and X is not a dendrite, then X contains a simple closed curve, which is a retract of X (cf. [5], p. 271). Consequently, the first homology group of X in the sense of E. Čech $H_1(X, Z)$ is not trivial, which yields a contradiction with Theorem 1 of [7], because $H_1(S^2, Z) = 0$.

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On lattices whose lattices of congruences are Stone lattices

by

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M. F. Janowitz proves in [3] that the lattice of congruences on any complete relatively complemented lattice is a Stone lattice and poses the question:—Find necessary and sufficient conditions on a complete lattice L for the lattice of congruences on L to become a Stone lattice. This note gives an answer to the above question. We also show that the lattice of congruences on any complete, weakly complemented, weakly modular lattice is a Stone lattice. This is a generalization of the result of M. F. Janowitz, proved by the fact that a complete, weakly complemented, weakly modular lattice is not always relatively complemented.

We further show that, in the case of a finite lattice L, the lattice of congruences on L is a Stone lattice if and only if, given any prime interval I of L, there exists one and only one minimal element in $L_p/\sim$ less than $\{I\}$ (where L_p denotes the set of all prime intervals of L and $\sim$ is the equivalence relation defined on L_p thus: $A \sim B$ if and only if A is a lattice translate of B and B is a lattice translate of A; and $\{I\}$ denotes the class containing I with respect to the relation $\sim$).

1. Complete lattices.

THEOREM 1. *Let L be a complete lattice. The lattice of congruences on L is a Stone lattice if and only if for any congruence θ on L there exist a finite number of elements $0 = b_1 < b_2 < \ldots < b_n = 1$ such that either (b_{i-1}, b_i) has no non-trivial lattice translate annulled by θ or every lattice translate of (b_{i-1}, b_i) has a non-trivial lattice translate annulled by θ.*

Proof. Follows from theorems 1 and 3 of [2].

COROLLARY. *Let L be a complete weakly modular lattice. The lattice of congruences on L is a Stone lattice if and only if for any congruence θ on L there exists a finite chain $0 = b_1 < b_2 < \ldots < b_n = 1$ such that either (b_{i-1}, b_i) consists of single point congruence classes under θ or every subinterval of (b_{i-1}, b_i) has a proper part annuled by θ.*

As a special case of theorem 1 we get,

THEOREM 2. *Let L be a complete, weakly complemented, weakly modular lattice. Then the lattice of congruences on L is a Stone lattice.*

Proof. Let θ be any congruence on L. Let I be the kernel of the congruence θ on L. Let $s = \bigvee_{x \in I} x$ (s exists as L is complete). I is a standard ideal of L since L is weakly complemented (cf. p. 56 of [1]). Therefore every element of $\alpha(s)$ (the principal α-ideal corresponding to the element s) is a single-point congruence class under θ. Now, since L is weakly modular, $x \equiv y(\theta')$ if and only if $(xy, x+y)$ consists of single-point congruence classes under θ. Thus θ' annuls $\alpha(s)$ (cf. corollary 1 p. 229 of [2]).

CLAIM. *$\alpha(s)$ is a congruence class under θ'.*

If not, an interval $(p, s)(p \lneqq s)$ is annulled by θ'. Now $px \leqslant x$ for all x in I. If $px \neq x$ for some x in I, then $p \equiv s(\theta')$ implies $px \equiv x(\theta')$; also $px \equiv x(\theta)$ as (px, x) belongs to I. This contradicts the fact $\theta \wedge \theta' = 0$. On the other hand, if $px = x$ for all x in I, then $p \geqslant x$ for all x in I, which implies $p \geqslant \bigvee_{x \in I} x = s$, a contradiction. Hence the claim.

Next, no interval in $\mu(s)$ (the principal μ-ideal generated by s) is annulled by θ'. Let, if possible, $p \lneqq q$ ($\leqslant s$) be such that $p \equiv q(\theta')$. Let p' be the complement of p in $(0, q)$ and p'' a complement of p' in $(0,1)$. $p \equiv q(\theta')$ implies $0 \equiv p'(\theta')$, which implies $p'' \equiv 1(\theta')$, which implies $p'' \geqslant s$, which (as $\alpha(s)$ is a congruence class under θ') implies $p's \leqslant p'p'' = 0$, which implies $p's = p' = 0$, a contradiction since $p' \neq 0$. Thus $\mu(s)$ consists of single-point congruence classes under θ' and hence is annulled by θ'' (since L is weakly modular). Hence $\theta' \vee \theta'' = 1$ for all θ on L.

As a corollary we get the theorem due to M. F. Janowitz (cf. Theorem 4.8 of [3]).

COROLLARY 1. *For any complete relatively complemented lattice, the lattice of congruences on L is a Stone lattice.*

Proof. Follows from the fact that a relatively complemented lattice is both weakly complemented and weakly modular.

It is interesting to note that a weakly complemented, weakly modular lattice is not necessarily relatively complemented, which shows that our result is more general than that of M. F. Janowitz.

Lattice L of Figure 1 is a simple, weakly complemented lattice and hence it is weakly modular; but it is not relatively complemented, since the element b has no complement in the interval $(a, 1)$.

2. Finite lattices. Let L be a finite lattice. Let L_p be the set of all prime intervals of L. Let A, B be in L_p. Define an equivalence relation $\sim$ on L thus: $A \sim B$ if and only if A is a lattice translate of B and B is a lattice translate of A. Consider $L_p/\sim$ and define $\{A\} \leqslant \{B\}$ if A is a lattice translate of B; then $\leqslant$ defines a partial order on $L_p/\sim$. Also, as L is finite, $L_p/\sim$ is a finite set. Hence there exist minimal elements in $L_p/\sim$.

THEOREM 3. *Let L be a finite lattice. The lattice of congruences on L is a Stone lattice if and only if, given any prime interval I in L, there exists one and only one minimal element in $L_p/\sim$, less than $\{I\}$ (where $\{I\}$ denotes the class containing I with respect to the relation $\sim$).*

Proof. Let L satisfy the condition of the theorem. Let J be any prime interval of L. It suffices to show that J is annulled by $\theta' \vee \theta''$ for any congruence θ on L.

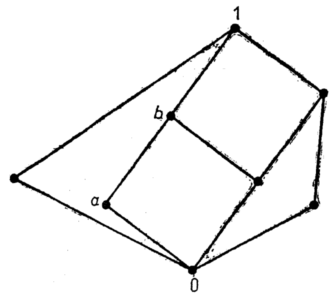

Fig. 1

Now, since L satisfies the condition, let $\{K\}$ be the minimal element of $L_p/\sim$ less than $\{J\}$. If K is not annulled by θ, then, since J has no non-trivial lattice translate annulled by θ; J is annulled by θ' (cf. [2]). On the other hand, if K is annulled by θ, then K is not annulled by θ' and hence J is annulled by θ'' (following the same argument as above for θ' instead of θ). This shows that any prime interval of L is annulled by $\theta' \vee \theta''$ in either case. Therefore $\theta' \vee \theta'' = 1$ for all θ on L, i.e., the lattice of congruences on L is a Stone lattice.

Conversely, let the lattice of congruences on L be a Stone lattice and let there exist, if possible, a prime interval J in L such that there exist two minimal elements $\{J_1\}$ and $\{J_2\}$ of $L_p/\sim$ less than $\{J\}$.

Consider θ on L generated by J_1. Let φ be the congruence generated by J_2; then $\theta \wedge \varphi = 0$ and so $\theta' \supset \varphi$. Now θ' cannot annul J, since J_1 is annulled by θ; also θ'' cannot annul J, since J_2 is annulled by θ'. Thus, for this congruence θ on L, $\theta' \vee \theta'' \lneqq 1$. So the lattice of congruences on L is not a Stone lattice.

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