Regulated semilattices and locally compact spaces

by

S. Leader and **L. Finkelstein** (New Brunswick, N. J.)

1. Introduction. In 1937 M. H. Stone [6] showed that every Boolean algebra lives in some totally disconnected, compact, Hausdorff space as a topological base of compact open subsets. In 1962 H. De Vries [2] extended Stone's theory to include all compact, Hausdorff spaces by equipping the Boolean algebra with a suitably axiomatized "compingency" relation $a \ll b$. The Boolean algebra is represented as a base of regular, open sets [3] with $a \ll b$ corresponding to $\emptyset \neq \bar{A} \subseteq B$. Stone's theory is the special case of De Vries' theory with $a \ll b$ taken to be $0 \neq a \leqslant b$.

We present here a generalization of De Vries' theory with the Boolean algebra replaced by a semilattice [7]. The latter involves only the meet. Join and complementation play no explicit role in our theory. Moreover, since our semilattice need not have an identity our theory applies to locally compact spaces. Our "regulator" differs from the "compingency" of De Vries in that we find it convenient to make $a \ll b$ correspond to $\bar{A} \subseteq B$ with the special case $A = \emptyset$ included.

A classical example of our theory is the construction of the real line from the semilattice of bounded, open intervals of rationals. In the other direction our theory is a special interesting case of a general theory presented in [4].

At the end of our paper we consider the case in which our semilattice is a lattice.

2. Regular semilattices. Recall [7] that a semilattice $(S, \cdot)$ is a set S with a binary operation $\cdot$ (denoted hereafter by juxtaposition) which is associative, commutative, and idempotent. We define the partial ordering $a \leqslant b$ in S to be $a = ab$. We shall call a semilattice $(S, \cdot)$ *regular* if the following two conditions hold:

(S_1) *There exists* 0 *in* S *such that* $0a = 0$ *for all* a *in* S.

(S_2) *If* $xa = 0$ *for all* x *in* S *such that* $xb = 0$, *then* $a = ab$.

(*) Supported by the National Science Foundation Undergraduate Research Participation Program GY-2545, GY-6162 while the latter author was a Henry Rutgers Scholar. The first author was also supported by the Research Council of Rutgers University.

In terms of the ordering, ab is the greatest lower bound of a, b. 0 is a lower bound of S. (S_2) is equivalent to

(S_2') *Given* $p \not\leqslant q$ *there exists* $c \neq 0$ *such that* $cq = 0$ *and* $c \leqslant p$.

Let A and B be subsets of S. We say B *S-covers* A if every member of S which annihilates B annihilates A. Thus (S_2) gives the equivalence of $a \leqslant b$ and b S-covers a. We say A *clings to* B if $ab \neq 0$ for all a in A and all b in B. An *ultrafilter* in a regular semilattice S is a subset F of S which is maximal with respect to the property

(F_0) $a_1, \ldots, a_n \in F$ *implies* $a_1 a_2 \ldots a_n \neq 0$.

Equivalently, an ultrafilter is a subset F of S such that

(F_1) $a, b \in F$ *implies* $ab \in F$

and

(F_2) $c \in F$ *if and only if* c *clings to* F.

3. Regulated semilattices. A *regulated semilattice* $(S, \cdot, \ll)$ consists of a regular semilattice $(S, \cdot)$ with a binary relation (the "regulator" $\ll$) on S subject to the following axioms:

(A_1) $a \ll b$ *implies* $a \leqslant b$.

(A_2) $0 \ll 0$.

(A_3) *If* $a \ll b$ *and* $c \ll d$ *then* $ac \ll bd$.

(A_4) *If* $a \ll b$ *and* $b \leqslant c$ *then* $a \ll c$.

(A_5) *Given* $p \ll q$ *with* $q \neq 0$ *there exist finite sequences* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *in* S *such that* $\{a_1, \ldots, a_n\}$ *S-covers* p, $b_i \neq 0$ *for some* i, *and* $a_i \ll b_i \ll q$ *for all* i.

(A_6) *Given* a, b, c *with* $a \ll b$ *there exist finite sequences* $d_1, \ldots, d_n$ *and* $e_1, \ldots, e_n$ *in* S *such that* $\{b, d_1, \ldots, d_n\}$ *S-covers* c, $ae_i = 0$ *for all* i, *and* $d_i \ll e_i$ *for all* i.

(A_7) *If* $\{a_1, \ldots, a_n\}$ *S-covers* p *and* $a_i \ll q$ *for all* i *then* $p \ll q$.

For A, B subsets of a topological space we define $A \Subset B$ to mean that the closure $\bar{A}$ of A is contained in the interior B° of B.

THEOREM 1. *Let* X *be a locally compact, Hausdorff space with a topological base* $\mathcal{S}$ *such that*

(i) *Every member of* $\mathcal{S}$ *is the interior of a compact subset of* X.

(ii) $A, B \in \mathcal{S}$ *implies* $A \cap B \in \mathcal{S}$.

(iii) $\emptyset \in \mathcal{S}$.

Then $(\mathcal{S}, \cap, \Subset)$ *is a regulated semilattice.*

Proof. (ii) makes $(\mathcal{S}, \cap)$ a semilattice. (iii) gives (S_1). (S_2) holds because $\mathcal{S}$ is a base whose members are regular, open sets by (i) ([3]). Now (A_1)–(A_4) clearly hold for $\Subset$ in any topological space. So does (A_7) once the reader has verified that $\{C_1, \ldots, C_n\}$ $\mathcal{S}$-covers C if and only if $C \subseteq \bar{C}_1 \cup \ldots \cup \bar{C}_n$. Using this equivalence the reader can readily derive (A_5) and (A_6) from (i) since locally compact, Hausdorff spaces are regular.

4. The representation theorem. Our main result is that every regulated semilattice can be represented in the form given by Theorem 1. The proof will be given by a series of lemmas.

THEOREM 2. *Given a regulated semilattice* $(S, \cdot, \ll)$ *there exists a unique, locally compact, Hausdorff space* X *with a base* $\mathcal{S}$ *of interiors of compact sets such that* $(\mathcal{S}, \cap, \Subset)$ *is isomorphic to* $(S, \cdot, \ll)$. X *is compact if and only if the semilattice* $(S, \cdot)$ *has a finite subset whose annihilator is trivial.*

LEMMA 1. *Let* F *be a subset of* S *satisfying* (F_1). *Let* r *cling to* F *and* $\{a_1, \ldots, a_n\}$ *S-cover* r. *Then some* a_i *clings to* F.

Proof. Suppose no a_i clings to F. Then we could choose b_i in F with $a_i b_i = 0$ for $i = 1, \ldots, n$. Let $b = b_1 \ldots b_n$. Then $b \in F$ by (F_1). Also $ba_i = 0$ for all i. Therefore, since $\{a_1, \ldots, a_n\}$ S-covers r, $br = 0$. But this contradicts the hypothesis that r clings to F.

Define an *end* E to be a nonempty set of nonzero elements of S such that

(E_1) $a, b \in E$ *implies* $ab \in E$.

(E_2) *If* $b \in E$ *then* $a \ll b$ *for some* a *in* E.

(E_3) *If* $p \ll q$ *and* p *clings to* E *then* $q \in E$.

LEMMA 2. *If* E *is an end, then* E *clings to* E.

Proof. Apply (E_1) and the condition $0 \notin E$.

LEMMA 3. *If* E *is an end,* $p \leqslant q$, *and* $p \in E$ *then* $q \in E$.

Proof. By (E_2) there exists r in E with $r \ll p$. By (A_4) $r \ll q$. So $q \in E$ by (E_3) and Lemma 2.

LEMMA 4. *Given* $q \neq 0$ *there exists an end* E *to which* q *belongs.*

Proof. Using (A_5) with $p = 0$, (A_2), and (A_4), choose $b \neq 0$ with $b \ll q$. By the Axiom of Choice b belongs to some ultrafilter F in S. We define E by

(4.1) $p \in E$ if and only if there exists a member r of F with $r \ll p$.

By setting $r = b$ in (4.1) we see that $q \in E$. Moreover, since $0 \notin F$, (A_1) and (4.1) imply $0 \notin E$. Hence, to show that E is an end we need only verify (E_1), (E_2), (E_3).

If $p_i \in E$ for $i = 1, 2$ then (4.1) yields r_i in F with $r_i \ll p_i$. By (A_3) $r_1 r_2 \ll p_1 p_2$. Also $r_1 r_2 \in F$ by (F_1). So $p_1 p_2 \in E$ by (4.1). Hence (E_1) holds.

Given p in E choose r in accordance with (4.1). By (A_5) choose an S-cover $\{a_1, \ldots, a_n\}$ of r together with $b_1, \ldots, b_n$ such that $a_i \ll b_i \ll p$. By Lemma 1 some a_i clings to F, hence belongs to F by (F_2). So by (4.1) the corresponding b_i belongs to E. So (E_2) holds.

Let $p \ll q$ and p cling to E. Apply (A_5) to get a_i and b_i. By (E_1) and Lemma 1 we may assume that a_1 clings to E. Choose any r in F. Apply (A_6) to a_1, b_1, r to get $d_i \ll e_i$ such that $\{b_1, d_1, \ldots, d_n\}$ S-covers r and $e_i a_i = 0$ for $i = 1, \ldots, n$. Then by (4.1) no d_i belongs to F since no e_i can belong to E because a_1 clings to E but $a_1 e_i = 0$. Therefore $b_1 \in F$ by Lemma 1 and (F_2). Since $b_1 \ll q$ from (A_5), $q \in E$ by (4.1). So (E_3) holds.

LEMMA 5. *The following are equivalent*:

(i) *q clings to every end to which p belongs.*

(ii) $p \leqslant q$.

Proof. Suppose (ii) false. Then choose c in accordance with (S_2'). By Lemma 4 there is some end E to which c belongs. Then $p \in E$ by Lemma 3. But q fails to cling to E since $cq = 0$. So (i) is false. Thus, (i) implies (ii). The converse follows from Lemmas 2 and 3.

Let X be the set of all ends from our regulated semilattice. Let $[q]$ be the subset of X consisting of all ends to which the element q of S belongs. That is,

(4.2) $E \in [q]$ *if and only if* $q \in E$.

LEMMA 6. $[ab] = [a] \cap [b]$.

Proof. Apply (E_1), Lemma 3 and (4.2).

LEMMA 7. $[q] = \emptyset$ *if and only if* $q = 0$.

Proof. $[0] = \emptyset$ by (4.2) since 0 belongs to no end. The converse follows from (4.2) and Lemma 4.

LEMMA 8. *The set $\mathcal{S}$ of all $[q]$ with q belonging to S is a base for a topology in X.*

Proof. Consider any member E of X. Let $\mathcal{C}$ be the set of all $[q]$ in $\mathcal{S}$ to which E belongs. We need only show $\mathcal{C}$ is nonvoid and directed downward by inclusion [5]. Since by definition every end is nonempty there exists some q belonging to E. By (4.2) $[q] \in \mathcal{C}$. Let $[a]$ and $[b]$ belong to $\mathcal{C}$. Then $[ab] \in \mathcal{C}$ by (4.2) and (E_1). So by Lemma 6 $\mathcal{C}$ is directed downward.

LEMMA 9. *In the topology given by Lemma 8 $\overline{[q]}$, the closure of $[q]$, is the set of all ends which cling to q.*

Proof. The proof follows from the following chain of equivalent conditions:

$E \in \overline{[q]}$

$E \in [x]$ implies $[x] \cap [q] \neq \emptyset$. (Lemma 8)

$E \in [x]$ implies $[xq] \neq \emptyset$. (Lemma 6)

$x \in E$ implies $xq \neq 0$. ((4.2), Lemma 7)

E clings to q. (Definition).

LEMMA 10. *$[q]$ is a regular open set*: $\overline{[q]}^{\circ} = [q]$.

Proof. The proof follows from the following chain of equivalent conditions.

$E \in \overline{[q]}^{\circ}$.

$E \in [p] \subseteq \overline{[q]}$ for some p. (Lemma 8)

There exists a member p of E such that q clings to every end to which p belongs. ((4.2), Lemma 9)

$p \leqslant q$ for some p in E. (Lemma 5)

$q \in E$. (Lemma 3)

$E \in [q]$. ((4.2)).

LEMMA 11. *If $p \ll q$ then $\overline{[p]} \subseteq [q]$.*

Proof. Apply Lemma 9, (E_3), and (4.2).

LEMMA 12. *The following are equivalent*:

(i) *$\{r_1, \ldots, r_n\}$ S-covers r.*

(ii) *If an end E clings to r then E clings to some r_i.*

(iii) $\overline{[r]} \subseteq \overline{[r_1]} \cup \ldots \cup \overline{[r_n]}$.

Proof. (i) implies (ii) by Lemma 1. Conversely if (i) is false we can choose q such that $qr \neq 0$ and $qr_i = 0$ for all i. By Lemma 4 qr belongs to some end E. So both q and r belong to E by Lemma 3. Hence E clings to r by Lemma 2. But E clings to no r_i since $qr_i = 0$. So (ii) is false. Lemma 9 gives the equivalence of (ii) and (iii).

LEMMA 13. *Given r there exist finite sequences p_j, q_j, r_j $(j = 1, \ldots, m)$ such that*

(i) *$\{r_1, \ldots, r_m\}$ S-covers r,*

(ii) *$r_j \ll q_j \ll p_j$ for all j.*

Moreover, given p with $r \ll p$ we can also fulfill

(iii) *$p_j \ll p$ for all j.*

Proof. In view of (A_2) we can ignore the trivial case $r = 0$. Given $r \neq 0$ apply (A_2) and (A_6) with $a = b = 0$ and $c = r$ to get $d_i \ll e_i$ for $i = 1, \ldots, n$ so that the d_i's S-cover r. If we are also given $r \ll p$ use (A_5) instead of (A_6) to gain the additional condition $e_i \ll p$. In either case for each $d_i \neq 0$ apply (A_5) to $d_i \ll e_i$ to get r_j and q_j for $m_{i-1} < j \leqslant m_i$ with $m_0 = 0$ such that for these j's the r_j's S-cover d_i and $r_j \ll q_j \ll e_i$. For these j's take $p_j = e_i$. Then we clearly have (i), (ii), and (iii) with $m = m_n$.

LEMMA 14. *$\overline{[c]}$ is compact for every c in S.*

Proof. Consider any ultrafilter $\mathcal{F}$ in the power set of X such that $\overline{[c]} \in \mathcal{F}$. We must show that $\mathcal{F}$ has a cluster point in X. Define E as follows:

(4.3) $p \in E$ if there exists $r \ll p$ with $[r]$ in $\mathcal{F}$.

We contend that E is an end.

Given p in E then r as given by (4.3) cannot be 0 by Lemma 7 since $\emptyset$ cannot belong to $\mathcal{F}$. Hence (4.3) and (A_1) imply $p \neq 0$.

To show E is nonvoid apply Lemma 13 with $r = c$. Then Lemma 12 yields some r_j with $\overline{[r_j]}$ in $\mathcal{F}$. By Lemma 11 and (ii) of Lemma 13 $[q_j] \in \mathcal{F}$. Hence (ii) of Lemma 13 and (4.3) imply $p_j \in E$.

To verify (E_1) let p and p' belong to E. Choose r and r' according to (4.3). Then $rr' \ll pp'$ by (A_3). Therefore, since $[rr'] \in \mathcal{F}$ by Lemma 6, $pp' \in E$ by (4.3).

To verify (E_2) let $p \in E$. Choose r according to (4.3). Apply Lemma 13. Then by Lemma 12 some $\overline{[r_j]}$ belongs to $\mathcal{F}$. Hence by Lemma 11 $[q_j] \in \mathcal{F}$. So by (ii) of Lemma 13 and (4.3) $p_j \in E$. Since $p_j \ll p$ by (iii) of Lemma 13 we get (E_2).

To verify (E_3) let $r \ll p$ and r cling to E. We contend $p \in E$. Apply Lemma 13. Then by Lemma 1 some r_j (say r_1) clings to E. From Lemma 13 we have $r_1 \ll q_1 \ll p_1 \ll p$. Choose p_0 in E and r_0 according to (4.3). Applying (A_6) with a, b, c replaced respectively by r_1, q_1, r_0 we obtain $d_i \ll e_i$ for $i = 1, \ldots, n$ such that $\{q_1, d_1, \ldots, d_n\}$ S-covers r_0 and $r_1 e_i = 0$ for all i. Since r_1 clings to E no e_i belongs to E. Hence no $\overline{[d_i]}$ belongs to $\mathcal{F}$ since by (4.3) and Lemmas 11, 12, and 13 we conclude: $\overline{[d]} \in \mathcal{F}$ and $d \ll e$ imply $e \in E$. So by Lemma 12 $\overline{[q_1]} \in \mathcal{F}$. Hence by Lemma 11 $[p_1] \in \mathcal{F}$. So $p \in E$ by (4.3).

So E is an end. Clearly $[a] \in \mathcal{F}$ whenever $a \in E$ by (4.3) and Lemma 11. That is, every basic neighborhood $[a]$ of the point E in X belongs to $\mathcal{F}$. So E is a cluster point of $\mathcal{F}$.

Lemma 15. *If $\overline{[p]} \subseteq [q]$ then $p \ll q$.*

Proof. For each end E to which q belongs we can use (E_2) to choose a in E with $a \ll q$. In terms of (4.2) each point E in $[q]$ belongs to some $[a]$ with $a \ll q$. Thus, using our hypothesis and Lemma 14, we can find a finite covering $\{[a_1], \ldots, [a_n]\}$ of $\overline{[p]}$ with $a_i \ll q$ for all i. By Lemma 12 $\{a_1, \ldots, a_n\}$ S-covers p. Hence $p \ll q$ by (A_7).

Lemma 16. *X is locally compact.*

Proof. Given any basic neighborhood $[b]$ of a point E of X we have $b \in E$ by (4.2). By (E_2) we get a in E with $a \ll b$. By (4.2) and Lemma 11 we have $E \in [a]$ and $\overline{[a]} \subseteq [b]$. Apply Lemma 14.

Lemma 17. *X is compact if and only if S has a finite subset whose annihilator is trivial.*

Proof. If X is compact it is covered by finitely many members of $\mathcal{S}$. The only basic set disjoint from every member of the covering is $\emptyset$. Conversely, let $\{r_1, \ldots, r_n\}$ in S be annihilated only by 0. So $\{r_1, \ldots, r_n\}$ S-covers every member r of S. By Lemma 12 every basic set $[r]$ is covered by $\overline{[r_1]} \cup \ldots \cup \overline{[r_n]}$ which therefore must be X. Hence X is compact by Lemma 14.

Lemma 18 (Uniqueness). *Let X, $\mathcal{S}$ and X', $\mathcal{S}'$ satisfy the conditions of Theorem 1. Let $(\mathcal{S}, \cap, \Subset)$ be isomorphic to $(\mathcal{S}', \cap, \Subset)$. Then there exists a homeomorphism between X and X' which induces the isomorphism between $\mathcal{S}$ and $\mathcal{S}'$.*

Proof. Given the isomorphism σ on $\mathcal{S}$ to $\mathcal{S}'$ let E_x be the end in $\mathcal{S}$ consisting of all members of $\mathcal{S}$ to which the point x belongs. Then σ maps E_x into an end σE_x in $\mathcal{S}'$. The members of σE_x intersect in a unique point y in X'. Define $f(x) = y$. Then $f(A) = \sigma(A)$ for every member A of $\mathcal{S}$. So f is a homeomorphism between X and X'.

5. Regular semilattices and totally disconnected spaces. Given a regular semilattice $(S, \cdot)$ one can attempt to introduce a regulator by defining

(5.1) $a \ll b$ to be $a \leqslant b$ $(ab = a)$.

Under (5.1) the conditions (A_1)–(A_5) and (A_7) are easily verified. However (A_6) need not hold. Under (5.1) we can reformulate (A_6) as follows:

(S_3) *Given a, c there exist $d_1, \ldots, d_n$ such that $\{a, d_1, \ldots, d_n\}$ S-covers c and $ad_i = 0$ for all i.*

As an example of a regular semilattice which violates (S_3) consider the set of all convex, open subsets of the plane under intersection. Then (S_3) is false for any pair a, c of distinct, overlapping discs.

However, even without (S_3) the bulk of the proof of Theorem 2 remains valid for arbitrary regular semilattices under (5.1). (A_6) is first used in the proof of Lemma 4. But under (5.1) ends are just ultrafilters. So Lemma 4 follows directly from the Axiom of Choice. The proofs of Lemmas 5 through 12 remain valid since (A_6) is not used at all there. Since under (5.1) ends are just ultrafilters (F_2) implies $[p]$ is closed. So Lemma 15 follows from Lemma 5. We thereby obtain the following theorem.

Theorem 3. *Given a regular semilattice $(S, \cdot)$ there exists a unique, totally disconnected, Hausdorff space X with a base $\mathcal{S}$ of open-closed sets such that $(\mathcal{S}, \cap)$ is isomorphic to $(S, \cdot)$ and every subset $\mathcal{F}$ of $\mathcal{S}$ with the finite intersection property (F_0) has nonvoid intersection in X. Every member of $\mathcal{S}$ is compact if and only if $(S, \cdot)$ satisfies (S_3). X is compact if and only if (S_3) holds and S has a finite subset whose annihilator is trivial.*

6. Regulated lattices. We introduce here a structure which lies between our regulated semilattices and the compingent Boolean algebras of De Vries [2].

We call $(S, \wedge, \vee, \ll)$ a *regulated lattice* if $(S, \wedge, \vee)$ is a distributive lattice, $(S, \wedge)$ is a regular semilattice, and $\ll$ is a binary relation on S subject to the following axioms:

(B_1) *$a \ll b$ implies $a \leqslant b$.*

(B_2) *$0 \ll a$ for all a.*

(B_3) *If $a \ll b$ and $c \ll d$ then $a \wedge c \ll b \wedge d$.*

(B_4) *If $a \ll b$ and $c \ll d$ then $a \vee c \ll b \vee d$.*

(B_5) *Given $p \ll q$ with $q \neq 0$ there exists $r \neq 0$ such that $p \ll r \ll q$.*

(B_6) *Given a, b, c with $a \ll b$ there exist d, e such that $d \ll e$, $d \leqslant c \leqslant b \vee d$, and $a \wedge e = 0$.*

PROPOSITION I. *Let $(S, \wedge, \vee)$ be a distributive lattice for which $(S, \wedge)$ is a regular semilattice. Then the following are equivalent:*

(i) *$\{r_1, \ldots, r_n\}$ S-covers r.*

(ii) *$r \leqslant r_1 \vee \ldots \vee r_n$.*

Proof. (ii) implies (i) by the distributive law. Conversely, if (ii) is false then (S_2') yields c with $cr_i = 0$ for all i and $cr = c \neq 0$ thereby contradicting (i).

PROPOSITION II. *The following are equivalent:*

(i) *$(S, \wedge, \vee, \ll)$ is a regulated lattice.*

(ii) *$(S, \wedge, \vee)$ is a distributive lattice and $(S, \wedge, \ll)$ is a regulated semilattice.*

Proof. Given (i) we need only verify (A_1)–(A_7) to prove (ii). (A_1), (A_3) are precisely (B_1), (B_3). (A_2) follows from (B_2). (A_4) follows from (B_2), (B_4). (B_5) gives (A_5) with $n = 1$, $a_1 = p$, and $b_1 = r$. (A_6) with $n = 1$ follows from (B_6) and Proposition I.

To prove (A_7) note that under Proposition I and (B_4) the hypothesis in (A_7) yields $p \leqslant a \ll q$ with $a = a_1 \vee \ldots \vee a_n$. To get the conclusion $p \ll q$ apply (A_2) and (B_6) with $a = b = 0$ and $c = p$ to get e such that $p \ll e$. Then by (B_3) $p = a \wedge p \ll q \wedge e \leqslant q$. Hence $p \ll q$ by (A_4). So (A_7) holds. Therefore (i) implies (ii).

Conversely, let (ii) hold. We must verify (B_1)–(B_6). (B_1), (B_3) are again just (A_1), (A_3). (B_2) follows from (A_2), (A_4). By (A_4) the hypothesis in (B_4) implies $a \ll b \vee d$ and $c \ll b \vee d$. Hence $a \vee c \ll b \vee d$ by (A_7) since under the distributive law $\{a, c\}$ S-covers $a \vee c$. So we have (B_4). (B_5) follows from (A_5) with $r = b_1 \vee \ldots \vee b_n$ under (A_4), (B_4), and (A_7). Finally, (B_6) follows from (A_6) with $d = c \wedge (d_1 \vee \ldots \vee d_n)$ and $e = e_1 \vee \ldots \vee e_n$ under Proposition I, (B_4), (A_7), and the distributive law.

PROPOSITION III. *Let $\mathcal{S}$ be any base of interiors of compact subsets of a locally compact, Hausdorff space X such that $\mathcal{S}$ is a distributive lattice under the partial ordering of set inclusion. Then $A \wedge B = A \cap B$ and $A \vee B = A \mathbf{U} B$ where $A \mathbf{U} B = \overline{A \cup B}^{\circ}$.*

Proof. Clearly $A \wedge B \subseteq A \cap B$. If equality failed to hold then since $A \wedge B$ and $A \cap B$ are regular open sets [3] and $\mathcal{S}$ is a base there would exist a nonempty member C of $\mathcal{S}$ such that $C \subseteq A \cap B$ and $C \cap (A \wedge B) = \emptyset$. Since $A \wedge B$ is the largest member of $\mathcal{S}$ contained in both A and B, the former relation implies $C \subseteq A \wedge B$. Hence the latter relation implies $C = \emptyset$, a contradiction. So $A \wedge B = A \cap B$ which we can denote hereafter by AB.

Let $D = A \vee B$, the smallest member of $\mathcal{S}$ containing both A and B. Clearly $A \cup B \subseteq D$. Hence, since D is regular and $A \mathbf{U} B$ is the smallest regular open set containing both A and B, $A \mathbf{U} B \subseteq D$. If equality failed to hold here then $D \not\subseteq \overline{A \cup B}$ from which we will derive a contradiction. Indeed, since X is a regular space and $\mathcal{S}$ is a base, we could choose P and Q in $\mathcal{S}$ such that $\emptyset \neq \overline{P} \subseteq Q \subseteq D \sim \overline{A \cup B}$. Then $\overline{P}$ and $\overline{D} \sim Q$ would be disjoint compact sets. So we could choose $R_1, \ldots, R_n$ in the base $\mathcal{S}$ to form a covering of $\overline{D} \sim Q$ that is disjoint from P. Then, since $A \cup B \subseteq D \sim Q$, $A \cup B \subseteq R_1 \cup \ldots \cup R_n \subseteq R_1 \vee \ldots \vee R_n$. Therefore $D \subseteq R_1 \vee \ldots \vee R_n$ by the definition of D. Hence, since our lattice is distributive and $PR_i = \emptyset$ for all i, $P = PD = \emptyset$ contradicting $P \neq \emptyset$.

THEOREM 4. *Let $\mathcal{S}$ be a topological base in a locally compact, Hausdorff space X such that*

(i) *Every member of $\mathcal{S}$ is the interior of a compact subset of X.*

(ii) *$A, B \in \mathcal{S}$ implies $A \cap B \in \mathcal{S}$ and $\overline{A \cup B}^{\circ} \in \mathcal{S}$.*

(iii) *$\emptyset \in \mathcal{S}$.*

Then $(\mathcal{S}, \cap, \mathbf{U}, \Subset)$ is a regulated lattice.

Proof. $(\mathcal{S}, \cap, \mathbf{U})$ is a distributive lattice since it is a sublattice of the Boolean algebra [3] of all regular open subsets of X. (B_1)–(B_4) are trivial.

To verify (B_5) we are given $\overline{P} \subseteq Q$ with Q nonempty. Since $\overline{P}$ is compact and $\mathcal{S}$ is a base in a regular space we can find nonempty $R_1, \ldots, R_n$ in $\mathcal{S}$ to cover $\overline{P}$ with $\overline{R}_i \subseteq Q$ for all i. Let $R = R_1 \mathbf{U} \ldots \mathbf{U} R_n$, a nonempty member of $\mathcal{S}$. Then $\overline{P} \subseteq R$ and $\overline{R} \subseteq Q$ which is the conclusion of (B_5).

To verify (B_6) let $A, B, C \in \mathcal{S}$ with $\overline{A} \subseteq B$. Then the compact set $\overline{C} \sim B$ is contained in the open set $X \sim \overline{A}$. So we can choose $D_1, \ldots, D_n$ and $E_1, \ldots, E_n$ in $\mathcal{S}$ such that $\overline{C} \sim B \subseteq D_1 \cup \ldots \cup D_n$ and $\overline{D}_i \subseteq E_i \subseteq X \sim \overline{A}$ for all i. Take $D = D_1 \mathbf{U} \ldots \mathbf{U} D_n$ and $E = E_1 \mathbf{U} \ldots \mathbf{U} E_n$ to get (B_6).

THEOREM 5. *Given a regulated lattice* $(S, \wedge, \vee, \leqslant)$ *there exists a unique, locally compact, Hausdorff space* X *with a base* $\mathcal{S}$ *of interiors of compact sets such that* $(\mathcal{S}, \cap, \cup, \Subset)$ *is isomorphic to* $(S, \wedge, \vee, \leqslant)$. X *is compact if and only if* $(S, \wedge)$ *has an identity.*

Proof. Theorem 2 under Proposition II gives X and $\mathcal{S}$ with $(\mathcal{S}, \cap, \Subset)$ isomorphic to $(S, \wedge, \leqslant)$. Since S is a distributive lattice under $\leqslant$, the isomorphism implies that $\mathcal{S}$ is a distributive lattice under set inclusion. Hence $\vee$ corresponds to $\cup$ by Proposition III. Finally, the compactness criterion follows from that of Theorem 2 by Proposition I.

A study of lattices along the lines of section 5 would lead us to nothing more than Stone's theory [6]. For we have the following result whose trivial proof we omit.

PROPOSITION IV. *Let* $(S, \wedge, \vee)$ *be a lattice with partial ordering* $\leqslant$. *Then* $(S, \wedge, \vee, \leqslant)$ *is a regulated lattice if and only if* $(S, \wedge, \vee)$ *is a Boolean ring.*

References

[1] G. Birkhoff, *Lattice Theory*, New York 1948.

[2] H. De Vries, *Compact Spaces and Compactifications: An Algebraic Approach*, Assen 1962.

[3] P. R. Halmos, *Lectures on Boolean Algebras*, Princeton 1963.

[4] S. Leader, *Regulated bases and completions of regular spaces*, Fund. Math. 67 (1970), pp. 279–287.

[5] H. L. Royden, *Real Analysis*, New York 1963.

[6] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans Amer. Math. Soc. 41 (1937), pp. 375–481.

[7] G. Szász, *Introduction to Lattice Theory*, Budapest 1963.

RUTGERS UNIVERSITY
THE STATE UNIVERSITY OF NEW JERSEY

Reçu par la Rédaction le 21. 10. 1969

Cardinal multiplication of structures with a reflexive relation

by

Ralph McKenzie * (Berkeley, Calif.)

Introduction. This paper is a sequel to investigations of Chang, Jónsson and Tarski (reported in [1, 2, 3]) dealing with refinement properties for the operation of cardinal multiplication (direct product, cartesian product) of relational structures. The results presented here extend, and almost complete that theory, insofar as it applies to structures having a reflexive binary relation.

Our main result is a Lemma (3.1) whose formulation is rather technical, but roughly states that a structure has the "strict refinement property" defined in [3], provided that indistinguishable elements of the structure are identified. The lemma is proved for structures of the form $\mathfrak{A} = \langle A, S\rangle$ in which S is a binary relation over A and the relations $S|\breve{S}$ and $\breve{S}|S$ are connected over A; in particular it applies if S is reflexive and connected over A. The lemma yields for structures in this class a reduction of the ordinary refinement property to a purely set theoretic question, which is easily answered in every specific case if the general continuum hypothesis is assumed (Theorem 4.4). Independently of the GCH, it follows that every finite structure of this class has the refinement property.(¹) Thus we obtain a useful description of the algebra of all finite reflexive isomorphism types—under operations of binary cardinal addition and multiplication—which has been suspected for some time: viz. this algebra is isomorphic to a "semi-ring" of polynomials, $Z^+[x]$ (Theorem 5.1).

Departing briefly from the main line of development, we prove in § 7 an interesting and unexpected form of the Cantor Bernstein theorem: (Corollary 7.2) *Let* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}_0$ *and* $\mathfrak{C}_1$ *be similar relational structures of an arbitrary similarity type and assume that* $\mathfrak{C}_0 \times \mathfrak{A} \cong B$ *and* $\mathfrak{C}_1 \times \mathfrak{B} \cong \mathfrak{A}$. *If, in addition,* $\mathfrak{A}$ *is denumerable and* $\mathfrak{C}_0$ *is finite then* $\mathfrak{A} \cong \mathfrak{B}$.

---

* The work reported here was supported by the National Science Foundation through grants GP-7578 and GP-6232X3.

(¹) This solves the central problem studied in [2].