorem 3 is equivalent to the assumption that Y satisfies the first axiom of countability (see [1], Problem U, p. 204). Non-metrizable closed images of complete metric spaces are not necessarily complete in the sense of Čech: the quotient space Y obtained by identification of the x-axis in the plane $X = R^2$ to a point does not satisfy the first axiom of countability (cf. [1], Example 2, p. 115) but has a countable grid, and hence (see [1], Problem T, p. 166) is not complete in the sense of Čech.

4. For open mappings an extension theorem analogous to Theorem 2 was proved by Mazurkiewicz in [4] for separable metric spaces X and Y (a simpler proof was given by Hausdorff in [2]). Whether this theorem holds without the assumption of separability is still an unsolved problem.

5. We say that a mapping $f: X \to Y$ is *open at a point* $y \in Y$ if for every $A \subset X$, the condition $y \in f(\mathrm{Int} A)$ implies that $y \in \mathrm{Int} f(A)$. One can easily show, that the set $O(f) \subset Y$ of all points y at which f is open is a G_δ-set if X is compact and a CA set (analytic complement) if X is separable and complete (the last evaluation cannot be improved and neither separability nor completness can be ommited). The Mazurkiewicz theorem (even in the case of compact X) cannot be proved by the method applied here, because no counterpart of Lemma 4 is valid for open mappings.

References

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