On the space of BV-ω functions

by

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1. Introduction. Let $\omega(t)$ be non-decreasing on $[0,1]$ and outside the interval it is defined by $\omega(t) = \omega(0)$ for $t < 0$ and $\omega(t) = \omega(1)$ for $t > 1$. Let S denote the set of points of continuity of $\omega(t)$ and D the set of its points of discontinuity. We use the following notations:

$$S_0 = \bigcup_i (\alpha_i, \beta_i), \quad S_1 = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots\},$$

$$S_2 = S \cap S_1 \quad \text{and} \quad S_3 = [0,1] \cap S - (S_0 \cup S_2),$$

where $\{(\alpha_i, \beta_i)\}$ is the set of pairwise disjoint open intervals in $[0,1]$ such that $\omega(t)$ is constant on each of them.

R. L. Jeffery [2] has defined the class $\mathfrak{U}$ of functions $x(t)$ in the following way: $x(t)$ is defined on $[0,1] \cap S$ such that $x(t)$ is continuous at each point of $[0,1] \cap S$ with respect to the set S. If $\alpha \in D$, then $x(t)$ tends to limits as t tends to $\alpha+$ and to $\alpha-$ over the points of the set S. For $t < 0$, $x(t) = x(0+)$ and for $t > 1$, $x(t) = x(1-)$. $x(t)$ may or may not be defined at the points of the set D.

A set of points $0 \leqslant t_0 < t_1 < t_2 < \ldots < t_n \leqslant 1$ with $\omega(t_i) < \omega(t_{i+1})$ $(i = 0, 1, \ldots, n-1)$ is said to be an ω-subdivision [1] of $[0,1]$. Let $x(t)$ be defined on $[0,1]$ and be in class $\mathfrak{U}$. The least upper bound of the sums

$$\sum_{i=1}^{n} |x(t_i+) - x(t_{i-1}-)|$$

for all possible ω-subdivisions $t_0, t_1, \ldots, t_n$ of $[0,1]$ with $t_i \in E \subset [0,1]$ is called the *total variation* of $x(t)$ on E relative to ω [1] and is denoted by $V_\omega(x; E)$. If $V_\omega(x; E) < +\infty$, then $x(t)$ is said to be of bounded variation relative to ω, BV-ω, on E.

In the present paper we assume that if α, β $(>\alpha)$ be any two points of D, then $\omega(\alpha) < \omega(t) < \omega(\beta)$ for all t in (α, β). We denote by X the set of all functions $x(t)$ in $\mathfrak{U}$ such that $x(t)$ is defined and continuous on D with respect to the set S and BV-ω on $[0,1]$. To each pair x, y of X we associate the real number $d(x,y)$ defined by

$$d(x,y) = \int_0^1 |x(t)-y(t)|\,d\omega + |T(x)-T(y)|$$

where the integral is taken in Lebesgue–Stieltjes sense and $T(x)$ stands for $V_\omega(x; [0,1])$. Since $d(x,y)=0$ implies that $x(t)=y(t)$ ω-almost everywhere in $[0,1]$, d is a pseudometric for X and (X,d) is a pseudometric space.

The purpose of the present paper is to study some properties of the space (x,d). Whenever we speak of completeness of (x,d), we make it a metric space by writing $x=y$ if $x(t)=y(t)$ ω-almost everywhere in $[0,1]$ and $T(x)=T(y)$.

2. Preliminary lemmas.

LEMMA 2.1. *Let $x(t) \in \mathfrak{U}$ and let $x(t)$ be* BV-ω *on $[0,1]$. If $c \in S_3 \cap (0,1)$, then*

$$V_\omega(x; [0,1]) = V_\omega(x; [0,c]) + V_\omega(x; [c,1]).$$

The lemma can be proved in the usual way.

LEMMA 2.2. *Let $x(t) \in \mathfrak{U}$ and let $x(t)$ be* BV-ω *on $[0,1]$. Then $x(t)$ can be expressed as $x(t)=\pi(t)-\nu(t)$, where $\pi(t)$ and $\nu(t)$ are non-decreasing and continuous on S_3.*

Proof. We define the function $\pi(t)$ on $[0,1]$ as follows:

$$\pi(0)=0, \quad \pi(t)=V_\omega(x; [0,t]) \quad \text{for } 0<t\leqslant 1.$$

Let t_1 and t_2 $(>t_1)$ be any two points on S_3. By lemma 2.1, we have

$$V_\omega(x; [0,t_2]) = V_\omega(x; [0,t_1]) + V_\omega(x; [t_1,t_2])$$

or

$$\pi(t_2) = \pi(t_1) + V_\omega(x; [t_1,t_2]) \geqslant \pi(t_1).$$

Therefore the function $\pi(t)$ is non-decreasing on S_3. Next, we define the function $\nu(t)$ on $[0,1]$ by

$$\nu(t) = \pi(t) - x(t).$$

Let t_1 and t_2 $(>t_1)$ be any two points of S_3. Then

$$\begin{aligned}\nu(t_2)-\nu(t_1) &= \{\pi(t_2)-x(t_2)\} - \{\pi(t_1)-x(t_1)\}\\ &= \{\pi(t_2)-\pi(t_1)\} - \{x(t_2)-x(t_1)\}\\ &= V_\omega(x; [t_1,t_2]) - \{x(t_2)-x(t_1)\} \geqslant 0.\end{aligned}$$

We now show that $\pi(t)$ is continuous on S_3. Let a be a point of $S_3 \cap (0,1)$ such that a is a limit point of S_3, on the right. Choose $\varepsilon>0$ arbitrarily. Take an ω-subdivision

$$a = t_0 < t_1 < t_2 < \ldots < t_n \leqslant 1$$

of $[a,1]$ such that

$$\sum_{i=1}^{n} |x(t_i+)-x(t_{i-1}-)| > V_\omega(x; [a,1]) - \varepsilon. \tag{1}$$

Since the sum on the left of (1) does not decrease on adding new points of ω-subdivision of $[a,1]$ belonging to the set S_3 and since such points can be taken arbitrary close to a we may suppose that $t_1 \in S_3$ and

$$|x(t_1)-x(t_0)| < \varepsilon.$$

Hence from (1),

$$\begin{aligned}V_\omega(x; [a,1]) &< 2\varepsilon + \sum_{i=2}^{n} |x(t_i+)-x(t_{i-1}-)|\\ &\leqslant 2\varepsilon + V_\omega(x; [t_1,1]),\end{aligned}$$

or

$$V_\omega(x; [a,t_1]) < 2\varepsilon,$$

or

$$\pi(t_1)-\pi(a) < 2\varepsilon.$$

Letting $t_1 \to a+$ over the points of the set S_3 and noting that $\varepsilon>0$ is arbitrary we obtain $\pi(a+)=\pi(a)$. If a is a limit point of S_3 on the left, then we can show that $\pi(a-)=\pi(a)$. So $\pi(t)$ is continuous on S_3. Since $\nu(t)=\pi(t)-x(t)$, it follows that $\nu(t)$ is also continuous on S_3. This proves the lemma.

LEMMA 2.3. *Let $x(t)$ be defined on $[0,1]$ and let $x(t) \in \mathfrak{U}$. If the set $E \subset [0,1]$ is such that* (i) *$S_2 \cup D \subset E$,* (ii) *$E \cap S_3$ is dense in S_3, and* (iii) *$E \cap S_0$ is dense in S_0, then*

$$V_\omega(x; E) = V_\omega(x; [0,1]).$$

Proof. We first suppose that $V_\omega(x; [0,1])$ is finite. Let D_ω: $(t_0, t_1, t_2, \ldots, t_n)$ be any ω-subdivision of $[0,1]$.

(i) If $t_0, t_1, t_2, \ldots, t_n$ all belong to E, then

$$V = \sum_{i=1}^{n} |x(t_i+)-x(t_{i-1}-)| \leqslant V_\omega(x; E). \tag{2}$$

(ii) Let none of $t_0, t_1, \ldots, t_n$ belong to E.

Then none of $t_0, t_1, \ldots, t_n$ belong to $S_2 \cup D$. Hence corresponding to $\varepsilon > 0$, chosen arbitrarily, we can find points

$$\xi_0 \in (t_0, t_1) \cap S \cap E,$$
$$\xi_1 \in (\xi_0, t_1) \cap S \cap E,$$
$$\xi_2 \in (t_1, t_2) \cap S \cap E,$$
$$\cdots\cdots\cdots\cdots$$
$$\xi_n \in (t_{n-1}, t_n) \cap S \cap E,$$

where $\omega(\xi_0) < \omega(\xi_1) < \omega(\xi_2) < \ldots < \omega(\xi_n)$, and such that

$$|x(t_i) - x(\xi_i)| < \varepsilon/2n.$$

The choice of the points ξ_0 and ξ_n remains the same whether t_0 and t_n do or do not coincide with the points 0 and 1 respectively.

We have, for each $i = 1, 2, 3, \ldots, n$

$$|x(t_i) - x(t_{i-1})| = |x(t_i) - x(\xi_i) + x(\xi_i) - x(\xi_{i-1}) + x(\xi_{i-1}) - x(t_{i-1})|$$
$$\leqslant |x(\xi_i) - x(\xi_{i-1})| + \varepsilon/n.$$

Then,

$$V = \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \leqslant \sum_{i=1}^{n} |x(\xi_i) - x(\xi_{i-1})| + \varepsilon.$$

Since the points $\xi_0, \xi_1, \xi_2, \ldots, \xi_n$ form an ω-subdivision of $[0,1]$ we see that

$$V \leqslant V_\omega(x; E) + \varepsilon.$$

This implies that

(3) $$V_\omega(x; [0,1]) \leqslant V_\omega(x; E).$$

(iii) If D_ω does not satisfy (i) and (ii), then some of the t_i's belong to E and some do not belong to E.

Without loss of generality we may suppose that only the point t_1 does not belong to E. Choose $\varepsilon > 0$ arbitrarily. Then we can find a point $\xi \in (t_0, t_1) \cap S \cap E$ such that

$$|x(\xi) - x(t_1)| < \varepsilon/2 \quad \text{and} \quad \omega(t_0) < \omega(\xi) < \omega(t_2).$$

We have

$$|x(t_1) - x(t_0-)| + |x(t_2+) - x(t_1)| < |x(\xi) - x(t_0-)| + |x(t_2+) - x(\xi)| + \varepsilon.$$

Since the points $t_0, \xi, t_2, \ldots, t_n$ form an ω-subdivision of $[0,1]$ we see that

(4) $$V < V_\omega(x; E) + \varepsilon.$$

From (2), (3) and (4) it follows that

$$V_\omega(x; [0,1]) \leqslant V_\omega(x; E).$$

But it is clear that

$$V_\omega(x; E) \leqslant V_\omega(x; [0,1]).$$

Hence we obtain

$$V_\omega(x; E) = V_\omega(x; [0,1]).$$

If $V_\omega(x; [0,1])$ is infinite, we can show as above that $V_\omega(x; E)$ is also infinite. This completes the proof.

DEFINITION 2.1. Let $x(t)$ be an element of X and D': $(t_0, t_1, \ldots, t_n)$ be any ω-subdivision of $[0,1]$. We denote by $B(t) = B(t; x, D')$ the function whose graph is the polygonal line joining the points $(t_i, x(t_i))$ $(i = 0, 1, 2, \ldots, n)$. We call $B(t)$ a *polygonal function associated with* $x(t)$.

LEMMA 2.4. *Let $x(t)$ be an element of X, then for every $\varepsilon > 0$ there is a polygonal function $B(t)$ in X associated with $x(t)$ such that $d(x, B) < \varepsilon$.*

Proof. Let $x(t)$ be an element of X and $\varepsilon > 0$ be chosen arbitrarily. There is an ω-subdivision D_ω: $(t_0, t_1, \ldots, t_p)$ of $[0,1]$ such that

$$\sum_{i=1}^{p} |x(t_i+) - x(t_{i-1}-)| > V_\omega(x; [0,1]) - \varepsilon/2.$$

Let E be an enumerable subset of $S_3 \cup D \cup D_\omega$ containing the set $D \cup D_\omega$ such that $E \cap S_3$ is dense in S_3. We now choose a sequence of ω-subdivisions

$$D_\omega^{(n)}: (0 \leqslant t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < \ldots < t_{r_n}^{(n)} \leqslant 1)$$

of $[0,1]$ such that

(i) $E = \bigcup_{n=1}^{\infty} D_\omega^{(n)}$,

(ii) $D_\omega \subset D_\omega^{(n)}$, $\quad (n = 1, 2, \ldots)$.

(iii) $D_\omega^{(n)} \subset D_\omega^{(n+1)}$,

Write $B_n(t) = B(t; x, D_\omega^{(n)})$. Since $D_\omega \subset D_\omega^{(n)}$ for all n we have

$$V_\omega(B_n; [0,1]) \geqslant \sum_{i=1}^{p} |x(t_i+) - x(t_{i-1}-)| > V_\omega(x; [0,1]) - \varepsilon/2.$$

But,

$$V_\omega(B_n; [0,1]) \leqslant V_\omega(x; [0,1]).$$

So,

$$|T(x) - T(B_n)| < \varepsilon/2.$$

It is clear that $B_n(t) \to x(t)$ at each point of E. We show that $B_n(t) \to x(t)$ at all points of $S_3 \cup E$. Let $\xi \in S_3 - E$. Choose $\eta > 0$ arbitrarily. By lemma 2.2 we can choose two points ξ', ξ'' in $E \cap S_3$ with $\xi' < \xi < \xi''$ such that

$$V_\omega(x; [\xi', \xi'']) < \eta/2 .$$

We can find the positive integer N such that $\xi' \in D_\omega^{(n)}$ when $n \geqslant N$. Then for all $n \geqslant N$,

$$\begin{aligned}|x(\xi) - B_n(\xi)| &\leqslant |x(\xi) - x(\xi')| + |x(\xi') - B_n(\xi')| + |B_n(\xi') - B_n(\xi)| \\ &\leqslant V_\omega(x; [\xi', \xi'']) + V_\omega(B_n; [\xi', \xi'']) \\ &\leqslant 2V_\omega(x; [\xi', \xi'']) < \eta .\end{aligned}$$

So, the sequence $\{B_n(t)\}$ converges to $x(t)$ ω-almost everywhere in $[0, 1]$. Also the sequence is uniformly bounded on $[0, 1]$ because $x(t)$ is bounded on $[0, 1]$.

Hence

$$\lim_{n\to\infty} \int_0^1 |x(t) - B_n(t)|\, d\omega = 0 .$$

So, there is a positive integer m such that

$$\int_0^1 |x(t) - B_n(t)|\, d\omega < \varepsilon/2 \quad \text{when } n \geqslant m .$$

Take any $n > m$ and write $B(t) = B_n(t)$. Then

$$d(x, B) < \varepsilon .$$

3. The space (X, d).

THEOREM 3.1. *The space* (X, d) *is separable.*

Proof. Let E be an enumerable set in $[0, 1]$ containing $S_2 \cup D \cup \{0, 1\}$ such that $E \cap S_3$ is dense in S_3 and $E \cap S_0$ is dense in S_0. We denote by X_0 the set of all polygonal functions $P(t)$ in (X, d) with corners at points (t, r) where $t \in E$ and r, rational numbers. Then clearly X_0 is an enumerable set. The proof of the theorem will be complete if we can show that X_0 is dense in (X, d).

Let $x(t)$ be an element of (X, d). Choose $\varepsilon > 0$ arbitrarily. By lemma 2.4, there is a polygonal function $B(t)$ in X associated with $x(t)$ such that $d(x, B) < \varepsilon$. Denote the abscissae of the corners of $B(t)$ by $t_0', t_1', \ldots, t_m'$. Then $t_0', t_1', \ldots, t_m'$ form an ω-subdivision of $[0, 1]$. We choose an $\eta > 0$ such that $4m\eta < \varepsilon$. We can choose a polygonal function $P(t)$ in X_0 with corners having abscissae at points $t_0, t_1, \ldots, t_m$ such that

$$|P(t) - B(t)| < \eta \quad \text{for all } t \text{ in } [0, 1] ,$$

and

$$|B(t_i) - B(t_i')| < \eta \quad \text{for } i = 0, 1, 2, \ldots, m .$$

We take the points $t_0, t_1, \ldots, t_m$ such that they also form an ω-subdivision of $[0, 1]$.

Now,

$$\int_0^1 |x(t) - P(t)|\, d\omega \leqslant \int_0^1 |x(t) - B(t)|\, d\omega + \int_0^1 |B(t) - P(t)|\, d\omega < \varepsilon + \|[0, 1]\|_\omega \cdot \varepsilon$$

and

$$|T(P) - T(B)| \leqslant |T(P) - \Sigma| + |\Sigma - T(B)| ,$$

where $\Sigma = \sum_{i=1}^{m} |B(t_i) - B(t_{i-1})|$.

$$\begin{aligned}|T(P) - \Sigma| &= \Big| \sum_{i=1}^{m} |P(t_i) - P(t_{i-1})| - \sum_{i=1}^{m} |B(t_i) - B(t_{i-1})| \Big| \\ &\leqslant \sum_{i=1}^{m} \{|P(t_i) - B(t_i)| + |P(t_{i-1}) - B(t_{i-1})|\} \\ &< 2m\eta < \varepsilon/2 .\end{aligned}$$

$$\begin{aligned}|T(B) - \Sigma| &= \Big| \sum_{i=1}^{m} |B(t_i) - B(t_{i-1})| - \sum_{i=1}^{m} |B(t_i') - B(t_{i-1}')| \Big| \\ &\leqslant \sum_{i=1}^{m} \{|B(t_i) - B(t_i')| + |B(t_{i-1}) - B(t_{i-1}')|\} \\ &< 2m\eta < \varepsilon/2 .\end{aligned}$$

So, $|T(P) - T(B)| < \varepsilon$, and

$$|T(x) - T(P)| \leqslant |T(x) - T(B)| + |T(B) - T(P)| < 2\varepsilon .$$

Therefore,

$$\begin{aligned}d(x, P) &= \int_0^1 |x(t) - P(t)|\, d\omega + |T(x) - T(P)| \\ &< \varepsilon + \varepsilon \cdot \|[0, 1]\|_\omega + 2\varepsilon = K \cdot \varepsilon .\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that X_0 is dense in (X, d).

THEOREM 3.2. *If there is an interval* $[a, b] \subset [0, 1]$ *such that* $\omega(t)$ *is strictly increasing on* $[a, b]$ *then* (X, d) *is not complete and no closed sphere in it is compact.*

Proof. Let M be any closed sphere in (X, d) with centre at a and radius r. Take a number c in $(a, b) \cap S$ and choose a positive integer n_0

with $c+(1/n_0)<b$. We define the sequence $\{a_n(t)\}$ $(n\geqslant n_0)$ on $[0,1]$ as follows:

$$a_n(t)=\begin{cases}0 & \text{for } 0\leqslant t\leqslant c,\\ Kn(t-c) & \text{for } c<t\leqslant c+(1/n),\\ K & \text{for } c+(1/n)<t\leqslant 1,\end{cases}$$

$n\geqslant n_0$ and $K>0$.

Let us write $x_n(t)=a_n(t)+a(t)$ for all t in $[0,1]$. Then $x_n(t)$ is an element of X $(n\geqslant n_0)$.

We have

$$d(x_n,a)=\int_0^1|x_n(t)-a(t)|\,d\omega+|T(x_n)-T(a)|$$

$$\leqslant\int_0^1|a_n(t)|\,d\omega+T(x_n-a)$$

$$\leqslant K\cdot[\omega(1)-\omega(0)]+K.$$

If we take $K<r/\{\omega(1)-\omega(0)+1\}$, then $d(x_n,a)<r$. So $x_n\in M$ $(n\geqslant n_0)$. If possible, let M be compact. Then there is a subsequence $\{x_{n_i}\}$ which converges to an element x of M.

We have

$$d(x_{n_i},x)=\int_0^1|x_{n_i}(t)-x(t)|\,d\omega+|T(x_{n_i})-T(x)|.$$

This implies that for any two numbers λ,μ in $[0,1]$,

$$\int_\lambda^\mu|x_{n_i}(t)-x(t)|\,d\omega\to 0\quad\text{as } i\to\infty.$$

Since

$$\int_a^c|x_{n_i}(t)-x(t)|\,d\omega=\int_a^c|a(t)-x(t)|\,d\omega,$$

letting $i\to\infty$ we have

$$\int_a^c|a(t)-x(t)|\,d\omega=0.$$

Since $a(t)$ and $x(t)$ are continuous on $[a,c]$ and $\omega(t)$ is strictly increasing on $[a,c]$ we obtain

(5) $$x(t)=a(t)\quad\text{for } a\leqslant t\leqslant c.$$

Choose $\varepsilon>0$ arbitrarily with $c+\varepsilon<b$ and take a positive integer i_0 such that $1/n_{i_0}<\varepsilon$. Then for all $i\geqslant i_0$

$$\int_{c+\varepsilon}^b|x_{n_i}(t)-x(t)|\,d\omega=\int_{c+\varepsilon}^b|K+a(t)-x(t)|\,d\omega.$$

Letting $i\to\infty$ we get

$$\int_{c+\varepsilon}^b|K+a(t)-x(t)|\,d\omega=0,$$

which gives that

$$x(t)=K+a(t)\quad\text{for } c+\varepsilon<t\leqslant b.$$

Since $\varepsilon>0$ is arbitrary we obtain

(6) $$x(t)=K+a(t)\quad\text{for } c<t\leqslant b.$$

From (5) and (6) we see that x is not an element of X. Thus we arrive at a contradiction. Hence M is not compact.

Next, we show that (X,d) is not complete. For this we consider the sequence $\{a_n\}$ defined above. For any two positive integers m and n, $(m>n\geqslant n_0)$

$$d(a_m,a_n)=\int_0^1|a_m(t)-a_n(t)|\,d\omega+|T(a_m)-T(a_n)|$$

$$=\int_c^{c+\frac{1}{n}}|a_m(t)-a_n(t)|\,d\omega$$

$$\leqslant K\big[\omega\big(c+(1/n)+\big)-\omega(c)\big]\to 0\quad\text{as } n\to\infty.$$

Hence $\{a_n\}$ is a Cauchy sequence in (X,d). Assuming that $\{a_n\}$ converges to a limit x in (X,d) we can show as above that

$$x(t)=\begin{cases}0 & \text{for } a\leqslant t\leqslant c,\\ K & \text{for } c<t\leqslant b.\end{cases}$$

This contradicts the fact that x is an element of (X,d). Hence the space (X,d) is not complete.

We now show by an example that for some $\omega(t)$, the space (X,d) is complete and every closed sphere in it is compact.

EXAMPLE 3.1. Let $\omega(t)=0$ for $0\leqslant t\leqslant \alpha$ and $\omega(t)=1$ for $\alpha<t\leqslant 1$, where $0<\alpha<1$.

We first show that the space (X,d) is complete. Let $\{x_n\}$ be any Cauchy sequence in (X,d). Since

$$d(x_m,x_n)=\int_0^1|x_m(t)-x_n(t)|\,d\omega+|T(x_m)-T(x_n)|$$

$$=|x_m(\alpha)-x_n(\alpha)|+|T(x_m)-T(x_n)|,$$

it follows that $\{x_n(\alpha)\}$ and $\{T(x_n)\}$ are Cauchy sequences.

Let

$$\lim_{n\to\infty} x_n(a) = l \quad \text{and} \quad \lim_{n\to\infty} T(x_n) = P .$$

We define the function $x(t)$ on $[0,1]$ as follows:

$$x(t) = Pt+l-Pa \quad \text{for} \quad 0 \leqslant t \leqslant 1 .$$

Then $x(t)$ is continuous on $[0,1]$ and $T(x) = P$. Clearly $x \in X$. We have

$$d(x_n, x) = \int_0^1 |x_n(t)-x(t)|\,d\omega + |T(x_n)-T(x)|$$

$$= |x_n(a)-x(a)| + |T(x_n)-P| \to 0$$

as $n\to\infty$. Hence the space (X, d) is complete.

Next, we show that every closed sphere in (X, d) is compact. Let M be any closed sphere in (X, d) with centre at x_0 and radius K. Take any sequence $\{x_n\}$ in M.

We have

$$d(x_n, x_0) = \int_0^1 |x_n(t)-x_0(t)|\,d\omega + |T(x_n)-T(x_0)|$$

$$= |x_n(a)-x_0(a)| + |T(x_n)-T(x_0)| .$$

This gives that

$$|x_n(a)| \leqslant K + |x_0(a)| \quad \text{and} \quad T(x_n) \leqslant K + T(x_0) \quad \text{for all } n .$$

Hence we can choose a subsequence $\{x_{n_i}\}$ such that $\{x_{n_i}(a)\}$ as well as $\{T(x_{n_i})\}$ converges.

Let

$$\lim_{i\to\infty} x_{n_i}(a) = l \quad \text{and} \quad \lim_{i\to\infty} T(x_{n_i}) = P .$$

We define $x(t)$ as in the previous case.

Then $x \in X$. Also $x(a) = l$ and $T(x) = P$.

We have

$$d(x_{n_i}, x) = |x_{n_i}(a)-x(a)| + |T(x_{n_i})-T(x)| \to 0$$

as $i\to\infty$. Thus $\{x_{n_i}\}$ converges to x. Again

$$d(x, x_0) \leqslant d(x, x_{n_i}) + d(x_{n_i}, x_0) \quad \text{for all } i .$$

So letting $i\to\infty$ we have $d(x, x_0) \leqslant K$ which gives that $x \in M$. Hence the sphere M is compact.

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