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A characterization of locally connected continua which are quasi-embeddable into E^2

by

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1. Introduction. We shall consider metrizable spaces only. A map f of a compactum X into a space Y is said to be an *ε-mapping* if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in f(X)$. A compact space X is said to be *quasi-embeddable* into Y if for every $\varepsilon > 0$ there is an ε-mapping $f\colon X \to Y$. The problem of finding a characterization of locally connected continua which can be quasi-embedded into S^2 (E^2) has been raised by Mardešić and Segal in [6] in connection with the following

THEOREM OF MARDEŠIĆ AND SEGAL. *If P is a connected polyhedron, hen the following statements are equivalent*:

(a) *P is embeddable into S^2*,

(b) *P is quasi-embeddable into S^2*,

(c) *P does not contain any homeomorphic images of the Kuratowski graphs K_1 and K_2 and any 2-umbrella.*

The graph K_1 is the 1-skelton of a 3-simplex with midpoints of a pair of non-adjacent edges joined by a segment and the graph K_2 is the 1-skelton of a 4-simplex. A *2-umbrella* is the one-point union of a disk and of an arc relative to an interior point of the disk and an end-point of the arc.

In [8] I have generalized that theorem, namely I have shown that the equivalence of (a), (b) and (c) holds for each locally connected continuum P satisfying the following condition: There is a number $\varepsilon > 0$ such that no simple closed curve $S \subset P$ with $\operatorname{diam} S < \varepsilon$ is a retract of P. Another similar generalization has been found by J. Segal (see [10]). He has shown the equivalence of (a) and (b) for locally connected continua which do not contain any homeomorphic images of the curves K_3 and K_4 (described by Kuratowski in [4]).

In this paper we shall prove the equivalence of (b) and (c) for arbitrary locally connected continua, i.e.

THEOREM 1. *A locally connected continuum X is quasi-embeddable into S^2 if and only if it does not contain any homeomorphic images of the graphs K_1 and K_2 and any 2-umbrella.*

Any set homeomorphic with S^2 will be said to be a simple surface. Theorem 1 immediately follows from

THEOREM 2. *A locally connected continuum X is quasi-embeddable into E^2 if and only if X satisfies the condition given in Theorem 1 and X is not a simple surface.*

The next part of the paper is devoted to the proof of Theorem 2.

2. Some lemmas. Any continuum containing more than one point will be called *cyclic* (in the sense of Whyburn) if it is not separated by any point. The notions of a set entirely arcwise connected and of a cyclic element will be useful for us. Recall that a subset A of a given locally connected continuum X is said to be *entirely arcwise connected* (in X) if $x, y \in A$ and $x \neq y$ imply that each arc (in X) joining x and y is contained in A. We shall base ourselves on the notion of a *cyclic element* and of the properties of cyclic elements given in [5] (§ 47). They were listed in [9], where the following two properties were also proved:

(2.1) *Given two different points $a, b \in X$, the least closed and entirely arcwise connected subset of X containing a and b is the union of an arc L joining a and b and of all the cyclic elements of X which have at least two points in common with L. Moreover, if E_1, E_2 are two cyclic elements having this property, then $E_i \cap L$ is a non-degenerate subarc L_i of L and $E_1 \cap E_2 = L_1 \cap L_2 = \dot{L}_1 \cap \dot{L}_2$. ($\dot{L}$ denotes the boundary of the arc L.)*

(2.2) *Let $X = \overline{\bigcup_{i=1}^{\infty} A_i}$, where $A_i \subset A_{i+1}$ and $A_i = \overline{A}_i \neq X$ is a set entirely arcwise connected. If the maximum of the diameters of the components of $X - A_i$ is equal to δ_i, then $\lim_{i=\infty} \delta_i = 0$.*

Now, we shall prove the following simple

LEMMA 1. *Suppose that $X \subset E^2$ is a locally connected continuum and that $x_0 \in X$ does not belong to the closure of any component of $E^2 - X$. Given an $\varepsilon > 0$, there is a simple closed curve $S \subset X - (x_0)$ such that $\operatorname{diam} S < \varepsilon$ and such that x_0 belongs to the bounded component of $E^2 - S$.*

Proof. By [5] (p. 363), there is only a finite number of components of $E^2 - X$, say $C_1, \ldots, C_l$, with diameter greater than $\varepsilon/3$. Let $Q_0 \subset E^2$ be any disk such that $\operatorname{diam} Q_0 < \varepsilon/3$, $Q_0 \cap \overline{C}_i = \emptyset$ for $i = 1, \ldots, l$ and such that $x_0 \in \mathring{Q}_0$. We can assume that $\dot{Q}_0 \cap (E^2 - X) \neq \emptyset$. Let A denote the union of Q_0 and of the closures of all components C of $E_2 - X$ such that $C \cap \dot{Q}_0 \neq \emptyset$. Since each set $\overline{C}$ is a locally connected continuum (cf. [5], p. 360) and since the diameters of these components converge to zero provided their number is infinite, it follows that A is a locally connected continuum. Evidently, $\operatorname{diam} A < \varepsilon$.

Let B denote the union of A and of all bounded components of $E^2 - A$. Then B is a locally connected continuum which does not separate E^2 and $\operatorname{diam} B = \operatorname{diam} A < \varepsilon$. Let Q denote the cyclic element of B which contains x_0. Since $B \supset A \supset Q_0$ and $x_0 \in \mathring{Q}_0$, it follows from [5] (p. 238, No. 10) that $Q_0 \subset Q$. Thus Q is a cyclic locally connected continuum which does not separate E^2. Consequently, we infer from [5] (p. 380, No.11) that Q is a disk such that $x_0 \in \mathring{Q}$. Evidently, $\dot{Q} \subset \operatorname{Bd}(B) \subset \operatorname{Bd}(A) \subset X$. Thus $S = \dot{Q}$ is the required simple closed curve.

The following result of S. Claytor (see [2], p. 632) is the main one which makes the proof of Theorem 2 possible.

LEMMA 2 (Claytor). *Each cyclic locally connected continuum which does not contain homeomorphic images of the graphs K_1 and K_2 is embeddable into S^2.*

3. A proof of Theorem 2. Since the necessity of the conditions given in Theorem 2 is clear (cf. [6], p. 637), it remains to prove that they are also sufficient. Thus, let X satisfy these conditions. First, notice that:

(3.1) *X does not contain any simple surface.*

Indeed, if $S \subset X$ is a simple surface then $X - S \neq \emptyset$. Since X is arcwise connected, it follows that X contains a 2-umbrella, which contradicts the assumption.

Given an $\varepsilon > 0$, we have to prove that there is an ε-mapping of X into E^2. We shall first show that the following additional assumption can be made:

(3.2) *There is a finite set $F \subset X$ such that the least closed and entirely arcwise connected subset of X containing F is equal to X.*

Indeed, let F_i denote any finite subset of X such that for each point $x \in X$ there is a point $y \in F_i$ such that $\varrho(x, y) < 1/i$, where $i = 1, 2, \ldots$ Let A_k denote the least closed and entirely arcwise connected subset of X containing the set $\bigcup_{i=1}^{k} F_i$. Assume that no A_k is equal to X. Then the sets A_k satisfy the assumptions of (2.2), and therefore there is an index k_0 such that the diameter of each component of $X - A_{k_0}$ is less than $\varepsilon/3$. Let r_0 denote the retraction of X onto A_{k_0} such that for each component C of $X - A_{k_0}$ we have $r_0(\overline{C}) = \overline{C} - C$ (cf. [5], p. 263, No. 5). Suppose that there is an $\varepsilon/3$-mapping f_0: $A_{k_0} \to E^2$. Then $f_0 r_0$ is an ε-mapping of X into E^2, which implies that the additional assumption (3.2) can be made. (Notice that in view of (3.1) the set $A_{k_0} \subset X$ satisfies analogical assumptions to those satisfied by X.)

Now, let us remark that:

(3.3) *Each cyclic element of X is embeddable into E^2.*

Indeed, let E be a non-degenerate (i.e. containing more than one point) cyclic element of X. Then $E \subset X$ is a cyclic space and Lemma 2 immediately implies that E is embeddable in S^2. By (3.1), we obtain (3.3)

If E is a cyclic element of X, then a point $x_0 \in E$ will be called a *Euclidean point* of E provided x_0 is mapped onto an interior point of the image of E under some imbedding of E into E^2. We shall prove that:

(3.4) *For every $\varepsilon > 0$ there is an ε-mapping f: $X \to E^2$ such that for each $y_0 \in f(X) - \overline{E^2 - f(X)}$ the set $f^{-1}(y_0)$ consists of exactly one point x_0 and there is a cyclic element E of X such that x_0 is a Euclidean point of E (which implies that $x_0 \in E - \overline{X - E}$, since X contains no 2-umbrella).*

The proof of (3.4) will be inductive with respect to the number m of the points of the set F mentioned in (3.2). Evidently, we can assume that $m > 1$. First, we shall consider

The case $m = 2$. Let F consist of the points a_1 and a_2. In virtue of (2.1), X is the union of an arc L joining these points and of a sequence (finite or not) $E_1, E_2, \ldots$ of the non-degenerate cyclic elements of X (cf. [5], p. 238, No. 9), where $E_i \cap L$ is a non-degenerate subarc L_i of L and $E_i \cap E_j = L_i \cap L_j = \dot{L}_i \cap \dot{L}_j$ for $i \neq j$. Notice that we can assume that the sequence $E_1, E_2, \ldots$ is finite. Otherwise, by [5] (p. 238, No. 9), there is an index n_0 such that the diameter of each element E_n with $n > n_0$ is less than $\varepsilon/3$. Evidently, there is a retraction r_0 of X onto $L \cup \bigcup_{i=1}^{n_0} E_i$ such that $r_0(E_n) = L_n$ for $n > n_0$. Further we reason as in the proof that (3.2) can be assumed. The condition given in (3.4) will be satisfied under the superposition of r_0 and of a suitable map of $r_0(X)$ into E^2 (provided it is satisfied under this map), because for each point $x_0 \in E_n - \overline{r_0(X) - E_n}$ with $n \leqslant n_0$ the set $r_0^{-1}(x_0)$ consists of the point x_0 only.

Thus we can and do assume that X has exactly n_0 non-degenerate cyclic elements and we shall prove (3.4) by induction with respect to n_0.

If $n_0 = 0$ then $X = L$ and (3.4) is trivial. Now, let $n_0 \geqslant 1$ and suppose (3.4) to be true for each space Y satisfying analogical conditions to those satisfied by X and having less than n_0 non-degenerate cyclic elements. Consider the cyclic element E_1 and let b_1, b_2 denote the end-points of the arc $L_1 = E_1 \cap L$. By (3.3), there is an embedding h: $E_1 \to E^2$. Let $E_1' = h(E_1)$, $b_i' = h(b_i)$ for $i = 1, 2$. Evidently, we can assume that $X - E_1 \neq \emptyset$ and let us assume that $b_1, b_2 \in \overline{X - E_1}$. (If b_1 or b_2 is an interior point of E_1, then there is no necessity of considering it.) Since X does not contain any 2-umbrella, it follows that $b_1', b_2' \in \overline{E^2 - E_1'}$. Let us assume that none of these points belongs to the closure of a component of $E^2 - E_1'$. If this is not true then the proof simplifies, because the retraction r: $E_1' \to E_1'$ we are going to construct may be assumed to be the identity outside a neighborhood of the point b_i' which satisfies this condition.

Let $\eta > 0$ be a number selected so that for each set $A \subset E_1'$ the inequality $\operatorname{diam} A < \eta$ implies that $\operatorname{diam} h^{-1}(A) < \varepsilon/2$. Apply Lemma 1 to the set $E_1' \subset E^2$ and to the points $b_1', b_2' \in E_1'$. Thus, there are two disks $Q_1, Q_2 \subset E^2$ such that $Q_1 \cap Q_2 = \emptyset$, $\operatorname{diam} Q_i < \eta$, $S_i = \dot{Q}_i \subset E_1'$ and such that $b_i' \in \mathring{Q}_i$ for $i = 1, 2$. Since $b_i' \in \overline{E^2 - E_1'}$, there is a retraction r_i: $Q_i \cap E_1' \to S_i$ $(i = 1, 2)$.

Consider the map r: $E_1' \to E_1'$ defined as follows:

$$r(x) = \begin{cases} x & \text{if} \quad x \in E_1' - Q_1 - Q_2\,, \\ r_1(x) & \text{if} \quad x \in Q_1 \cap E_1'\,, \\ r_2(x) & \text{if} \quad x \in Q_2 \cap E_1'\,. \end{cases}$$

It is easily seen that r is an η-mapping, which retracts E_1' onto $E_1' - \mathring{Q}_1 - \mathring{Q}_2$. The points $r(b_i')$ belong to the closure of a component of $E^2 - r(E_1')$ either and for each point $x \in r(E_1') - \overline{E^2 - r(E_1')}$ we have $r^{-1}(x) = (x)$.

Now, for $i = 1, 2$ let Y_i denote the closure of the component of $X - E_1$ bounded by the point b_i. It is clear that Y_i is the least closed and entirely arcwise connected subset of X (and of itself also) containing the points a_i and b_i. Since Y_i has less than n_0 non-degenerate cyclic elements, it follows that it satisfies the induction hypothesis. Thus there is an $\varepsilon/2$-mapping f_i: $Y_i \to E^2$ satisfying the analog of (3.4). Let $\hat{Y}_i = f_i(Y_i)$, $\hat{b}_i = f_i(b_i)$. Since $b_i \in Y_i \cap \overline{X - Y_i}$ and X does not contain any 2-umbrella, it follows from (3.4) applied to Y_i that $\hat{b}_i \in \overline{E^2 - \hat{Y}_i}$. By [5] (p. 17), there is a number $\delta > 0$ such that $B \subset \hat{Y}_i$ and $\operatorname{diam} B < \delta$ imply $\operatorname{diam} f_i^{-1}(B) \leqslant \varepsilon/2$. By the same method as that used to construct the retraction r, we can construct a δ-mapping s_i which retracts $\hat{Y}_i$ into $\hat{Y}_i$ such that the point $s_i(\hat{b}_i)$ belongs to the closure of a component of $E^2 - s_i(\hat{Y}_i)$ and such that for each point $x \in s_i(\hat{Y}_i) - \overline{E^2 - s_i(\hat{Y}_i)}$ we have $s_i^{-1}(x) = (x)$. Then it is easy to observe that there is a homeomorphism h_i of $s_i(\hat{Y}_i)$ into Q_i such that

(3.5) $$h_i s_i(\hat{Y}_i) \cap S_i = r(b_i') = h_i s_i(\hat{b}_i)\,.$$

Now, let

$$X' = r(E_1') \cup h_1 s_1(\hat{Y}_1) \cup h_2 s_2(\hat{Y}_2)\,.$$

It follows from the construction that $h_1 s_1(\hat{Y}_1) \cap h_2 s_2(\hat{Y}_2) = \emptyset$ and $h_i s_i(\hat{Y}_i) \cap r(E_1') = r(b_i')$. Define f: $X \to X'$ as follows:

$$f(x) = \begin{cases} rh(x) & \text{if} \quad x \in E_1\,, \\ h_i s_i f_i(x) & \text{if} \quad x \in Y_i, \ i = 1, 2\,. \end{cases}$$

Since $Y_1 \cap Y_2 = \emptyset$ and $Y_i \cap E_1 = (b_i)$, we infer from (3.5) that f is a map. If $y \in f(E_1)$ and $y \neq f(b_i)$ for $i = 1, 2$, then $\operatorname{diam} f^{-1}(y) < \varepsilon/2$ because of $f^{-1}(y) = h^{-1}r^{-1}(y)$, because of r being an η-mapping and with respect to the definition of η. If $y \in f(Y_i)$ and $y \neq f(b_i)$, then $\operatorname{diam} f^{-1}(y) \leqslant \varepsilon/2$, because of $f^{-1}(y) = f_i^{-1}s_i^{-1}h_i^{-1}(y)$, because of h_i being a homeomorphism, s_i being a δ-mapping and with respect to the definition of δ. Finally, if $y = f(b_i)$ then $f^{-1}(y) = [(f|E_1)^{-1}(y)] \cup [(f|Y_i)^{-1}(y)]$, $\operatorname{diam}(f|E_1)^{-1}(y) < \varepsilon/2$ and $\operatorname{diam}(f|Y_i)^{-1}(y) \leqslant \varepsilon/2$, which implies that $\operatorname{diam} f^{-1}(y) < \varepsilon$, because $[(f|E_1)^{-1}(y)] \cap [(f|Y_i)^{-1}(y)] = (b_i)$. Thus f is an ε-mapping of X onto $X' \subset E^2$.

Suppose that y_0 is an interior point of X' in E^2. Then $y_0 \neq f(b_i)$ for $i = 1, 2$. If $y_0 \in f(E_1) = r(E_1')$, then $r^{-1}(y_0) = (y_0)$ is an interior point of E_1', which implies that $f^{-1}(y_0) = h^{-1}r^{-1}(y_0)$ is a Euclidean point of E_1. If $y_0 \in f(Y_i) = h_i s_i(\hat{Y}_i)$, then the definitions of h_i and s_i imply that $s_i^{-1}h_i^{-1}(y_0)$ is an interior point of $\hat{Y}_i$. Since f_i satisfies the analog of (3.4), we infer that $f^{-1}(y_0) = f_i^{-1}s_i^{-1}h_i^{-1}(y_0)$ satisfies the condition given in (3.4). Thus the induction step and therefore the proof of (3.4) in the case of $m = 2$ is completed.

The induction step. Now, suppose that the set $F \subset X$ mentioned in (3.2) consists of m points, where $m > 2$, and assume (3.4) to be true for each space satisfying analogical assumptions to those satisfied by X with the corresponding set having less than m of points.

Fix a point $a \in F$ and let Y denote the least closed and entirely arcwise connected subset of X containing the set $F-(a)$. We can assume that $a \notin Y$ and let C denote the component of $X-Y$ containing a. Then $\bar{C}-C$ consists of exactly one point b (cf. [5], p. 232, No. 4). Let Z denote the least closed and entirely arcwise connected subset of X containing a and b. Then $Z \subset \bar{C}$. Since the set $Y \cup Z$ is closed, entirely arcwise connected (cf. [5], p. 232, No. 8) and it contains F, it follows that $Y \cup Z = X$, whence $Z = \bar{C}$.

Now, it is clear that the sets Y and Z both satisfy the induction hypothesis, and therefore there are two $\varepsilon/4$-mappings, $f_1\colon Y \to E^2$, $f_2\colon Z \to E^2$, which satisfy the conditions analogical to those of (3.4). By these conditions, $f_1(b) \in \overline{E^2 - f_1(Y)}$ and $f_2(b) \in \overline{E^2 - f_2(Z)}$. Further proceedings are similar to the induction step for the case of $m = 2$. We construct two retractions $r_1\colon f_1(Y) \to f_1(Y)$, $r_2\colon f_2(Z) \to f_2(Z)$ such that $r_i f_i$ are $\varepsilon/2$-mappings, the point $r_1 f_1(b)$ ($r_2 f_2(b)$) belongs to the closure of a component of $E^2 - r_1 f_1(Y)$ ($E^2 - r_2 f_2(Z)$) and which are such that $r_i^{-1}(x) = (x)$ for each interior point x of $r_1 f_1(Y)$ or $r_2 f_2(Z)$ in E^2. Next, we find a homeomorphism h of $r_2 f_2(Z)$ into E^2 such that $h r_2 f_2(Z) \cap r_1 f_1(Y) = h r_2 f_2(b) = r_1 f_1(b)$ and we construct an ε-mapping f of $X = Y \cup Z$ onto $h r_2 f_2(Z) \cup r_1 f_1(Y) \subset E^2$ satisfying (3.4).

4. Compacta quasi-homeomorphic with S^2. Given two compact spaces X, Y, X is said to be *Y-like* (cf. [7]) if for every $\varepsilon > 0$ there is an ε-mapping of X onto Y. The spaces X and Y are said to be *quasi-homeomorphic* if X is Y-like and Y is X-like. It has been proved by Ganea (see [3]) that each ANR which is quasi-homeomorphic with S^2 is a simple surface. The following corollary to Theorem 2 generalizes Ganea's result:

THEOREM 3. *A compactum X is quasi-homeomorphic with S^2 if and only if it is a simple surface.*

Proof. Let X be a compactum which is quasi-homeomorphic with S^2. Since X is a continuous image of S^2, it follows that X is a locally connected continuum. Since X is S^2-like, X does not contain any homeomorphic images of the graphs K_1 and K_2 and any 2-umbrella. Suppose that X is not a simple surface. Then, by Theorem 2, X is quasi-embeddable into E^2.

Fix $S^2 = [x \in E^3 |\ |x| = 1]$. Since S^2 is X-like, there is a map f of S^2 onto X such that $\operatorname{diam} f^{-1}(x) < 1$ for each $x \in X$. Choose a number $\eta > 0$ such that, for each set $A \subset X$, $\operatorname{diam} A < \eta$ implies $\operatorname{diam} f^{-1}(A) \leqslant 1$. Let g be an η-mapping of X into E^2. Then gf is a map of S^2 into E^2 such that no pair of antipodal points of S^2 is mapped onto the same point. This contradicts Borsuk's well known antipodal point theorem (see [1]).

Remark. Using other methods Mardešić and Segal have proved the following theorem (see [7], p. 163): Let X be a locally connected continuum which is either cyclic or 2-dimensional. Then X is S^2-like if and only if X is a simple surface. Let us notice that the methods of Mardešić and Segal permit us to prove the following sharper

THEOREM 4. *A locally connected continuum is S^2-like if and only if it is either a dendrite (containing more than one point) or a simple surface.*

Proof. First, let us show that the segment $I = \langle 0, 1\rangle$ is S^2-like. The proof is similar to that of Example 9 in [7]. Let g denote a map of the set $I \times S^1$ onto S^2 which collapses the sets $(0) \times S^1$ and $(1) \times S^1$ to points and which is one-to-one otherwise. Let n be a natural number such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and let I_i denote the segment $\left\langle \frac{i-1}{n}, \frac{i}{n} \right\rangle$ for $i = 1, 2, \ldots, n$. Let f_i be a map of I_i onto $I_i \times S^1$. We can assume that the maps f_i agree, and then they determine an ε-mapping f of I onto $I \times S^1$. Then gf is an ε-mapping of I onto S^2.

Now, if X is a dendrite (containing more than one point), then X can be ε-mapped onto a tree and each tree is S^2-like, as can be proved by a method similar to that used in the case of a segment.

On the other hand, if X is a locally connected continuum which is S^2-like and $\dim X \geqslant 2$, then $\dim X = 2$ (cf. [5], p. 64) and, by the above mentioned result of Mardešić and Segal, X is a simple surface. If $\dim X = 1$

and X is not a dendrite, then X contains a simple closed curve, which is a retract of X (cf. [5], p. 271). Consequently, the first homology group of X in the sense of E. Čech $H_1(X, Z)$ is not trivial, which yields a contradiction with Theorem 1 of [7], because $H_1(S^2, Z) = 0$.

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On lattices whose lattices of congruences are Stone lattices

by

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M. F. Janowitz proves in [3] that the lattice of congruences on any complete relatively complemented lattice is a Stone lattice and poses the question:—Find necessary and sufficient conditions on a complete lattice L for the lattice of congruences on L to become a Stone lattice. This note gives an answer to the above question. We also show that the lattice of congruences on any complete, weakly complemented, weakly modular lattice is a Stone lattice. This is a generalization of the result of M. F. Janowitz, proved by the fact that a complete, weakly complemented, weakly modular lattice is not always relatively complemented.

We further show that, in the case of a finite lattice L, the lattice of congruences on L is a Stone lattice if and only if, given any prime interval I of L, there exists one and only one minimal element in $L_p/\sim$ less than $\{I\}$ (where L_p denotes the set of all prime intervals of L and $\sim$ is the equivalence relation defined on L_p thus: $A \sim B$ if and only if A is a lattice translate of B and B is a lattice translate of A; and $\{I\}$ denotes the class containing I with respect to the relation $\sim$).

1. Complete lattices.

THEOREM 1. *Let L be a complete lattice. The lattice of congruences on L is a Stone lattice if and only if for any congruence θ on L there exist a finite number of elements $0 = b_1 < b_2 < \ldots < b_n = 1$ such that either (b_{i-1}, b_i) has no non-trivial lattice translate annulled by θ or every lattice translate of (b_{i-1}, b_i) has a non-trivial lattice translate annulled by θ.*

Proof. Follows from theorems 1 and 3 of [2].

COROLLARY. *Let L be a complete weakly modular lattice. The lattice of congruences on L is a Stone lattice if and only if for any congruence θ on L there exists a finite chain $0 = b_1 < b_2 < \ldots < b_n = 1$ such that either (b_{i-1}, b_i) consists of single point congruence classes under θ or every subinterval of (b_{i-1}, b_i) has a proper part annuled by θ.*