theories of fields.) It seems likely that in order to make an advance on the problem one will have to use techniques like Ehrenfeucht's condition, or Keisler's finite cover property [1, 4].

When working on this paper we proved the following result, which may be useful.

THEOREM 3. *Suppose $\mathcal{L}$ is countable, and $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{L}$-structures such that $Th(\mathcal{M}_1)$ and $Th(\mathcal{M}_2)$ are totally transcendental. Then $Th(\mathcal{M}_1 \oplus \mathcal{M}_2)$ is totally transcendental.*

This result fails if we replace "totally transcendental" by "ω_1-categorical". To see this, take $\mathcal{M}_1$ as $\boldsymbol{Q}$, $\mathcal{M}_2$ as $\bigoplus_{i\in I} \boldsymbol{Z}(p)$ where I is infinite and p is prime, and use Lemma 4.

The result also fails for infinite direct sums and products. Thus, $Th(Z(p^n))$ is totally transcendental, but, by Theorem 1, neither

$$Th\big(\bigoplus_n Z(p^n)\big),$$

nor

$$Th\big(\prod_n Z(p^n)\big)$$

is totally transcendental.

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Some theorems about the embeddability of ANR-sets into decomposition spaces of E^n

by

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1. Introduction. This paper is a continuation of my earlier paper [18], in which the following general theorem has been proved:

THEOREM A ([18], p. 290). *If X is a connected* ANR *containing no n-umbrella and if the cyclic elements of X are embeddable into E^n, then X is embeddable into an n-dimensional Cartesian divisor of E^{n+1}.*

As a corollary to this theorem and to Claytor's results ([6] and [7]) the following theorem has been deduced:

THEOREM B ([18], p. 291). *If X is a connected* ANR *which does not contain any 2-umbrellas and any homeomorphic images of the graphs of Kuratowski, then X is embeddable into S^2.*

This theorem gives a positive answer to a problem of Mardešić and Segal ([13], p. 637). In [18] some historical remarks concerning Theorems A and B have been given, which we do not repeat here. The following remarks concern the terminology. Only metrizable separable spaces are considered. The ANR-spaces are always assumed to be compact. We base our considerations on the definition and the propositions concerning cyclic elements given in [12], § 47, which have been recalled in [18]. Therefore, we do not repeat them here, although, in general we give references to respective propositions proved in [12], § 47. By an n-umbrella we mean a one-point union of a (topological) n-ball Q and of an arc I relative to a point $p \in \mathring{Q}$ and a point $q \in \dot{I}$. By a graph we mean any space which is a homeomorphic image of a compact, at most 1-dimensional polyhedron. A connected, acyclic graph (i.e. a graph which is an AR-set) is called a tree. The graphs of Kuratowski (which are called *primitive skew curves* by Mardešić and Segal) are the following polyhedra K_1 and K_2 (cf. [11]): K_1 is the 1-skelton of a 3-simplex in which the midpoints of a pair of non-adjacent edges are joined by a segment, K_2 is the 1-skelton of a 4-simplex. Given a space X, any space Y is called a *Cartesian divisor* of X if there is a space Z such that the product $Y \times Z$ is homeomorphic with X.

If $\mathfrak{D}$ is a decomposition of a space X, then $\mathfrak{D}$ is called a null-decomposition if for every $\eta > 0$ there exist (at most) finitely many elements of $\mathfrak{D}$ of diameter greater than η. It is clear that each null-decomposition of any space X into compact sets is upper semi-continuous.

For each set A, the boundary of A will be denoted by $\mathrm{Bd}(A)$ and the diameter of A by $\delta(A)$. The set of the points which locally separate a connected space X will be denoted by L_X.

In [10] A. Kosiński has introduced the concept of strongly cyclic elements of a space X and has formulated (without proof) some of their properties under the assumption that $X \in \mathrm{ANR}$. In Section 4 of this paper we shall recall the definition of Kosiński and we shall prove more properties, assuming only that X belongs to a class α. For the definition of this class see Section 2. The following theorem, which corresponds to Theorem A formulated above, is the main theorem of this paper:

THEOREM 1. *If X is a connected* ANR *such that, for every strongly cyclic element E of X and for every graph $G \subset X$ such that $E \cap G$ is a finite set (also if $E = \emptyset$ or $G = \emptyset$), the union $E \cup G$ is embeddable into E^n, then X is embeddable into the space $E^n/\mathfrak{D}$, where $\mathfrak{D}$ is a null-decomposition of E^n such that all the non-degenerate elements of $\mathfrak{D}$ are trees and almost all of them are arcs.*

Remark 1. It can easily be noticed (cf. the beginning of Section 8) that in the case of $n = 1$ we can replace the space $E^n/\mathfrak{D}$ in Theorem 1 simply by E^n. In the case of $n = 2$ this is also true, since Moore's well-known theorem [16] implies that $E^2/\mathfrak{D} \underset{\mathrm{top}}{=} E^2$. In the case of $n = 3$ (as for the Cartesian divisor of E^{n+1} in Theorem A), this is not true by my example [17] of two crumpled cubes in E^3, the one-point union of which relative to some boundary points is not embeddable into E^3. However, for $n > 3$ such an example does not exist. Indeed, if $X, Y \subset E^n$ are two disjoint AR-sets and $x_0 \in \mathrm{Bd}(X)$, $y_0 \in \mathrm{Bd}(Y)$, then there exists an infinite polygonal arc $L \subset E^n$ for which $L \cap (X \cup Y) = (x_0) \cup (y_0) = \dot{L}$ and which is locally tame at each point $x \in \mathring{L}$. Thus, the set of the points $x \in L$ such that L fails to be locally tame at x does not contain any homeomorphic image of the Cantor set, which implies—by Cantrell's result [5]—that L is tame in E^n provided $n > 3$. Consequently, $E^n/L \underset{\mathrm{top}}{=} E^n$, and therefore, the one-point union $X \cup Y/(x_0) \cup (y_0)$ is embeddable in E^n.

In the proof of Theorem A given in [18] we have constructed an embedding of X in the decomposition space $E^n/\mathfrak{D}$, where $\mathfrak{D}$ is an upper semi-continuous decomposition of E^n with only countably many non-degenerate elements, each of which is an arc (see [18], p. 296, Lemma). It can easily be seen from the proof given in [18] that each of these arcs can be constructed so that it is locally tame except a sequence of points containing at most one accumulation point. Similarly, in the present proof of Theorem 1 (see Sections 8–10) one can easily construct each tree $T \subset E^n$ belonging to the constructed decomposition $\mathfrak{D}$ of E^n such that each arc $L \subset T$ is locally tame except a sequence of points containing at most a finite number of accumulation points. Thus, the Cantrell theorem [5] implies again that each one of these trees is tame in E^n provided $n > 3$. Consequently, the answer to the following problem seems to be positive:

PROBLEM 1. *Given an $n > 3$, can the space $E^n/\mathfrak{D}$ in Theorem 1 (as well as the Cartesian divisor of E^{n+1} in Theorem* A*) be replaced by E^n?*

Remark 2. The following theorem has been proved by S. Armentrout [1]: If $\mathfrak{D}$ is a point-like decomposition of E^n with only countably many non-degenerate elements, then the space $E^n/\mathfrak{D}$ can be embedded in E^{n+1}. Recall that an upper semi-continuous decomposition $\mathfrak{D}$ of E^n is said to be point-like if each element A of $\mathfrak{D}$ is a continuum such that $E^n - A$ is homeomorphic with $E^n - (p)$, where p is a point of E^n. Thus, by the preceding remark, we infer that for each $n > 3$ the decomposition space $E^n/\mathfrak{D}$ in Theorem 1 can be constructed so that it is embeddable in E^{n+1}. If $n = 3$, then we can assume that $E^3 \subset E^4$ and we can extend trivially the decomposition $\mathfrak{D}$ of E^3 to the decomposition $\hat{\mathfrak{D}}$ of E^4, whose elements are the elements of $\mathfrak{D}$ considered as subsets of E^4 and the one-point sets contained in $E^4 - E^3$. Then, by the preceding remark and the Cantrell theorem [5], the trees belonging to $\mathfrak{D}$ can be constructed so that they are tame in E^4. Consequently, $E^4/\hat{\mathfrak{D}}$ (and therefore also $E^3/\mathfrak{D}$) is embeddable in E^5. Thus, we obtain the following

COROLLARY TO THEOREM 1. *If X is a connected* ANR *satisfying the assumptions of Theorem 1, then X is embeddable in E^{n+1} for $n > 3$ and X is embeddable in E^{n+2} for $n = 3$.*

Remark 3. Gilman and Martin (see [8]) have proved that if $\mathfrak{D}$ is an upper semi-continuous decomposition of E^n with only countably many non-degenerate elements each of which is an arc, then $(E^n/\mathfrak{D}) \times E^1 \underset{\mathrm{top}}{=} E^{n+1}$. Recently, Meyer (see [15]) has generalized this theorem, replacing the assumption that the non-degenerate elements of $\mathfrak{D}$ are arcs by the assumption that they are (finite) brooms and, as he privately says, he believes that the theorem also holds if the non-degenerate elements are trees. Thus, the answer to the following problem seems to be positive.

PROBLEM 2. *Is the space $E^n/\mathfrak{D}$ (where $\mathfrak{D}$ is a decomposition of E^n as described in Theorem 1) a Cartesian divisor of E^{n+1}?*

The structure of this paper is as follows: In Section 2, we shall give some useful definitions (specially, of the class α) and we shall prove some easy propositions. In Section 3 we shall prove some properties of the set L_X in any cyclic space $X \in \alpha$. These properties will be used in Section 4, where we shall recall Kosiński's definition of strongly cyclic elements of X and we shall prove some of their properties for $X \in \alpha$. In Section 5, we shall give a topological characterization of strongly cyclic polyhedra which are embeddable in S^2. The results of Section 5 will be used in Section 6, where we shall give a topological characterization of all polyhedra which are embeddable in S^2. The result of Section 6 is an improvement of Mardešić and Segal's theorem (see [13] and [14]), which characterizes the polyhedra which are embeddable in S^2 among all polyhedra. In Section 7, assuming Theorem 1 to be true, we shall give a characterization of ANR-sets which are embeddable in S^2. This characterization is an improvement of Theorem B, formulated at the beginning of the paper. In the proof of that characterization, in contrast with the proof of Theorem B given in [18], we do not use Claytor's results ([6] and [7]), because in the paper we can and do base our argument on the proof of Mardešić and Segal's theorem given in [14], in which Claytor's results are not used, either. The last three sections are devoted to the proper proof of Theorem 1 (although the results of all previous sections except Section 7 are used in that proof). In Section 8 we shall reduce Theorem 1 to a lemma. The proof of this lemma for two cases (the second of which is the general one) will be given successively in Sections 9 and 10.

2. Preliminary definitions and propositions. First, we shall give the definitions of four classes of spaces $\alpha, \alpha_0, \alpha'$ and α_0'.

DEFINITION OF THE CLASS α. A locally connected continuum X belongs to the class α if and only if there is a number $\varepsilon > 0$ such that no simple closed curve $S \subset X$ with $\delta(S) < \varepsilon$ is a retract of X.

DEFINITION OF THE CLASS α_0. A locally connected continuum X belongs to the class α_0 if and only if no simple closed curve $S \subset X$ is a retract of X.

It is clear that the class α contains all connected ANR-sets and, more generally, all locally connected continua which are semi-locally 1-connected. (Recall that a space X is semi-locally 1-connected if for each point $x_0 \in X$ there is a neighbourhood U of x_0 in X such that every map of the pair (S^1, s_0) into (U, x_0) is homotopic to the constant map in (X, x_0).) However, there is a space $X \in \alpha$, which is not semi-locally 1-connected. Actually, consider the 2-dimensional projective space P^2. Then $\pi_1(P^2) = Z_2 \neq 0$. Nevertheless, no simple closed curve $S \subset P^2$ is a retract of P^2, because the group Z is not a direct divisor of the group $H_1(P^2, Z) = Z_2$ (cf. [4], p. 42). Let Y_n be a homeomorphic image of P^2 such that $\delta(Y_n) < \frac{1}{n}$ and let $y_n \in Y_n$. Form the disjoint union

$$Y = \bigcup_{n=1}^{\infty} Y_n$$

and let X denote the compact metric space which we obtain from Y by the identification of all points y_n and by a suitable definition of the metric. Let us identify Y_n with its image under the identification map. Then the sets Y_n are the non-degenerate cyclic elements of X and therefore retracts of X. It is now clear that X is not semi-locally 1-connected, however, $X \in \alpha$; moreover, no simple closed curve $S \subset X$ is a retract of X.

Analogically, it is clear that the class α_0 contains all AR-sets and, more generally, all locally connected continua X such that the group Z is not a direct divisor of the first homology group $H_1(X, Z)$ of X in the sence of E. Čech. It is also clear that each retract of a space $X \in \alpha$ (of a space $X \in \alpha_0$) also belongs to this class.

Recall that a connected space X (containing more than one point) is said to be cyclic (in the sense of Whyburn) if it is not separated by any point.

DEFINITION OF THE CLASSES α' AND α_0'. A space X belongs to the class α' (to the class α_0') if and only if $X \in \alpha$ ($X \in \alpha_0$) and X is a cyclic space.

It is clear that each non-degenerate (i.e. containing more than one point) cyclic element of any space $X \in \alpha$ (of any space $X \in \alpha_0$) belongs to the class α' (to the class α_0'). Therefore, the properties of the spaces $X \in \alpha'$ (of the spaces $X \in \alpha_0'$) which we shall prove in Sections 3 and 4 may be understood as the properties of the non-degenerate cyclic elements of the spaces $X \in \alpha$ (of the spaces $X \in \alpha_0$).

Now, we shall prove four simple propositions, which will be useful in the subsequent sections.

(2.1) *Let X be a locally connected space and let $F_i = \overline{F_i} \subset X$ for $i = 1, 2, \ldots$ Then* $\mathrm{Bd}(\bigcap_{i=1}^{\infty} F_i) \subset \overline{\bigcup_{i=1}^{\infty} \mathrm{Bd}(F_i)} \cap \bigcap_{i=1}^{\infty} F_i$.

Using the inclusion given in [12], (p. 168), we have:

$$\mathrm{Bd}(\bigcap_{i=1}^{\infty} F_i) = \mathrm{Bd}(X - \bigcap_{i=1}^{\infty} F_i) = \mathrm{Bd}(\bigcup_{i=1}^{\infty} X - F_i) \subset \overline{\bigcup_{i=1}^{\infty} \mathrm{Bd}(X - F_i)}$$

$$= \overline{\bigcup_{i=1}^{\infty} \mathrm{Bd}(F_i)}\,,$$

which implies the required inclusion, because F_i are closed sets.

(2.2) *Let X be a locally connected continuum and let F be a finite subset of X containing more than one point and such that $X - F$ is connected. Then there is a tree $T \subset X$ such that $T \supset F$, $T - F$ is connected and such that, for each $x \in T$, every component of $T - (x)$ intersects F (i.e. such that the set of the end-points of T is equal to F).*

One can easily prove this proposition by induction with respect to the number of the points of the set F.

(2.3) *Let X be a locally connected continuum and let $U \subset X$ be a region (i.e. a subset of X both open and connected) such that $\mathrm{Bd}(U)$ is a finite set. Then $\overline{U}$ is a retract of X.*

By [12] (p. 170), $\overline{U}$ is a locally connected continuum. By (2.2), there is a tree $T \subset \overline{U}$ (which can degenerate to a point) such that $T \supset \mathrm{Bd}(U)$. Since $T \in \mathrm{AR}$, there is a retraction r of the set $(X-U) \cup T$ onto T. Since $\big((X-U) \cup T\big) \cap \overline{U} = \mathrm{Bd}(U) \cup T = T$, it follows that the function $s\colon X \to \overline{U}$ defined as

$$s(x) = \begin{cases} x & \text{if} \quad x \in \overline{U}, \\ r(x) & \text{if} \quad x \in (X-U) \cup T \end{cases}$$

is a retraction of X onto $\overline{U}$.

(2.4) *Let X be a cyclic locally connected continuum. Suppose that $U \subset X$ is a region, $x_n \in U$ for $n = 1, 2, \ldots$ and let U_n be a component of $U-(x_n)$. If $U_n \cap U_m = \emptyset$ for $n \neq m$ and $\lim_{n=\infty} x_n = x_0$, then $x_0 \notin U$.*

On the contrary, suppose that $x_0 \in U$. Let V be any open neighbourhood of x_0 in X such that $\overline{V} \subset U$. We can assume that, for $n = 1, 2, \ldots$, $x_n \in V$, whence $U_n \cap V \neq \emptyset$. Since X is a cyclic space, no U_n is a component of $X-(x_n)$, and therefore $U_n \cap (X-V) \neq \emptyset$. Consequently, $U_n \cap \mathrm{Bd}(V) \neq \emptyset$, because U_n is connected. Now, this is impossible, since $\mathrm{Bd}(V)$ is compact and the sets U_n $(n = 1, 2, \ldots)$ are disjoint.

3. Some properties of the set L_X in any space $X \in \alpha'$. In this section we shall consider a fixed space $X \in \alpha'$. Let us fix for the space X a number $\varepsilon > 0$ with the property mentioned in the definition of the class α'.

(3.1) *If $U \subset X$ is a region with $\delta(U) < \varepsilon$ and $x_0 \in U \cap L_X$, then x_0 separates U. Moreover, if V is a region such that $x_0 \in V \subset U$ and x_0 separates V between two points $x_1, x_2 \in V-(x_0)$, then x_0 separates U between these points.*

Evidently, it suffices to prove the second statement of (3.1). Suppose, contrary to this statement, that the points x_1 and x_2 belong to one component U_0 of $U-(x_0)$. Since U_0 is a region, there is an arc $J \subset U_0$ such that $\dot{J} = (x_1) \cup (x_2)$. Since V is also a region containing x_1 and x_2, there is an arc $I \subset V$ such that $\dot{I} = (x_1) \cup (x_2)$. Then $x_0 \in \mathring{I}$, because x_0 separates V between the points x_1 and x_2. Thus, replacing the arcs I and J by their sub-arcs if necessary, we can assume that the set $C = I \cup J$ is a simple closed curve and $x_0 \in \mathring{I}$. Since x_0 separates V between the points x_1 and x_2,

there are two open sets V_1 and V_2 such that $V-(x_0) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, $x_1 \in V_1$ and $x_2 \in V_2$. Denote the components of $I-(x_0)$ by I_1 and I_2. We can assume that $I_i \subset V_i$ for $i = 1, 2$. Now, let F denote a closed neighbourhood of x_0 in X such that $F \subset V$ and $F \cap J = \emptyset$. Let $F_i = F \cap V_i$ for $i = 1, 2$. Then $F-(x_0) = F_1 \cup F_2$, $F_1 \cap F_2 = \emptyset$, $\overline{F}_i = F_i \cup (x_0)$, $I_1 \cap F_2 = \emptyset$ and $I_2 \cap F_1 = \emptyset$.

Since $x_0 \in \mathrm{Int}(F)$, it follows that $\mathrm{Bd}(F)$ is a compact subset of $F_1 \cup F_2$. Let $F_i' = \mathrm{Bd}(F) \cap F_i = \mathrm{Bd}(F) \cap \overline{F}_i$. Since $F_i' \cap I_i = F_i' \cap \overline{I}_i$ is a compact subset of I_i, there is an arc I_i' such that $F_i' \cap I_i \subset I_i' \subset I_i$. Evidently, there is a retraction r_i of $F_i' \cup I_i'$ onto I_i'. Since F_1' and F_2' are compact disjoint subsets of F and $F_i' \cap I = F_i' \cap I_i \subset I_i'$, it follows that the map $r\colon \mathrm{Bd}(F) \cup I = F_1' \cup F_2' \cup I \to I$ defined as follows

$$r(x) = \begin{cases} r_1(x) & \text{if} \quad x \in F_1', \\ r_2(x) & \text{if} \quad x \in F_2', \\ x & \text{if} \quad x \in I \end{cases}$$

can be extended to a retraction $\overline{r}$ of $F \cup I$ onto I.

Now, since $I_1' \subset I_1$, $I_2' \subset I_2$ and J are arcs lying on the simple closed curve $C = I \cup J$ such that $x_0 \notin I_1' \cup I_2' \cup J$, we infer that there is an arc $K \subset C-(x_0)$ such that $K \supset I_1' \cup I_2' \cup J$. The function $s\colon \mathrm{Bd}(F) \cup K \to K$ defined as

$$s(x) = \begin{cases} \overline{r}(x) & \text{if} \quad x \in \mathrm{Bd}(F), \\ x & \text{if} \quad x \in K \end{cases}$$

is a map, because $\mathrm{Bd}(F) \cap K \subset F \cap C \subset I$, since $F \cap J = \emptyset$. Thus, the map s can be extended to a retraction $\overline{s}$ of the set $\overline{X-F} \cup K$ onto K.

Finally, consider the function $t\colon X \to C$ defined as follows:

$$t(x) = \begin{cases} \overline{r}(x) & \text{if} \quad x \in F \cup I, \\ \overline{s}(x) & \text{if} \quad x \in \overline{X-F} \cup K. \end{cases}$$

To prove that t is a map, notice that $(F \cup I) \cap (\overline{X-F} \cup K) \subset \mathrm{Bd}(F) \cup C$. If $x \in \mathrm{Bd}(F)$, then $\overline{s}(x) = s(x) = \overline{r}(x)$. If $x \in C \cap (F \cup I)$, then $\overline{r}(x) = r(x) = x$, because $C \cap (F \cup I) = (I \cup J) \cap (F \cup I) = I$, as $F \cap J = \emptyset$. If $x \in C \cap (\overline{X-F} \cup K)$, then $\overline{s}(x) = s(x) = x$, because $C \cap (\overline{X-F} \cup K) = K$, as $C-K \subset \mathrm{Int}(F)$ (since $C-K$ is a connected set containing $x_0 \in \mathrm{Int}(F)$ and $(C-K) \cap \mathrm{Bd}(F) \subset (I-I_1'-I_2') \cap (F_1' \cup F_2') \subset [F_1' \cap (I-I_1')] \cup [F_2' \cap (I-I_2')] = \emptyset$). Since $C = I \cup J \subset I \cup K \subset C$, which implies $I \cup K = C$, and since $t(x) = x$ for $x \in I \cup K$, the map t is a retraction of X onto C. Thus, we have obtained a contradiction with the definition of the number ε, because $C = I \cup J \subset U$ and $\delta(U) < \varepsilon$.

(3.2) *Let $U \subset X$ be a region such that $\delta(U) < \varepsilon$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points such that $\lim_{n=\infty} x_n = x_0 \in U$ and $x_0 \neq x_n \in L_X$ for $n = 1, 2, \ldots$ Then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{x_n\}_{n=1}^{\infty}$ such that for each $k > 1$ the point x_{n_k} separates U between the points $x_{n_{k-1}}$ and x_0.*

By (2.4), the set $U-(x_0)$ cannot have infinitely many components, and therefore, replacing the sequence $\{x_n\}_{n=1}^{\infty}$ by a subsequence if necessary, we can assume that all points x_n lie in one component of $U-(x_0)$. We also assume that all points x_n are distinct.

We shall construct the sequence $y_k = x_{n_k}$, $k = 1, 2, \ldots$, by induction. Let $y_1 = x_1$. Consider an $m > 1$ and suppose that the points $y_1, \ldots, y_{m-1}$ have been defined. Set $n_0 = n_{m-1}$. Suppose that the point y_m cannot be found, i.e., that there is no $n > n_0$ such that x_n separates U between y_{m-1} and x_0. Since $x_n \in L_X \cap U$ and $\delta(U) < \varepsilon$, it follows from (3.1) that x_n separates U. Thus, for each $n > n_0$, there is a component U_n of $U-(x_n)$ containing neither y_{m-1} nor x_0. Since $\lim_{n=\infty} x_n = x_0$, we can assume, replacing the sequence $x_{n_0+1}, x_{n_0+2}, \ldots$ by a subsequence if necessary, that for each fixed $n_1 > n_0$ the set U_{n_1} does not contain any point x_n for $n > n_1$. Thus, if $q > p > n_0$, then U_p is a connected subset of $U-(x_q)$, whence either $U_p \subset U_q$ or $U_p \cap U_q = \emptyset$. Since $\lim_{n=\infty} x_n = x_0 \in U$, it follows from (2.4) that there exist only finitely many regions U_n (where $n > n_0$) which are disjoint to one another. Consequently, replacing again the sequence $x_{n_0+1}, x_{n_0+2}, \ldots$ by a subsequence if necessary, we can assume that $U_n \subset U_{n+1}$ for each $n > n_0$. Then $x_n \in U_{n+1}$, because $U_n \cup (x_n)$ is a connected subset of $U-(x_{n+1})$.

Let $U_0 = \bigcup [U_n |\ n > n_0]$. Then U_0 is a subregion of U which contains neither $y_{m-1} = x_{n_0}$ nor x_0, but which contains all points x_n for $n > n_0$. Since U—as a region in X—is locally connected, considering U as a space, we have $\mathrm{Bd}(U_0) \subset \overline{\bigcup_{n>n_0} \mathrm{Bd}(U_n)} - U_0$ (cf. [12], p. 168), whence $\mathrm{Bd}(U_0) = (x_0)$, because $\mathrm{Bd}(U_n) = (x_n) \subset U_0$. Consequently, U_0 is a component of $U-(x_0)$ such that $x_{n_0+1} \in U_0$ and $x_{n_0} \notin U_0$, which contradicts the assumption that all points x_n are contained in one component of $U-(x_0)$, which has been made at the beginning of the proof. Thus the point $y_m = x_{n_m}$ with the required property can be found, which completes the proof.

(3.3) *Let be given an $\eta > 0$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points such that $\lim_{n=\infty} x_n = x_0$ and $x_0 \neq x_n \in L_X$ for $n = 1, 2, \ldots$ Then there are three indices k, l, m such that $1 \leqslant k < l < m$ and such that the set $(x_l) \cup (x_0)$ separates X between the points x_k and x_m, the diameter of the component of $X-(x_l)-(x_0)$ containing x_m being less than η.*

Let $U \subset X$ be a region with $\delta(U) < \min(\varepsilon, \eta)$ containing x_0. By (3.2), replacing the sequence $\{x_n\}_{n=1}^{\infty}$ by a subsequence if necessary, we can assume that $x_n \in U$ for $n = 1, 2, \ldots$ and that for each $n > 1$ the point x_n separates U between the points x_{n-1} and x_0. It follows that for each $n \geqslant 1$ the points x_n and x_{n+1} lie in the same component of $U-(x_0)$, and therefore there is a component C of $U-(x_0)$ containing all these points. Since, for $n = 1, 2, \ldots$, $x_n \in C \cap L_X$ and $\delta(C) \leqslant \delta(U) < \varepsilon$, we infer from (3.1) that x_n separates C. By (2.4), $C-(x_n)$ has only a finite number of components, and therefore there is a component U_n of $C-(x_n)$ such that $x_0 \in \overline{U}_n$. Then the boundary of U_n in U is equal to $(x_n) \cup (x_0)$. Since, for each $n \geqslant 1$, x_{n+1} separates U between the points x_n and x_0, it follows that $x_{n+1} \in U_n$ and $x_{n-1} \notin U_n$.

We shall prove that there is an index $n_0 > 1$ such that $\overline{U}_{n_0} \subset U$, i.e., such that the boundary of U_{n_0} in X is equal to $(x_{n_0}) \cup (x_0)$. Then, setting $k = n_0 - 1$, $l = n_0$ and $m = n_0 + 1$, we shall immediately obtain (3.3). For this purpose let $F_n = \overline{U}_n \cap U$ and $F = \bigcap_{n=1}^{\infty} F_n = U \cap \bigcap_{n=1}^{\infty} \overline{U}_n$. Then none of the points x_n $(n = 1, 2, \ldots)$ belongs to F. Considering U as a space and using (2.1), we have $\mathrm{Bd}(F) \subset \overline{\bigcup_{n=1}^{\infty} x_n} \cap F = (x_0)$. Consequently, $F = (x_0)$, because F cannot contain any component of $U-(x_0)$. Now, observe that the formula $x_{n+1} \in U_n$ implies that $U_{n+1} \subset U_n$, because $U_{n+1} \cup (x_{n+1})$ is a connected subset of $C-(x_n)$ and U_n is a component of $C-(x_n)$. Consequently, the sets $\overline{U}_n$ form a decreasing sequence of continua, and therefore $\bigcap_{n=1}^{\infty} \overline{U}_n$ is a continuum. Since, as we have proved, $F = U \cap \bigcap_{n=1}^{\infty} \overline{U}_n = (x_0)$ and U is a neighbourhood of x_0, we conclude that $\bigcap_{n=1}^{\infty} \overline{U}_n = (x_0)$. Thus, there is an index $n_0 > 1$ such that $\overline{U}_{n_0} \subset U$, which completes the proof.

(3.4) *L_X is a closed subset of X.*

Suppose that, on the contrary, there is a sequence of distinct points $x_n \in L_X$, $n = 1, 2, \ldots$, such that $\lim_{n=\infty} x_n = x_0 \in X - L_X$. By (3.3), there is an index $l > 1$ such that the set $(x_l) \cup (x_0)$ separates X. Since X is a cyclic space, the point x_0 belongs to the boundary of each component of $X-(x_l)-(x_0)$. Thus, if $U \subset X$ is a region such that $x_0 \in U$ and $x_l \notin U$, then x_0 separates U. Consequently, $x_0 \in L_X$.

(3.5) *If E is a component of $X - L_X$, then the set* $\mathrm{Bd}(E)$ *contains (at most) finitely many points.*

Suppose that (3.5) is not true. Then there is a sequence of distinct points $x_n \in \mathrm{Bd}(E)$, $n = 1, 2, \ldots$, such that $\lim_{n=\infty} x_n = x_0 \in \mathrm{Bd}(E)$. By (3.4), E is open and therefore $\mathrm{Bd}(E) \subset L_X$. Thus $x_n \in L_X$ and, by (3.3), there are three indices k, l, m such that $1 \leqslant k < l < m$ and such that the set $(x_l) \cup (x_0)$ separates X between the points x_k and x_m. Now, this is impossible, because $E \cup (x_k) \cup (x_m)$ is a connected subset of $X - (x_l) - (x_0)$.

4. The strongly cyclic elements of a space $X \in \alpha$. As mentioned in Introduction, A. Kosiński in [10] has defined the strongly cyclic elements of a space X and has formulated some of their properties for $X \in \mathrm{ANR}$. Now, we shall recall the definition of Kosiński, first assuming only that X is a locally connected continuum.

The strongly cyclic elements (abbreviated to s.c.e.'s) of a space X are the following sets:

1° For every point $x \in L_X$, the set (x).

2° For every point $a \in X - L_X$, the set E_a consisting of all points $x \in X$ such that no finite subset F of $X-(x)-(a)$ separates X between the points x and a.

It is clear that the s.c.e.'s of X cover X and that this covering is a refinement of the covering of X by cyclic elements. Notice also that in the definition of the set E_a we can restrict ourselves to the consideration of the sets F which separate X irreducibly between x and a. Such a set F is contained in the closure of each component of $X - F$, which implies that $F \subset L_X$, because if $x \in F$, then x separates each region $U \subset X$ such that $x \in U \subset (X-F) \cup (x)$. Thus, in the sequel we shall assume that the set F considered in 2° is a subset of L_X. It follows that $b \in E_a - L_X$ implies that $E_b = E_a$. Indeed, if $x \in E_a$, then for every finite set $F \subset L_X - (x)$ the points a, x and b lie in one component of $X - F$, which implies $x \in E_b$ and, similarly, $x \in E_b$ implies $x \in E_a$.

A connected space X containing more than one point will be called *strongly cyclic* if X is not separated by any finite set $F \subset X$. Thus $L_X = \emptyset$ implies that X is strongly cyclic, but not conversely. The s.c.e. of X which contain more than one point will be called *the true strongly cyclic elements* and abbreviated to t.s.c.e.'s

The following proposition is obvious.

(4.1) *If $a \in X - L_X$ and A is a strongly cyclic subset of X containing the point a, then $A \subset E_a$.*

Remark 1. We shall prove next (see (4.11)) that the t.s.c.e.'s of a space $X \in \alpha'$ are subsets of X maximal with respect to the property of being strongly cyclic spaces. The following example shows that this is not so for arbitrary locally connected cyclic continua. Let $X = D_1 \cup D_2 \cup \bigcup_{i=1}^{\infty} I_i$, where D_i are discs such that $D_1 \cap D_2 = \dot{D}_1 \cap \dot{D}_2 = (x_0)$, $\mathrm{Lim}_{i=\infty} I_i = (x_0)$ and I_i, $i = 1, 2, \ldots$, are disjoint arcs such that $\mathring{I}_i \cap (D_1 \cup D_2) = \emptyset$, one end-point of I_i belongs to $\dot{D}_1 - (x_0)$ and the other to $\dot{D}_2 - (x_0)$. Then $D_1 \cup D_2$ is a t.s.c.e. of X which is evidently separated by x_0. However, a positive answer to the following question seems to be probable:

PROBLEM. *Let X be any locally connected continuum. Is each s.c.e. of X a retract of X? (If $X \in \alpha$, then the answer is positive by* (4.4).)

Remark 2. It can easily be proved that a connected polyhedron P (containing more than one point) is strongly cyclic if and only if for each triangulation $\mathcal{C}$ of P each 1-simplex of $\mathcal{C}$ is a face of a 2-simplex of $\mathcal{C}$ and if P is strongly connected in dimension 2, this means that for each triangulation $\mathcal{C}$ of P and for each pair of two simplexes $\sigma, \sigma' \in \mathcal{C}$ there is in $\mathcal{C}$ a finite sequence of two simplexes the first of which is σ and the second σ' such that simplexes with successive indices intersect in a common 1-dimensional face. Thus, the t.s.c.e. of a connected polyhedron P are the subpolyhedra of P maximal with respect to that property.

In the sequel of this section we shall consider a fixed space $X \in \alpha'$ and we shall assume that the respective number $\varepsilon > 0$ is fixed. Since the cyclic elements of a space $X \in \alpha$ belong to α', the results are applicable to such (non-degenerate) cyclic elements.

(4.2) *The t.s.c.e.'s of X coincide with the closures of the components of $X - L_X$.*

Let $a \in X - L_X$ and let E denote the component of $X - L_X$ containing the point a. We shall prove that $\bar{E} = E_a$. Indeed, if $x \in \bar{E}$, then no finite subset of $L_X - (x)$ can separate X between x and a, whence $x \in E_a$. On the other hand, if $x \in X - \bar{E}$, then, by (3.4), the set $\mathrm{Bd}(E)$ separates X between x and $a \in E = \mathrm{Int}(E)$. This set is finite by (3.5), and therefore $x \notin E_a$, which completes the proof.

(4.3) *If E is a t.s.c.e. of X, then the set $E \cap L_X$ is finite and it does not separate E, and we have $E \cap L_X \supset \mathrm{Bd}(E) \cup L_E$. If $E \neq X$, then $\overline{\overline{\mathrm{Bd}(E)}} \geqslant 2$.*

It follows immediately from (3.4), (3.5) and (4.2) that the set $E \cap L_X$ is finite, does not separate E and contains $\mathrm{Bd}(E)$. Now, if $x \in E - L_X$, then $E - L_X$ is a neighbourhood of x in X (and in E), and therefore $x \notin L_E$, as $x \notin L_X$. Thus $L_E \subset E \cap L_X$. If $E \neq X$, then $\mathrm{Bd}(E) \neq \emptyset$, and it contains more than one point, because X is a cyclic space.

(4.4) *If E is a t.s.c.e. of X, then E is a retract of X. Consequently, E is a locally connected continuum (moreover, $E \in \alpha$) and if $X \in \mathrm{ANR}$, then also $E \in \mathrm{ANR}$.*

This is an immediate consequence of (2.3), (4.2) and (4.3).

(4.5) *If $\hat{E}$ is the union of a finite number of t.s.c.e.'s of X, then $X-\hat{E}$ has (at most) a finite number of components, the boundary of each one being a finite subset of $\hat{E}$ and the closure of each one being a retract of X.*

If C is a component of $X-\hat{E}$, then, by (4.3), $\mathrm{Bd}(C)$ is finite as a subset of the finite set $\mathrm{Bd}(\hat{E})$ and, by (2.3), $\bar{C}$ is a retract of X. Suppose that there is an infinite sequence $C_1, C_2, \ldots$ of components of $X-\hat{E}$. Since the set $\mathrm{Bd}(\hat{E})$ has only a finite number of subsets, we can assume that $\mathrm{Bd}(C_i) = \mathrm{Bd}(C_j)$ for all i and j. Since X is a cyclic space, $\mathrm{Bd}(C_i)$ contains at least two points. Let

$$\eta = \min[\varrho(x, y)|\ x \neq y;\ x, y \in \mathrm{Bd}(C_i)].$$

Since $\bar{C}_i$ is connected, there is a point $x_i \in C_i$ such that $\varrho(x, x_i) \geqslant \eta/2$ for each point $x \in \mathrm{Bd}(C_i)$. Let x_0 denote the limit of a subsequence of the sequence $\{x_i\}_{i=1}^{\infty}$. Then $x_0 \in \mathrm{Ls}(C_i) - \mathrm{Ls}(\mathrm{Bd}(C_i))$, which is a contradiction by [12], p. 169.

(4.6) *If $E_1, E_2, \ldots$ is a sequence of distinct t.s.c.e.'s of X, then $\lim_{n=\infty} \delta(E_n) = 0$. Consequently, the set of the t.s.c.e.'s of X is at most countable.*

First notice that the set $\bigcup_{i=1}^{\infty} \mathrm{Bd}(E_i)$ must be infinite. Indeed, if it is not so, then, replacing the sequence $\{E_i\}_{i=1}^{\infty}$ by a subsequence if necessary, we can assume that $\mathrm{Bd}(E_i) = \mathrm{Bd}(E_1)$ for all i. Thus the sets $\mathrm{Int}(E_i)$, $i = 2, 3, \ldots$, are distinct components of $X-E_1$, which is impossible by (4.5).

Now, suppose that (4.6) is not true. Then we can assume that there is a number $\eta > 0$ such that $\delta(E_i) \geqslant \eta$ for $i = 1, 2, \ldots$ Since each set $\mathrm{Bd}(E_i)$ is finite and the set $\bigcup_{i=1}^{\infty} \mathrm{Bd}(E_i)$ is infinite, replacing the sequence $\{E_i\}_{i=1}^{\infty}$ by a subsequence if necessary, we can assume that there is a sequence of points $x_i \in \mathrm{Bd}(E_i) - \bigcup_{j<i} \mathrm{Bd}(E_j)$, $i = 1, 2, \ldots$ Thus, $x_i \neq x_j$ for $i \neq j$ and, by (4.3), $x_i \in L_X$. We can assume that $\lim_{i=\infty} x_i = x_0$. By (3.3), there are three indices k, l, m such that $1 \leqslant k < l < m$ and such that the set $(x_l) \cup (x_0)$ separates X between the points x_k and x_m, the diameter of the component C of $X-(x_l)-(x_0)$ containing the point x_m being less than η. By (3.4), $x_0 \in L_X$ and, by (4.2), $E_m-(x_0)-(x_l)$ is a connected subset of $X-(x_l)-(x_0)$ containing the point x_m. Consequently, $E_m \subset \bar{C}$, and therefore $\delta(E_m) \leqslant \delta(C) < \eta$, which yields a contradiction.

(4.7) *Each simple closed curve $S \subset X$ such that $\delta(S) < \varepsilon$ is contained in a t.s.c.e. of X.*

First, notice that the set $S \cap L_X$ must be finite. Indeed, if it is not so, then there is a sequence of distinct points $x_i \in S \cap L_X$, $i = 1, 2, \ldots$, such that $\lim_{i=\infty} x_i = x_0 \in S$. By (3.3), there are three indices k, l, m such that $1 \leqslant k < l < m$ and such that the set $(x_l) \cup (x_0)$ separates X between the points x_k and x_m. Let A and K denote respectively the closures of the components of $X-(x_l)-(x_0)$ and of $S-(x_l)-(x_0)$ containing x_k. Evidently, there are a retraction r_1 of A onto K and a retraction r_2 of $\overline{X-A}$ onto $\overline{S-K}$. Since $A \cap \overline{X-A} = (x_l) \cup (x_0) = K \cap \overline{S-K}$, the retractions r_1 and r_2 determine a retraction r of X onto S, which contradicts the definition of the number ε, because $\delta(S) < \varepsilon$.

Now, suppose that (4.7) is not true. Then, by (4.2), there are two components L_1 and L_2 of $S-L_X$ which lie in different components of $X-L_X$ and for which $\bar{L}_1 \cap \bar{L}_2 \neq \emptyset$. Let $x_0 \in \bar{L}_1 \cap \bar{L}_2$ and let E denote the component of $X-L_X$ containing L_1. Thus, if $V \subset X$ is any region such that $V \cap \mathrm{Bd}(E) = (x_0)$, then x_0 separates V between some two points of which one belongs to L_1 and the other to L_2. On the other hand, since $\delta(S) < \varepsilon$, there is a region $U \subset X$ such that $U \supset S$ and $\delta(U) < \varepsilon$. Then x_0 cannot separate U between any two points belonging to $S-(x_0)$. Thus we have obtained a contradiction with (3.1).

(4.8) *If $Y \in \alpha_0'$ (specially, if Y is a cyclic AR), then $L_Y = \emptyset$ and the only s.c.e. of Y is equal to Y. Consequently, if E is a t.s.c.e. of X such that $\delta(E) < \varepsilon$, then $E \in \alpha_0'$, $L_E = \emptyset$ and $L_X \cap E = \mathrm{Bd}(E)$.*

Suppose that, contrary to (4.8), there is a point $y \in L_Y$. Since, for the space Y, we can assume that $\varepsilon = \delta(Y)+1$, it follows from (3.1) that y separates Y. Now, this contradicts the assumption that Y is a cyclic space. If E is a t.s.c.e. of X such that $\delta(E) < \varepsilon$, then, by (4.4), $E \in \alpha_0$ and, by (4.3), E is a cyclic space, whence $E \in \alpha_0'$. By (4.3), $\mathrm{Bd}(E) \subset E \cap L_X$. If $x \in \mathrm{Int}(E)$, then $x \notin L_E$ implies that $x \notin L_X$.

(4.9) *Let $E_1, E_2, \ldots$ denote the sequence (finite or not) of all t.s.c.e.'s of X. Then, for each i, there is a tree $T_i \subset E_i$ such that the set of the end-points of T_i is equal to $\mathrm{Bd}(E_i)$ (if $\mathrm{Bd}(E_i) = \emptyset$ then $T_i = \emptyset$), the set $G = L_X \cup \bigcup_i T_i$ being a graph (or the empty set). Consequently, L_X is a closed subset of a graph $G \subset X$ and, if $X = L_X$, then X is a graph.*

(2.2) and (4.3) imply the existence of the trees T_i. Let

$$G_0 = [L_X - \bigcup_i \mathrm{Int}(E_i)] \cup \bigcup_i T_i .$$

If r_i is a retraction of E_i onto T_i, then, by (4.6), the function $r\colon X \to G_0$ defined as

$$r(x) = \begin{cases} x & \text{if} \quad x \in L_X - \bigcup_i \operatorname{Int}(E_i), \\ r_i(x) & \text{if} \quad x \in E_i \end{cases}$$

is a retraction of X onto G_0. Consequently, G_0 is a locally connected continuum. Since $G_0 \cap E_i = T_i$, it follows from (4.7) that G_0 contains no simple closed curve S such that $\delta(S) < \varepsilon$, and therefore G_0 is a local dendrite. Since X is a cyclic space, X has no end-point. Consequently, one can also easily show that G_0 has no end-point, and therefore G_0 must be a graph.

Now, it follows from (4.3), (4.6) and (4.8) that the set $L_X \cap \bigcup_i \operatorname{Int}(E_i)$ is finite. Since G is the union of G_0 and of that set, we conclude that G is a graph.

(4.10) *If E is a t.s.c.e. of X and if the diameter of E is sufficiently small, then the set $\operatorname{Bd}(E) = E \cap L_X$ contains exactly two points.*

By (4.3) and (4.8), if $\delta(E)$ is sufficiently small, then $\operatorname{Bd}(E) = E \cap L_X$ and $\overline{\overline{\operatorname{Bd}(E)}} \geqslant 2$. If (4.10) is not true, then infinitely many trees T_i described in (4.9) must contain ramification points and therefore the graph G from (4.9) contains infinitely many ramification points, which yields a contradiction.

(4.11) *The t.s.c.e.'s of X coincide with the subsets of X which are maximal with respect to the property of being strongly cyclic spaces.*

Let A be a strongly cyclic subset of X. Since no graph contains a strongly cyclic subset, it follows from (4.9) that $A - L_X \neq \emptyset$. Fix a point $a \in A - L_X$. Then, by (4.1), $A \subset E_a$. On the other hand, it follows from (4.3) that E_a is a strongly cyclic space.

(4.12) *If $X \in$ ANR, then all the s.c.e.'s of X are ANR-sets and almost all are AR-sets.*

It follows from [4], p. 101 that there is a number $\delta > 0$ such that each subset of X of diameter less than δ is contractible in X. By (4.6), this holds for almost all s.c.e.'s of X. By (4.4), all s.c.e. of X are retracts of X, and therefore almost all of them are contractible in themselves. Consequently, (4.12) follows from [4], p. 96.

Conversely, we shall prove that:

(4.13) *If $X \in \alpha$ (we do not assume that X is a cyclic space) and if all s.c.e.'s of X are ANR-sets and almost all are AR-sets, then X is also an ANR-set.*

Let Z be a non-degenerate cyclic element of X. Let $G \subset Z$ denote the graph constructed for Z as described in (4.9). Consider an arc $I \subset G$. Let $A(I)$ denote the union of I and of all t.s.c.e.'s E of Z such that $E \in$ AR and $E \cap G = E \cap I$. It follows from (4.9) that $E \cap I$ is a subarc of I and if $E, E' \subset A(I)$ are two different t.s.c.e.'s of Z such that $E \cap E' \neq \emptyset$, then $E \cap E'$ is a point which is an end-point of both arcs $E \cap I$ and $E' \cap I$. Thus, each t.s.c.e. $E \subset A(I)$ is a cyclic element of $A(I)$. Consequently, $A(I)$ is a locally connected continuum all cyclic elements of which are AR-sets. Hence, Borsuk's theorem [3] implies that $A(I) \in$ AR. Let A denote the subset of Z which is the union of G and of all t.s.c.e.'s E of Z such that $E \in$ AR and $E \cap G$ is an arc. Then there is a finite number of arcs $I_1, \ldots, I_n \subset G$ such that $I_i \cap I_j = \dot{I}_i \cap \dot{I}_j$, for $i \neq j$ and such that $A = G \cup \bigcup_{i=1}^{n} A(I_i)$. Since $A(I_i) \cap A(I_j) = I_i \cap I_j$, for $i \neq j$, we infer from the addition theorem for ANR-sets (see [4], p. 90) that $A \in$ ANR. It follows from (4.9), (4.10) and from the assumptions of (4.13) that there is only a finite number of t.s.c.e.'s of Z which are not contained in A. Since Z is the union of A and of all these remaining t.s.c.e.'s of Z and since each such a t.s.c.e. intersect A on a subgraph of G, we conclude (using once more the addition theorem for ANR-sets) that $Z \in$ ANR.

Thus all non-degenerate cyclic elements of X are ANR-sets. It follows from [12] (p. 238, No. 9 and p. 263, No. 15) that almost all of them belong to α_0', and therefore, by (4.8), they are t.s.c.e.'s of X. Consequently, all cyclic elements of X are ANR-sets and almost all are AR-sets, which, by an easy extension of Borsuk's theorem [3] (see [18], p. 292), implies that X is an ANR-set.

Remark. Let us notice that (4.13) without the assumption that $X \in \alpha$ is false. Actually, all s.c.e.'s of the space X from the Remark 1 below (4.1) are AR-sets, but X is not an ANR.

5. A topological characterization of strongly cyclic polyhedra which are embeddable in S^2. First, notice that:

(5.1) *If X is a cyclic locally connected continuum which does not contain any homeomorphic images of the graph of Kuratowski K_1, then X contains no 2-umbrella.*

Otherwise, suppose that there are a disk $Q \subset X$ and an arc $L \subset X$ such that $Q \cap L = (p_0)$ and $p_0 \in \mathring{Q} \cap \dot{L}$. Since X is a cyclic space, the set $X - (p_0)$ is arcwise connected, and therefore there is an arc $K \subset X - (p_0)$ such that $\dot{K} = (p_1) \cup (p_2)$, where $p_1 \in L - (p_0)$ and $p_2 \in Q - (p_0)$. Replacing the arc K by a subarc if necessary, we can assume that $\mathring{K} \cap (L \cup Q) = \emptyset$. Then it is easily seen that the set $Q \cup L \cup K \subset X$ contains a homeomorphic image of the graph K_1, which yields a contradiction.

Suppose that S^2 has the polyhedral structure determined by a homeomorphism of S^2 with the boundary of a 3-simplex. The following theorem is the main result of this section.

THEOREM 2. *Let X be a strongly cyclic space. Then X is homeomorphic with a polyhedron $P \subset S^2$ if and only if X satisfies the following conditions:*

1° $X \in \alpha$.

2° X *contains no homeomorphic images of the graphs K_1 and K_2.*

The necessity of these conditions is obvious. In order to prove that they are also sufficient we shall first establish a lemma, which provides a characterization of a disk. This lemma can easily be deduced from Claytor's results [6] and [7], but we shall give here an independent proof.

LEMMA 1. *Let X be a cyclic locally connected continuum which does not contain any homeomorphic images of the graph K_1. Suppose that there is a simple closed curve $S \subset X$ such that S does not separate X and S is not a retract of X. Then X is a disk (whose boundary is equal to S).*

Proof. An arc $L \subset X$ will be said to span S if $\dot{L} \subset S$ and $\mathring{L} \subset X - S$. First, notice that X contains at least one such an arc. Indeed, it follows easily from the assumptions that $X - S$ is connected and non-empty that $\mathrm{Bd}(X-S)$ contains at least two points. By [12], p. 194, the set of the points belonging to $\mathrm{Bd}(X-S)$ which are accessible from $X-S$ is a dense subset of $\mathrm{Bd}(X-S)$. Thus, there are two different points $x_1, x_2 \in S$ which are accessible from $X-S$. Since $X-S$ is arcwise connected, it is easy to see that there is an arc $L \subset X$ such that $\dot{L} = (x_1) \cup (x_2)$ and $\mathring{L} \subset X-S$, which implies that L spans S. By the van Kampen characterization of a disk (see [9], p. 80), it remains to show that each arc $L \subset X$ spanning S irreducibly separates X.

First, we shall prove that:

(5.2) *Each arc $L \subset X$ spanning S separates X between the components of $S-L$.*

Suppose that (5.2) does not hold. Then there is a component C of $X-L$ containing $S-L$. Since C is arcwise connected, there is an arc $L_1 \subset C \subset X-L$ joining a point belonging to one component of $S-L$ with a point belonging to the other. Since $L_1 \cap S = L_1 \cap (S-L)$, replacing the arc L_1 by a subarc if necessary, we can assume that the arc L_1 spans S. Thus $\mathring{L} \cup \mathring{L}_1 \subset X-S$. Since S does not separate X, there is an arc $L_2 \subset X-S$ joining a point belonging to $\mathring{L}$ with a point belonging to $\mathring{L}_1$. Replacing also the arc L_2 by a subarc if necessary, we can assume that $\mathring{L}_2 \cap (L \cup L_1) = \emptyset$. It is easily seen that the set $S \cup L \cup L_1 \cup L_2 \subset X$ is homeomorphic with the graph K_1, which yields a contradiction.

Next, we shall prove that:

(5.3) *Each arc $L \subset X$ spanning S irreducibly separates X between the components of $S-L$.*

Let S_1 and S_2 denote the components of $S-L$. Suppose that (5.3) is not true. Thus, there exists a set $L' = \bar{L}' \subset L \neq L'$ which separates X between S_1 and S_2. Replacing L' by a larger set (which is also a both proper and closed subset of L) if necessary, we can assume that both $L_1 = S_1 \cup L'$ and $L_2 = S_2 \cup L'$ are arcs. Let C_1 denote the component of $X-L'$ containing S_1. Then there is a retraction r_1 of the set $C_1 \cup L' = \bar{C}_1 \cup L'$ onto the arc $L_1 = S_1 \cup L'$. Since $X - C_1 \supset S_2 \cup L' = L_2$, there is a retraction r_2 of the set $X - C_1 = \overline{X - C_1}$ onto the arc L_2. Since $(X-C_1) \cap (C_1 \cup L') = L' = L_1 \cap L_2$, it follows that the function $r\colon X \to L_1 \cup L_2$ defined as

$$r(x) = \begin{cases} r_1(x) & \text{if} \quad x \in C_1 \cup L', \\ r_2(x) & \text{if} \quad x \in X - C_1 \end{cases}$$

is a retraction of X onto $L_1 \cup L_2 = S_1 \cup S_2 \cup L' = S \cup L'$. Since S is a retract of $S \cup L'$, we infer that S is a retract of X, which contradicts the assumption of the lemma.

Thus, by (5.2) and (5.3), there are two different components C_1, C_2 of $X-L$ such that $C_i \supset S_i$ and $\mathrm{Bd}(C_1) = L = \mathrm{Bd}(C_2)$. In order to complete the proof, we must show that L is also the common boundary for the remaining components of $X-L$ (if they exist). Hence, if it is not so, then there exists a component C_3 of $X-L$ such that $L \neq \mathrm{Bd}(C_3) \subset L$. Since X is a cyclic space, $\mathrm{Bd}(C_3)$ contains at least two points. Moreover, since S does not separate X, $\mathrm{Bd}(C_3) - S \neq \emptyset$. By [12], p. 194, each point belonging to a dense subset of $\mathrm{Bd}(C_3)$ is accessible from C_3. We conclude that there exists an arc $I_3 \subset \bar{C}_3$ such that $\mathring{I}_3 \subset C_3 \subset (X-L)-(S_1 \cup S_2) = X-L-S$, $\dot{I}_3 \subset \mathrm{Bd}(C_3) \subset L$ and $\dot{I}_3 - S \neq \emptyset$. Let L' denote the component of $L - I_3$ bounded by both end-points of I_3. Since, for $i = 1, 2$, $S_i \subset C_i$ and $L' \subset \mathrm{Bd}(C_i)$, we infer from [12], p. 194 that there is an arc $I_i \subset \bar{C}_i$ such that one end-point of I_i belongs to S_i and the other to L' and such that $\mathring{I}_i \subset C_i - S_i \subset X-L-S-I_3$. Evidently, $(I_1 - L) \cap (I_2 - L) = \emptyset$. It is easy to see from our construction that the graph $S \cup L \cup \bigcup_{i=1}^{3} I_i$ contains a subgraph homeomorphic with the graph K_1, which contradicts the assumption of the lemma. Thus the proof is complete.

LEMMA 2. *If $X \in \alpha_0'$ and X does not contain any homeomorphic images of the graph K_1, then X is either a disk or a simple surface (i.e. a set homeomorphic with S^2).*

Proof. The definition of the class α_0' and (3.1) imply that $L_X = \emptyset$. It follows that no pair of points $x, y \in X$ separates X. Suppose that X is not a simple surface. Let us apply the Bing characterization of the 2-sphere S^2 (see [2]), which says that a locally connected continuum Y is a simple surface if and only if no pair of points $x, y \in Y$ separates Y but every simple closed curve $S \subset Y$ separates Y. Thus, there is a simple closed curve $S \subset X$ which does not separate X. Since $X \in \alpha_0'$, it follows that all assumptions of Lemma 1 are satisfied and therefore X is a disk.

Now, we pass to:

The proof of theorem 2. Let X be a strongly cyclic space satisfying 1° and 2° and let us fix for X a suitable number $\varepsilon > 0$. We can assume that X is not a simple surface. It follows that:

(5.4) *X does not contain any simple surface.*

Indeed, if there is a simple surface $S \subset X$, then $X - S \neq 0$ and the arcwise connectedness of X implies that X contains a 2-umbrella, which contradicts (5.1).

We shall prove that:

(5.5) *If $S \subset X$ is a simple closed curve such that $\delta(S) < \varepsilon$, then there is exactly one component C of $X - S$ such that $\overline{C} = C \cup S$ is a disk (whose boundary is S).*

It follows from (5.4) that there could not exist two components C_1, C_2 of $X - S$ with the property described. In order to prove that such a component exists it suffices to show the existence of a component C of $X - S$ such that S is not a retract of $C \cup S$. Indeed, the remaining assumptions of Lemma 1 will be satisfied by $C \cup S$ in virtue of 2° and because the cyclicity of X implies the cyclicity of $C \cup S$, since a component of $(C \cup S) - (x)$ disjoint with $S - (x)$, i.e. contained in the open set C, could not exist for any $x \in C \cup S$.

Thus, let us suppose that, for each component C of $X - S$, S is a retract of $C \cup S$. In order to obtain a contradiction with the definition of the number ε we have to show that S is a retract of X. We shall assume that the sequence of the components $C_1, C_2, \ldots$ of $X - S$ is infinite; if it is not so, then the proof is simpler.

Since $S \in \mathrm{ANR}$, there is a neighbourhood U of S in X such that S is a retract of U. Let $r_0\colon U \to S$ be such a retraction. Since X is a locally connected continuum, there is an index n_0 such that $C_n \subset U$ for each $n > n_0$. For every $n \leqslant n_0$, let r_n be a retraction of $C_n \cup S$ onto S. It is easy to see that the function $r\colon X \to S$ defined as

$$r(x) = \begin{cases} r_n(x) & \text{if} \quad x \in C_n \cup S, \quad \text{where } 1 \leqslant n \leqslant n_0, \\ r_0(x) & \text{if} \quad x \in \bigcup\limits_{n > n_0} C_n \cup S \end{cases}$$

is a retraction of X onto S, which completes the proof of (5.5).

Now, if $S \subset X$ is a simple closed curve such that $\delta(S) < \varepsilon$, the component of $X - S$ described in (5.5) will be denoted by $E(S)$ and will be called the Euclidean component of $X - S$. Evidently, $E(S)$ is homeomorphic with E^2 and is open in X.

Next, we shall prove that:

(5.6) *If there is a set $X_0 \subset X$ containing at most one point and such that for every point $x \in X - X_0$ there is a simple closed curve $S \subset X$ such that $\delta(S) < \varepsilon$ and $x \in E(S)$, then X is a 2-manifold (without boundary), and therefore a polyhedron.*

Since the case $X_0 = \emptyset$ is trivial, we shall assume that X_0 consists of exactly one point x_0. Since $X - (x_0) \neq \emptyset$, there is at least one simple closed curve $S \subset X$ such that $\delta(S) < \varepsilon$. By the van Kampen characterization of 2-manifolds (see [9], p. 83), we have to show that each such a simple closed curve irreducibly separates X. If S is such a simple closed curve and S does not separate X, then $X - S = E(S)$, which implies that $X = E(S) \cup S$ is a disk and the assumption of (5.6) is not satisfied by any point $x \in S - (x_0)$. Thus, S separates X. If S does not irreducibly separate X, then there is a component C of $X - S$ different from $E(S)$ such that $S \neq \mathrm{Bd}(C) \subset S$. Consequently, there is a point $x_1 \in \mathrm{Bd}(C) - (x_0)$ such that $\mathrm{ord}_{x_1} \mathrm{Bd}(C) \leqslant 1$. By the assumption of (5.6), there is a simple closed curve $S_1 \subset X$ such that $\delta(S_1) < \varepsilon$, and $x_1 \in E(S_1)$. Then there is a region $U \subset X$ such that $x_1 \in U \subset \overline{U} \subset E(S_1)$ and such that the set $\mathrm{Bd}(U) \cap \mathrm{Bd}(C)$ contains at most one point. Since U is homeomorphic with a plane region and since $\mathrm{Bd}(C)$ is a both proper and closed subset of S, it follows that the set $U - \mathrm{Bd}(C)$ is connected. Since $x_1 \in U \cap \mathrm{Bd}(C)$, $U \cap C \neq \emptyset$ and therefore $U - \mathrm{Bd}(C) \subset C$. Thus $U \cap E(S) = \emptyset$, which is impossible, because U is a neighbourhood of $x_1 \in S \subset \overline{E(S)}$. We conclude that S irreducibly separates X, which completes the proof of (5.6).

Now, we are in a position to prove that X is always a 2-dimensional polyhedron. For this purpose we shall prove that:

(5.7) *For each point $x_0 \in X$ there is a neighbourhood of x_0 in X which is the union of a finite collection of disks $D_1, \ldots, D_n$ such that $i \neq j$ implies $D_i \cap D_j = (x_0)$.*

By (5.6), we can assume that there is a point $x_1 \neq x_0$ such that for each simple closed curve $S \subset X$, where $\delta(S) < \varepsilon$, $x_1 \notin E(S)$. Since X is a strongly cyclic space, we infer from (4.3) and (4.11) that the set L_X is finite. Evidently, (x_0) is a continuum, the complement of which is connected. We infer from a proposition given in [12], p. 189 that there is a continuum $H \subset X - (x_0)$ such that $H \supset (x_1) \cup \big(L_X - (x_0)\big)$, $\delta(X - H) < \varepsilon$ and such that $X - H$ is a region. Applying once more this proposition now to the continuum H, we infer that there is a locally connected continuum $F \subset X-H$ such that $x_0 \in \mathrm{Int}(F)$ and such that $X-F$ is a region. Thus, $\delta(F) < \varepsilon$ and $(x_1) \cup (L_X-(x_0)) \subset X-F$. Let U denote the component of the set $\mathrm{Int}(F)$ containing x_0. By (2.4), the set $U-(x_0)$ has a finite number of components. Every such component is a region disjoint with L_X, and therefore it is not separated by any point. It follows from [12], p. 238 that each such a component is contained in a cyclic element of F. This cyclic element contains x_0, because it is a closed subset of F. We conclude that there is a finite number of cyclic elements of F, say $D_1, \ldots, D_n$, such that $\bigcup_{i=1}^{n} D_i \supset U$ and each D_i contains x_0. Thus $\bigcup_{i=1}^{n} D_i$ is a neighbourhood of x_0 and, by [12], p. 236, $D_i \cap D_j = (x_0)$ for $i \neq j$.

In order to complete the proof of (5.7) it remains to show that each set D_i is a disk. By [12], p. 236 and p. 238, each D_i is a cyclic locally connected continuum. By assumption 2° of the theorem, D_i does not contain any homeomorphic images of the graphs K_1 and K_2. By (5.4), D_i is not a simple surface. Thus, if we prove that no simple closed curve $S \subset D_i$ is a retract of D_i, Lemma 2 will yield the conclusion that D_i is a disk.

Consider any simple closed curve $S \subset D_i$. Since $D_i \subset F$ and $\delta(F) < \varepsilon$, we infer that $\delta(S) < \varepsilon$. Since $E(S)$ is open in X and since $E(S) \cap X-F = (E(S) \cup S) \cap X-F = \overline{E(S)} \cap X-F$, the set $E(S) \cap X-F$ is a both open and closed subset of $X-F$. Since $X-F$ is a region, the inequality $E(S) \cap X-F \neq \emptyset$ implies $X-F \subset E(S)$. But this inclusion is impossible, because $x_1 \in X-F$. Consequently, $\overline{E(S)} \subset F$. Since D_i is a cyclic element of F containing S, we conclude from [12], p. 238 that $\overline{E(S)} = E(S) \cup S \subset D_i$. Thus, S is not a retract of D_i (because it is not a retract of $\overline{E(S)}$), which completes the proof of (5.7).

As A. Kosiński has proved (see [10], p. 26), the property of being a 2-dimensional polyhedron is a local one for the class of compacta. Thus, (5.7) implies that X is a 2-dimensional polyhedron. Now, the assumptions of the theorem, (5.1) and the theorem of Mardešić and Segal [14] imply that X is embeddable in S^2, which completes the proof.

6. A topological characterization of polyhedra which are embeddable in S^2. First, we shall prove the following

THEOREM 3. *A space X is homeomorphic with a cyclic polyhedron $P \subset S^2$ if and only if X satisfies the following conditions*:

1° $X \in \alpha'$,

2° *X has a finite number of t.s.c.e.'s, i.e. the set $X-L_X$ has a finite number of components.*

3° *X does not contain any homeomorphic images of the graphs K_1 and K_2.*

Proof. Since the necessity of these conditions is trivial, we shall only prove that they are sufficient. Let X satisfy 1°, 2° and 3°. Then, by (4.4), (4.11) and by Theorem 2, each t.s.c.e. of X is a (2-dimensional) polyhedron. By (4.2) and (4.3), two different t.s.c.e.'s of X intersect in a (at most) finite subset of L_X. It follows from 2° that the union $\hat{E}$ of all t.s.c.e.'s of X is a polyhedron. By (4.5) and (4.9), the set $X-\hat{E}$ has a finite number of components, the closure of each being a graph (and therefore a polyhedron) and the boundary of each being a finite subset of $\hat{E}$. It follows that X is a polyhedron. By 1°, 3° and (5.1), X contains no homeomorphic images of K_1 and K_2 and no 2-umbrella. Thus, the Mardešić–Segal embedding theorem for polyhedra (see [14]) implies that X is embeddable in S^2, which completes the proof.

THEOREM 4. *A space X is homeomorphic with a connected polyhedron $P \subset S^2$ if and only if X satisfies the following conditions*:

1° $X \in \alpha$,

2° *X has a finite number of t.s.c.e.'s and a finite number of end-points,*

3° *X contains no 2-umbrella and no homeomorphic images of the graphs K_1 and K_2.*

Proof. As previously, we shall only prove that these conditions are sufficient. It follows from Theorem 3 that each cyclic element of X is a polyhedron. Since, by 2°, X has only a finite number of non-degenerate cyclic elements and since, by [12], p. 236, two different cyclic elements intersects in a set which contains at most one point, it follows that the union $\hat{Z}$ of the non-degenerate cyclic elements of X is a polyhedron. If C is a component of $X-\hat{Z}$, then, by [12], p. 231 and p. 239, $\overline{C}$ is a locally connected continuum. By [12], p. 238, No. 10, $\overline{C}$ contains no simple closed curve, and therefore it is a dendrite. Each end-point of $\overline{C}$ either is an end-point of X or belongs to a non-degenerate cyclic element of X. Since, by [12], p. 236, No. 6, every arc joining two different points belonging to one cyclic element is contained in that cyclic element, it follows that no two different end-points of $\overline{C}$ belong to the same cyclic element of X. Consequently, 2° implies that $\overline{C}$ has a finite number of end-points, and therefore it is a tree. Each point $x \in \mathrm{Bd}(C) = \overline{C}-C$ is an end-point of $\overline{C}$, because otherwise $\overline{C}$ would contain a simple closed curve passing through that point, which is impossible. Thus, $\mathrm{Bd}(C)$ is a finite subset of $\hat{Z}$. Consequently, in order to conclude that X is a polyhedron it remains to show that the set $X-\hat{Z}$ has a finite number of components.

Indeed, by 2°, there is only a finite number of components of $X-\hat{Z}$ containing end-points of X. If C is a component of $X-\hat{Z}$ containing no end-point of X, then $\overline{C}$ contains at least two end-points, which belong to different non-degenerate cyclic elements of X. It follows from [12], p. 236, No. 6, that the boundary of no component of $X-\hat{Z}$ different from C can contain a pair of points belonging to the same pair of non-degenerate cyclic elements of X. Thus, by 2°, $X-\hat{Z}$ has a finite number of components and therefore X is a polyhedron. Now, 3° and the Mardešić–Segal theorem [14] imply that X is embeddable in S^2, which completes the proof.

COROLLARY. *A space X is homeomorphic with a contractible polyhedron $P\subset S^2$ if and only if X satisfies* 2° *and, instead of* 1° *and* 3°, *the following two conditions*:

1′ $X\in\alpha_0$,

3′ *X contains neither a 2-umbrella, nor any homeomorphic images of the graphs K_1 and K_2, and X is not a simple surface.*

Proof. Let X satisfy 1′, 2° and 3′. Then, by Lemma 2 (see Section 5), each non-degenerate cyclic element of X is a disk, because if a cyclic element of X different from X were a simple surface, then X would contain a 2-umbrella. Thus, each cyclic element of X is an AR-set, which implies (see [3]) that X is also an AR-set. Thus, by Theorem 4, X is a contractible polyhedron embeddable in S^2, which completes the proof.

Other corollaries to Theorem 4 concerning the topological characterization of arbitrary polyhedra which are embeddable (or quasi-embeddable) in S^2 or E^2 can be obtained by easy modifications of conditions 1°, 2° and 3° in Theorem 4. Namely, conditions 1° and 2° now have to be satisfied by the components of X (X is assumed to be a locally connected compactum). The suitable modifications of 3° are the same as the modifications of condition (c) in Theorem 1 of [13], given in Theorems 4 and 5 of that paper. These corollaries are obtained from Theorem 4 in the same way as Theorems 4 and 5 are obtained from Theorem 1 in [13].

Remark. In the proof that any space X satisfying the assumptions of Theorem 3 is a polyhedron, we have applied only conditions 1°, 2° and the fact that each t.s.c.e. of X is a polyhedron. In the proof that any space X satisfying the assumptions of Theorem 4 is a polyhedron, we have applied conditions 1° and 2° and the fact that each non-degenerate cyclic element of X is a polyhedron. Since each non-degenerate cyclic element of any space X satisfying conditions 1° and 2° of Theorem 4 satisfies conditions 1° and 2° of Theorem 3, it follows that the following proposition is true:

(6.1) *If X satisfies the conditions* 1° *and* 2° *of Theorem* 4 *and if each t.s.c.e. of X is a polyhedron, then X is also a polyhedron.*

7. A characterization of ANR-sets which are embeddable in S^2. In this section, assuming Theorem 1 (as formulated in Section 1) to be true, we shall deduce some corollaries concerning the case of $n=2$.

THEOREM 5. *A connected space X is homeomorphic with an* ANR-*set $Y\subset S^2$ if and only if X satisfies the following two conditions*:

1° $X\in\alpha$.

2° *X contains neither a 2-umbrella nor any homeomorphic images of the graphs K_1 and K_2.*

Proof. The necessity of 1° and 2° is trivial. Thus, let us assume that X satisfies these conditions. We can and do assume that X is not a simple surface. Then X does not contain any simple surface, because otherwise X would contain a 2-umbrella. Each t.s.c.e. E of X satisfies the assumptions of Theorem 2, and therefore, it is a polyhedron embeddable into E^2. If $\delta(E)<\varepsilon$, then E satisfies the assumptions of Lemma 2 of Section 5, and therefore E is a disk. Thus, all s.c.e.'s of X are ANR-sets and almost all are AR-sets, which implies—by (4.13)—that $X\in$ ANR.

If E is a s.c.e. of X and $G\subset X$ is a graph such that $E\cap G$ is a finite set, then $E\cup G$ is a polyhedron. It follows from 2° and from the theorem of Mardešić and Segal [14] that $E\cup G$ is embeddable in E^2, because $E\cup G\subset X$ is not a simple surface. Consequently, X satisfies the assumptions of Theorem 1 and therefore there is an upper semi-continuous decomposition $\mathfrak{D}$ of E^2 whose all elements are trees and for which X is embeddable in $E^2/\mathfrak{D}$. By Moore's well-known theorem [16], $E^2/\mathfrak{D}$ is homeomorphic with E^2 and therefore X is embeddable in E^2, which completes the proof.

COROLLARY. *A space X is a homeomorphic with an* AR-*set $Y\subset S^2$ if and only if X satisfies the following two conditions*:

1′ $X\in\alpha_0$,

2′ *X contains neither a 2-umbrella nor any homeomorphic images of the graphs K_1 and K_2 and X is not a simple surface.*

Proof. Let Z be a non-degenerate cyclic element of X, where X satisfies 1′ and 2′. Then, by [12], p. 263, Z is a retract of X and therefore $Z\in\alpha_0'$. Since Z, as a subset of X, satisfies 2′, it follows from Lemma 2 of Section 5 that Z is a disk. Thus, all cyclic elements of X are AR-sets, which implies (by Borsuk's theorem [3]) that $X\in$ AR. By Theorem 5, X is embeddable in S^2.

Other corollaries to Theorem 5, yielding a characterization of arbitrary ANR-sets which are embeddable in E^2 or S^2, are easy to obtain if one omits the connectivity assumption in 1° and if one modifies 2° in the same way as in [18] (in the deduction of Corollaries 1 and 2 from Theorem 3; see [18], p. 291).

Remark. It follows from Theorem 5 that any space $X\in\alpha$ is embeddable in S^2 (E^2) if and only if it is quasi-embeddable in S^2 (E^2); more exactly, X is quasi-embeddable in S^2 if and only if X satisfies 2°, and X is quasi-embeddable in E^2 if and only if X satisfies $2'$ (cf. [13] and [18], p. 291). We shall prove in the next paper [19] that the same conditions characterize arbitrary locally connected continua which are quasi embeddable in S^2 or E^2.

8. Reduction of Theorem 1 to a lemma. First, notice that Theorem 1 is true for $n = 1$. The assumptions of Theorem 1 imply in this case that X contains no simple closed curve and X has no ramification point. Thus X is an arc (or a point), which evidently is embeddable in E^1.

Therefore, in the sequel we shall assume that $n > 1$. The subsets of E^n will be marked with "primes". In this section we shall prove that Theorem 1 follows from

LEMMA A. *Let X be a space of the class α' satisfying the assumptions of Theorem 1 (except the assumption that $X \in$ ANR). Then there is a space $X' \subset E^n$ belonging to α' and a map g from X' onto X such that:*

1° *All non-degenerate inverse sets $g^{-1}(x)$ are trees and almost all are arcs.*

2° *For every $\eta > 0$ there is only a finite number of points $x \in X$ such that $\delta(g^{-1}(x)) \geqslant \eta$.*

3° *The t.s.c.e.'s of X' are in a one-to-one correspondence with the t.s.c.e.'s of X, so that, for each t.s.c.e. E' of X', the map $g|E'$ is a homeomorphism of E' onto the corresponding t.s.c.e. of X.*

Remark 1. By 3° and by (4.13), $X \in$ ANR implies that $X' \in$ ANR.

Remark 2. In the same way as that followed in deriving Theorem A from Lemma B given below (cf. [18] (p. 296)), one can derive from Lemma A the following corollary, which can be named the embeddability theorem for the spaces of the class α':

COROLLARY TO LEMMA A. *If X is a space of the class α' satisfying the assumptions of Theorem 1 (except the assumption that $X \in$ ANR), then X is embeddable in the space $E^n/\mathfrak{D}$, where $\mathfrak{D}$ is a null-decomposition of E^n such that all the non-degenerate elements of $\mathfrak{D}$ are trees and almost all are arcs.*

In the proof that Lemma A implies Theorem 1, we shall make use of the following lemma, which has been proved in [18] (p. 296).

LEMMA B. *If X is a connected ANR containing no n-umbrella and if the cyclic elements of X are embeddable in E^n, then there exist a locally connected continuum $X' \subset E^n$ and a map g from X' onto X such that:*

$1'$ *The non-degenerate inverse sets $g^{-1}(x)$ are arcs,*

$2'$ *Identical with 2° of Lemma A,*

$3'$ *The non-degenerate cyclic elements of X' are in a one-to-one correspondence with the non-degenerate cyclic elements of X such that if Z' corresponds to Z, then the map $g|Z'$ is a homeomorphism of Z' onto Z.*

We shall first prove that:

(8.1) *The set X' and the map $g\colon X' \to X$ in Lemma B can be chosen so that they satisfy also the following two conditions:*

$4'$ *If Z is a non-degenerate cyclic element of X and $x_0 \in \operatorname{Int}(Z)$, then $g^{-1}(x_0)$ is a point.*

$5'$ *Given a finite number of non-degenerate cyclic elements of X, $Z_1, \ldots, Z_k$, if Z_i corresponds to Z'_i, then, for every point $x \in Z_i - \bigcup [Z_j | 1 \leqslant j \leqslant k,\ j \neq i]$ such that $g^{-1}(x)$ is an arc, the point $(g|Z'_i)^{-1}(x)$ is one of its end-points.*

First, we are going to prove that $4'$ can be fulfilled. For this purpose, suppose that X' and g satisfy Lemma B. Let Z be a non-degenerate cyclic element of X, $x_0 \in \operatorname{Int}(Z)$ and let Z' correspond to Z. Suppose that $I' = g^{-1}(x_0)$ is an arc. In virtue of $3'$, $I' \cap Z' = (g|Z')^{-1}(x_0)$ is a point x'_0. Thus, each component of $I' - (x'_0)$ is contained in a component of $X' - Z'$, which is bounded by x'_0 in virtue of [12], p. 232, No. 4. If C' is such a component of $X' - Z'$, then $I' \cap C'$ is a both closed and open subset of C'. Indeed, since $g(I' \cap C') = (x_0) \subset \operatorname{Int}(Z)$, there is a neighbourhood U' of $I' \cap C'$ in C' such that $g(U') \subset \operatorname{Int}(Z)$. If there is a point $x' \in U'$ such that $g(x') = x \neq x_0$, then $g^{-1}(x)$ cannot be connected, because $(x'_0) = \operatorname{Bd}(C') \not\subset g^{-1}(x)$ and because $(g|Z')^{-1}(x) \in g^{-1}(x) - C'$. Since this contradicts $1'$, we conclude that $C' = I' \cap C'$.

Now, define X'_0 as the subset of X' which arises if, for each non-degenerate cyclic element Z' of X', one removes from X' each component C' of $X' - Z'$ of the form considered above. Then, it is clear that the set $X'_0 \subset E^n$ and the map $g|X'_0$ satisfy Lemma B together with $4'$.

Now, we shall prove that condition $5'$ can also be satisfied. We shall proceed by induction with respect to k.

If $k = 0$, then $5'$ presents no novelty. Now, suppose that $k \geqslant 1$ and that $5'$ can be fulfilled for each Y satisfying the assumptions of Lemma B and such that the number of the distinguished non-degenerate cyclic elements of Y is less than k. Consider the cyclic element Z_1 of X. By the assumption of Lemma B there is an embedding h_1 of Z_1 in E^n. Let $Z'_1 = h_1(Z_1)$. Order in a sequence $a_1, a_2, \ldots$ (finite or not) all points belonging to $\operatorname{Bd}(Z_1)$ and let $a'_i = h_1(a_i)$. Notice that each point a'_i belongs to the boundary of Z'_1 in E^n. Indeed, a_i would otherwise be an interior point of a topological n-ball $Q \subset Z_1$ and, since $a_i \in \operatorname{Bd}(Z_1)$, X would contain an n-umbrella. Since $X \in$ ANR, it follows that $Z'_1 \underset{\text{top}}{=} Z_1 \in$ ANR and therefore each point a'_i is accessible from $E^n - Z'_1$ (cf. [4], p. 217). We infer that for each i there are an arc $I'_i \subset E^n$ and a geometric n-ball $Q'_i \subset E^n$ such that:

$$(I'_i \cup Q'_i) \cap Z'_1 = (a'_i) \subset \dot{I}'_i - Q'_i, \qquad I'_i \cap Q'_i = \dot{I}'_i \cap \dot{Q}'_i \quad \text{is a point;}$$

if $i \neq j$, then $(Q'_i \cup I'_i) \cap (Q'_j \cup I'_j) = \emptyset$ and if the sequence $a'_1, a'_2, \ldots$ is infinite, then $\lim_{i=\infty} \delta(I'_i \cup Q'_i) = 0$.

Let A_i denote the closure of the union of all components C of $X - Z_1$ such that $\mathrm{Bd}(C) = (a_i)$. Since A_i is a retract of X and since the distinguished cyclic elements of A_i are those ones from $Z_2, \ldots, Z_k$ which are contained in A_i, it follows that A_i satisfies the inductive hypothesis. Thus, there is a locally connected continuum $A'_i \subset \mathring{Q}'_i$ and a map g_i from A'_i onto A_i which satisfy the analogues of 1'–5'. If $g_i^{-1}(a_i)$ is a point, define b'_i to be this point and, if $g_i^{-1}(a_i)$ is an arc, define b'_i to be one of its end-points. Since $a_i \in \mathrm{Bd}(A_i)$ (in X), by 3' and since X contains no n-umbrella, it follows that b'_i is a boundary point of A'_i in E^n. Since $A_i \in \mathrm{ANR}$ and since, for locally connected continua, the property of being an ANR depends on cyclic elements (cf. [18], p. 292), it follows from 3' that $A'_i \in \mathrm{ANR}$. Thus, we can assume that b'_i belongs to the boundary of the unbounded component of $E^n - A'_i$ and it is accessible from this component. Consequently, there is an arc J'_i such that $\mathring{J}'_i \subset \mathring{Q}'_i - A'_i$, one end-point of J'_i is equal to b'_i and the second one fills up the set $Q'_i \cap I'_i$. Now, define X' by the formula:

$$X' = Z'_1 \cup \bigcup_i I'_i \cup J'_i \cup A'_i.$$

It easily follows from the construction that X' is a locally connected continuum and that the non-degenerate cyclic elements of X' are Z'_1 and the non-degenerate cyclic elements of the sets A'_i.

Define $g\colon X' \to X$ by the formula:

$$g(x') = \begin{cases} h_1^{-1}(x') & \text{if} \quad x' \in Z'_1, \\ a_i & \text{if} \quad x' \in I'_i \cup J'_i, \\ g_i(x') & \text{if} \quad x' \in A'_i. \end{cases}$$

Then g is a map, because $(I'_i \cup J'_i) \cap Z'_1 = (a'_i)$, $(I'_i \cup J'_i) \cap A'_i = (b'_i)$, $g_i(b'_i) = a_i = h_1^{-1}(a'_i)$ and because the sets $I'_i \cup J'_i \cup A'_i$ are disjoint and their diameters converge to 0 (as well as the diameters of the sets A_i), whenever the sequence $a_1, a_2, \ldots$ is infinite. It follows from the definition of I'_i, J'_i and b'_i that the set

$$g^{-1}(a_i) = I'_i \cup J'_i \cup g_i^{-1}(a_i)$$

is an arc, the point $a'_i = (g|Z'_1)^{-1}(a_i)$ being one of its end-points. Since, for a point $x \in X$ which is different from each point a_i, the inverse set $g^{-1}(x)$ is equal either to $h_1(x)$ (if $x \in Z_1$) or to $g_i^{-1}(x)$ (if $x \in A_i$), and since h_1 is a homeomorphism and A'_i, $g_i\colon A'_i \to A_i$ satisfy 1'–5' with respect to A_i, it follows that X' and $g\colon X' \to X$ satisfy 1'–5' (with respect to X). Thus, the induction step, and therefore the proof of (8.1), is completed.

Now, we pass to

The proof that Lemmas A and B imply Theorem 1. Let X satisfy the assumptions of Theorem 1. In order to obtain the theorem, we shall proceed as follows: First, making use of Lemma A, we shall construct a space Y satisfying the assumptions of Lemma B and a map f from Y onto X. Next, making use of Lemma B together with (8.1), we shall construct a space $Y' \subset E^n$ and a map g from Y' onto Y. Finally, we shall construct the desired decomposition space of E^n such that X is embeddable in it collapsing to a point each set of the form $h^{-1}(x)$, where $x \in X$ and $h = fg$.

Let $Z_1, Z_2, \ldots$ be a sequence (finite or not) consisting of all non-degenerate cyclic elements of X (cf. [12], p. 238). There is only a finite number of Z_i which are not AR-sets (cf. [18], p. 292). By (4.8), $Z_i \in \mathrm{AR}$ implies that Z_i is a t.s.c.e. of X, and therefore—by the assumption of Theorem 1—Z_i is embeddable in E^n. Thus, there is a number $m_0 \geqslant 0$ such that exactly m_0 of the elements Z_m are not embeddable in E^n. Reordering the sequence $Z_1, Z_2, \ldots$ if necessary, we can assume that these are the elements Z_m, where $m \leqslant m_0$.

Now, we shall prove by induction with respect to m_0 that:

(8.2) *There are a connected space $Y \in \mathrm{ANR}$ and a map f from Y onto X such that:*

1° All cyclic elements of Y are embeddable in E^n.

2° The non-degenerate cyclic elements of Y can be ordered in a sequence $\hat{Z}_1, \hat{Z}_2, \ldots$ (which has as many of the elements as the sequence $Z_1, Z_2, \ldots$) such that:

(a) For each $m \leqslant m_0$ the set $\hat{Z}_m$ (which can be assumed to be a subset of E^n by 1°) and the map $f|\hat{Z}_m$ satisfy Lemma A with respect to Z_m.

(b) For each $m > m_0$ the map $f|\hat{Z}_m$ is a homeomorphism of $\hat{Z}_m$ onto Z_m.

(c) For each $x \in X$ such that $f^{-1}(x)$ is not a point, there is an index $m \leqslant m_0$ such that $x \in Z_m$ and if $m_1, \ldots, m_k$ are all such indexes, then $f^{-1}(x) = \bigcup_{i=1}^{k} (f|\hat{Z}_{m_i})^{-1}(x)$ is a tree.

(d) For each point $y \in Y$, if there is an index m such that $y \in \mathrm{Bd}(\hat{Z}_m)$, then $f(y) \in \mathrm{Bd}(Z_m)$ and if $T_m = (f|\hat{Z}_m)^{-1}(f(y))$ is a non-degenerate tree then $\mathrm{ord}_y T_m = 1$ and $T_m \cap \mathrm{Bd}(\hat{Z}_m) = (y)$.

First, let $m_0 = 0$. Then all cyclic elements of X are embeddable in E^n, and assuming $Y = X$ and $f = \text{identity}$, we see that (8.2) is satisfied.

Now, let $m_0 \geqslant 1$ and suppose (8.2) to be true for each set Z satisfying the assumption of Theorem 1 and which has less than m_0 cyclic elements which are not embeddable in E^n. Since the s.c.e.'s of a cyclic element of X are at the same time s.c.e.'s of X (cf. [12], p. 232, No. 5), it follows that the cyclic element Z_1 of X satisfies the assumptions of Lemma A. Thus, (since Z_1 is an ANR, as X is one) there are a cyclic ANR $\hat{Z}_1$ embeddable in E^n and a map g_1 from $\hat{Z}_1$ onto Z_1 which satisfy the lemma with respect to Z_1. Order in a sequence $a_1, a_2, \ldots$ (finite or not) all points which belong to $\mathrm{Bd}(Z_1)$ and, for each i, choose a point $\hat{a}_i \in g_1^{-1}(a_i)$ such that if $g_1^{-1}(a_i)$ is a tree containing more than one point, then $\hat{a}_i$ is one of its end-points.

Let A_i denote the closure of the union of all components C of $X-Z_1$ such that $\mathrm{Bd}(C)=(a_i)$. It is easily seen that the sets A_i satisfy the inductive hypothesis. Thus, there are a set $\hat{A}_i$ and a map f_i: $\hat{A}_i \to A_i$ which satisfy (8.2) with respect to A_i. Evidently, the non-degenerate cyclic elements of A_i are those among Z_i which are contained in A_i. The corresponding cyclic elements of $\hat{A}_i$ will be denoted by $\hat{Z}_i$. We shall define the point $\hat{b}_i \in f_i^{-1}(a_i)$. Assume that the set $f_i^{-1}(a_i)$ contains more than one point. Then, by (8.2), 2° (c), it is equal to $\bigcup_{j=1}^{k} T_{ij}$, where $T_{ij} = (f_i|\ \hat{Z}_{m_{ij}})^{-1}(a_i)$ and $m_{i1}, \ldots, m_{ik}$ are all indexes m such that $2 \leqslant m \leqslant m_0$ and $a_i \in Z_m$ (which implies that $Z_m \subset A_i$). Each T_{ij} is a tree and, by (8.2), 2° (d), the set $T_{ij} \cap \mathrm{Bd}(\hat{Z}_{m_{ij}})$ consists at most of one point, which (if it exists and does not fill up T_{ij}) is an end-point of T_{ij}. If $k=1$, define $\hat{b}_i \in T_{i1} \cap \mathrm{Bd}(\hat{Z}_{m_{i1}})$ if this set is non-empty, otherwise define $\hat{b}_i$ to be an arbitrary end-point of T_{i1}. If $k>1$ then, since $\bigcup_{j=1}^{k} T_{ij}$ is connected and since two different cyclic elements can intersect only at their boundary points, there is exactly one point $\hat{c}_i \in \bigcap_{j=1}^{k} T_{ij}$. In this case define $\hat{b}_i = \hat{c}_i$.

Now, form the disjoint union

$$\hat{Z}_1 \cup \bigcup_i \hat{A}_i$$

and, if the sequence $\hat{A}_1, \hat{A}_2, \ldots$ is infinite, suppose that a metric in this disjoint union is defined such that $\lim_{i=\infty} \delta(\hat{A}_1) = 0$. Let Y denote the compact metric space we obtain from this disjoint union by the identification of the points $\hat{a}_i$ and $\hat{b}_i$ for each i and by the suitable definition of a metric. Since $\hat{Z}_1$ and all $\hat{A}_i$ are connected ANR-sets, it follows that Y is a locally connected continuum and it can be seen that the non-degenerate cyclic elements of Y are the images of the sets $\hat{Z}_i$ under the identification map. We infer from [18] (p. 292) and from the analogue of (8.2), 2° (a) and (b) with respect to A_i, $\hat{A}_i$ and f_i that Y is a connected ANR.

Define f: $Y \to X$ by the formula (where we identify $\hat{Z}_1$ and $\hat{A}_i$ with their images by the identification map)

$$f(y) = \begin{cases} g_1(y) & \text{if} \quad y \in \hat{Z}_1, \\ f_i(y) & \text{if} \quad y \in \hat{A}_i. \end{cases}$$

Since $g_1(\hat{a}_i) = a_i = f_i(\hat{b}_i)$ and since $\lim_{i=\infty} \delta(\hat{A}_i) = 0 = \lim_{i=\infty} \delta(A_i)$ whenever the sequence $A_1, A_2, \ldots$ is infinite, it follows that f is a map. Since $\hat{Z}_1$ and g_1 satisfy Lemma A with respect to Z_1 and $\hat{A}_i$ and f_i satisfy (8.2) with respect to A_i, it follows from the construction (especially, from the definition of $\hat{b}_i$) that Y and f satisfy (8.2) (with respect to X). Thus, the induction step, and therefore the proof of (8.2), is completed.

Now, observe that Y satisfies the assumptions of Lemma B. Since other assumptions are contained in (8.2), it remains to show that Y does not contain any n-umbrella. Thus, suppose that $Q \subset Y$ is a topological n-ball and $I \subset Y$ is an arc such that $Q \cap I = \mathring{Q} \cap \dot{I}$ is a point y. Since $n>1$, there is a cyclic element $\hat{Z}$ of Y such that $\hat{Z} \supset Q$ (cf. [12], p. 238, No. 10). It follows from (4.11) that there is a t.s.c.e. $\hat{E}$ of $\hat{Z}$ (and therefore also of Y) such that $\hat{E} \supset Q$. By (8.2), 2° (a) and (b), $f(Q) \subset f(\hat{E})$ is a (topological) n-ball and $f(\hat{E})$ is a t.s.c.e. of X. It follows from (8.2), 1° that $y \in \mathrm{Bd}(\hat{Z})$. Therefore, by (8.2), 2° (d), $f(y) \in \mathrm{Bd}\big(f(\hat{Z})\big)$ and $f(\hat{Z})$ is the cyclic element of X containing $f(\hat{E})$. Consequently, there is an arc $J \subset X$ such that $f(\hat{E}) \cap J = f(y)$ and $f(y) \in \dot{J}$. Since $f(y)$ is an interior point of $f(Q) \subset f(\hat{E})$, it follows that $f(\hat{E}) \cup J$ is not embeddable in E^n, which contradicts the assumption of Theorem 1.

Thus, Lemma B together with (8.1) (where in 5′ we assume $\hat{Z}_m$, for $m \leqslant m_0$, to be the distinguished cyclic elements of Y) can be applied to Y. Consequently, there are a connected ANR $Y' \subset E^n$ and a map g from Y' onto Y, which satisfy the analogues of 1′–5′. Then

$$h = fg \tag{8.3}$$

is a map of Y' onto X. Let $\mathfrak{D}$ denote the decomposition of E^n into the sets of the form $h^{-1}(x)$ for $x \in X$ and the individual points $x' \in E^n - Y'$. Then X is embedded in a natural way in the space $E^n/\mathfrak{D}$. We shall prove that $\mathfrak{D}$ is the decomposition required in Theorem 1, i.e., we shall show that:

(8.4) *$h^{-1}(x) = g^{-1}\big(f^{-1}(x)\big)$ is always a tree, there is only a finite number of points $x \in X$ such that $h^{-1}(x)$ is neither an arc nor a point and for every $\eta > 0$ there is only a finite number of points $x \in X$ such that $\delta\big(h^{-1}(x)\big) \geqslant \eta$.*

Considering the sets $h^{-1}(x)$, we shall distinguish the following four cases: (1) $x \in \mathrm{Int}(Z_{m_1})$ for an $m_1 \leqslant m_0$, (2) $x \in \mathrm{Bd}(Z_{m_1}) - \bigcup [Z_m |\ m \leqslant m_0,$

$m \neq m_1$] for an $m_1 \leqslant m_0$, (3) x belongs to at least two from the sets Z_m, where $m \leqslant m_0$ and (4) $x \notin \bigcup [Z_m |\ m \leqslant m_0]$. We shall denote by $\hat{Z}'_m$ the non-degenerate cyclic element of Y' which corresponds to $\hat{Z}_m$ (cf. 3′ in Lemma B). Thus, we have the one-to-one correspondences $\hat{Z}'_m \leftrightarrow \hat{Z}_m \leftrightarrow Z_m$ between the non-degenerate cyclic elements of Y', Y and X.

First, let case (1) hold. Then, in virtue of (8.2), 2° (d), $f^{-1}(x) \subset \operatorname{Int}(\hat{Z}_{m_1})$. It follows from Lemma B, 3′ and (8.1) 4′ that $h^{-1}(x) = g^{-1}(f^{-1}(x))$ is the image of $f^{-1}(x)$ under the homeomorphism $(g|\hat{Z}'_{m_1})^{-1}$. We conclude from (8.2), 2° (a) that, in the case (1), (8.4) is satisfied.

Now, let case (2) hold. It follows from (8.2), 2° (c), (d) and from the connectivity of Y that $f^{-1}(x) \subset \hat{Z}_{m_1} - \bigcup [\hat{Z}_m |\ m \leqslant m_0,\ m \neq m_1]$ is a tree and there is exactly one point $y \in f^{-1}(x) \cap \operatorname{Bd}(\hat{Z}_{m_1})$, which is an end-point of this tree if it is non-degenerate. We infer from Lemma B and from (8.1) that the set $h^{-1}(x) = g^{-1}(f^{-1}(x))$ is the union of the image of $f^{-1}(x)$ under the homeomorphism $(g|\hat{Z}'_{m_1})^{-1}$ and of the arc $g^{-1}(y)$, the point $y' = (g|\hat{Z}'_{m_1})^{-1}(y)$ being an end point of this arc if it is non-degenerate. Since $(g|\hat{Z}'_{m_1})^{-1}(f^{-1}(x)) \cap g^{-1}(y) = (y')$, it follows that $h^{-1}(x)$ is a tree, which is an arc if the former set is an arc. We infer from (8.2), 2° (a) and from Lemma B, 2′ that (8.4) is satisfied in this case.

Next, let case (3) hold. Let $m_1, \ldots, m_k$ be all indexes $m \leqslant m_0$ such that $x \in Z_m$ and let $T_i = (f|\hat{Z}_{m_i})^{-1}(x)$ for $1 \leqslant i \leqslant k$. Then, by (8.2), 2° (c), $f^{-1}(x) = \bigcup_{i=1}^{k} T_i$ is a tree. We infer from (8.2), 2° (d), from the connectivity of $f^{-1}(x)$ and from the fact that two different cyclic elements can intersect only at their boundary points that there is exactly one point $y_i \in T_i \cap \operatorname{Bd}(\hat{Z}_{m_i})$ and $y_1 = y_2 = \ldots = y_k$. Next, it follows from Lemma B and (8.1), 4′ that $h^{-1}(x) = g^{-1}(f^{-1}(x))$ is the union of the images of T_i under the homeomorphisms $(g|\hat{Z}'_{m_i})^{-1}$ and of the arc $g^{-1}(y_1)$. Since the trees $(g|\hat{Z}'_{m_i})^{-1}(T_i)$ can intersect one another and the arc $g^{-1}(y_1)$ only in the points $(g|\hat{Z}'_{m_i})^{-1}(y_1)$ (and they do intersect the arc), we conclude that $h^{-1}(x)$ is a tree. Since two different cyclic elements can intersect only at one point (cf. [12], p. 236), we see that the number of the points belonging to case (3) is finite, which completes the proof of (8.4) in this case.

Finally, let case (4) hold. Then, by (8.2), 2° (c), $f^{-1}(x)$ is a point and, by Lemma B, 1′, $h^{-1}(x) = g^{-1}(f^{-1}(x))$ is an arc. Lemma B, 2′ implies that (8.4) is satisfied again. Thus, we conclude that the decomposition $\mathfrak{D}$ satisfies the requirements of Theorem 1, which completes the reduction of the theorem to Lemma A.

It remains to prove Lemma A, which will be done in Sections 9 and 10.

Remark. It follows from the preceding proof (cf. (8.2), (8.3) and Lemma B, 3′) that:

(8.5) *There is a one-to-one correspondence between the t.s.c.e.'s of Y' and X such that if $\hat{E}'$ corresponds to E, then the map $h|\hat{E}'$ is a homeomorphism of $\hat{E}'$ onto E.*

9. A proof of Lemma A in the case where X has only a finite number of t.s.c.e.'s. First, let X satisfy the assumptions of Lemma A for $n = 2$. Then X satisfies the assumptions of Theorem 3 (see Section 6) and therefore X is homeomorphic to a polyhedron $P \subset S^2$. If X were a simple surface, then it would have a t.s.c.e. non-embeddable in E^2, which contradicts the assumption. Thus, X is embeddable in E^2.

Now, let X satisfy the assumptions of Lemma A for a given $n > 2$. We shall prove that:

(9.1) *There are a space $X' \subset E^n$ and a map $g\colon X' \to X$ satisfying the conclusion of Lemma* A, *where conditions* 1° *and* 2° *are replaced by the following one:*

I. *If $x \in X$ and $g^{-1}(x)$ is not a point, then there is a t.s.c.e. E of X such that $x \in \operatorname{Bd}(E)$ and $g^{-1}(x)$ is a tree* (I *implies* 1° *and* 2° *in virtue of* (4.3)).

Suppose that X has exactly m of t.s.c.e.'s.

We shall prove (9.1) by induction with respect to m.

First, let $m = 0$. Then, by (4.9), X is a graph, which evidently is embeddable in E^n and (9.1) is clear.

Now, let $0 < m < \infty$ and suppose (9.1) to be true for every space Y satisfying the assumptions of Lemma A, where Y has less than m of t.s.c.e.'s. Choose a fixed t.s.c.e. E_1 of X. Evidently, we can assume that $X - E_1 \neq \emptyset$. By (4.5), the set $X - E_1$ has a finite number of components. Denote the closures of these components by $A_1, \ldots, A_l$. By (4.5), the set $A_i \cap E_1$ is finite and therefore, by (2.2), there is a tree $T_i \subset A_i$ such that the set of the end-points of T_i is equal to $E_1 \cap A_i$. By the assumption of Lemma A there is an embedding h of the set $E_1 \cup \bigcup_{i=1}^{l} T_i$ in E^n. Let $E'_1 = h(E_1)$. Since each set $h(T_i) - E'_1$ is connected, it follows that:

(9.2) *There is a component C'_i of $E^n - E'_1$ such that $\overline{C'_i} \supset h(A_i \cap E_1)$ and if $a_{i1}, \ldots, a_{ik(i)}$ are all points of the set $A_i \cap E_1$, then each point $a'_{ij} = h(a_{ij})$ is accessible from C'_i. There are* (*geometric*) *n-balls $Q'_i \subset C'_i$ $(1 \leqslant i \leqslant l)$ such that $i \neq j$ implies $Q'_i \cap Q'_j = \emptyset$.*

Now, apply (2.2) to the set E_1 and its finite subset $A_i \cap E_1$. Thus, there is a tree $\hat{T}_i \subset E_1$ such that the set of the end-points of $\hat{T}_i$ is equal to $A_i \cap E_1$. Since X is a cyclic space, $x \in A_i$ implies that every component of $A_i - (x)$ intersects the set $\operatorname{Bd}(A_i) = A_i \cap E_1$. It follows that $Y_i = A_i \cup \hat{T}_i$ is a cyclic space. Since $\hat{T}_i \in \mathrm{AR}$ and $\hat{T}_i \cap A_i = \operatorname{Bd}(A_i)$, we infer that Y_i is a retract of X. Consequently, $Y_i \in \alpha'$. Evidently, the t.s.c.e.'s of Y_i are the t.s.c.e.'s of X which are contained in A_i, and therefore Y_i satisfies the inductive hypothesis. Thus, there are a set $Y'_i \subset \mathring{Q}'_i$ and a map $g_i\colon Y'_i \to Y_i$ which satisfy the analogue of (9.1) with respect to Y_i. Let $A'_i = g_i^{-1}(A_i)$ and $\hat{T}'_i = g_i^{-1}(\hat{T}_i - A_i)$. Since the sets A_i and $\hat{T}_i - A_i$ are both connected and g_i is a monotonic map, it follows from [12], (p. 123) that A'_i is a continuum and $\hat{T}'_i$ is a region in Y'_i disjoint with A'_i. Thus, we can assume that $\hat{T}'_i$ is contained in the unbounded component of $E^n - A'_i$. Since each point a_{ij} (see (9.2)) is an end-point of $\hat{T}_i$, it follows that the set $(\hat{T}_i - A_i) \cup (a_{ij})$ (and its counter-image under g_i also) is connected. Consequently, there is a point $b'_{ij} \in \overline{\hat{T}'_i} \cap g_i^{-1}(a_{ij})$. It follows from Lemma A, 3° (with respect to Y_i, Y'_i and g_i) that the set $\hat{T}'_i$ is contained in the complement of the union of all t.s.c.e.'s of $Y'_i \in \alpha'$. Hence, by (4.9), $\overline{\hat{T}'_i}$ is a graph. We infer that each point b'_{ij} is accessible from $\hat{T}'_i$, and therefore from the unbounded component of $E^n - A'_i$ also. Since $A'_i \subset Y'_i \subset \mathring{Q}'_i$ and $n > 2$, we conclude from this and from (9.2) that

(9.3) *There are some collections of arcs* $I'_{i1}, \ldots, I'_{ik(i)} \subset E^n$ $(i = 1, \ldots, l)$ *such that* $\dot{I}'_{ij} = (a'_{ij}) \cup (b'_{ij})$, $\mathring{I}'_{ij} \cap (E'_i \cup \bigcup_{i=1}^{l} A'_i) = \emptyset$ *and* $\mathring{I}'_{ij} \cap \mathring{I}'_{pq} = \emptyset$ *if either* $i \neq p$ *or* $j \neq q$.

Now, define X' by the formula:

(9.4)
$$X' = E'_1 \cup \bigcup_{i=1}^{l} \bigcup_{j=1}^{k(j)} A'_i \cup I'_{ij} \,.$$

Now, observe that the set $\overline{\hat{T}'_i} - \hat{T}'_i$ can consist only of the points b'_{ij}, because $g_i(\overline{\hat{T}'_i}) = \hat{T}_i$ contains no simple closed curve and (in virtue of (9.1), I with respect to Y_i, Y'_i and g_i) $g_i|\hat{T}'_i$ is one-to-one. Thus $g_i|\overline{\hat{T}'_i}$ is one-to-one, and therefore it is a homeomorphism of $\overline{\hat{T}'_i}$ onto $\hat{T}_i$. Since $Y'_i \in \alpha'$, we infer that $A'_i = Y'_i - \hat{T}'_i$ belongs to α and that $x' \in A'_i$ implies that every component of $A'_i - (x')$ intersects the set $\overline{\hat{T}'_i} - \hat{T}'_i$. $E'_1 \in \alpha'$, because it is a homeomorphic image of E_1 which is a t.s.c.e. of $X \in \alpha'$. Since each simple closed curve of sufficiently small diameter contained in X' must be contained either in E'_1 or in one of the sets A'_i, we conclude from (9.4) that $X' \in \alpha'$. Since $Y'_i = A'_i \cup \hat{T}'_i$, it is evident from (9.4) that E'_1 and the t.s.c.e.'s of all sets Y'_i are the t.s.c.e.'s of X'.

Now, define $g\colon X' \to X$ by the formula:

$$g(x') = \begin{cases} h^{-1}(x') & \text{if} \quad x' \in E'_1\,, \\ g_i(x') & \text{if} \quad x' \in A'_i\,, \\ a_{ij} & \text{if} \quad x' \in I'_{ij}\,. \end{cases}$$

It follows from (9.2), (9.3), (9.4) and from the definition of the points b'_{ij} that g is a map. Since $A_1, \ldots, A_l$ are the closures of all components of $X - E_1$, the maps g_i are onto and $A'_i = g_i^{-1}(A_i)$, we infer that g maps X' onto X. Since h^{-1} is a homeomorphism, Y'_i and g_i satisfy (9.1) with respect to Y_i and since the t.s.c.e.'s of Y_i are the t.s.c.e.'s of X contained in A_i, we see that X' and g satisfy condition 3° of Lemma A. Condition (9.1), I is clearly satisfied for each point $x \in X - \mathrm{Bd}(E_1)$. If $x \in \mathrm{Bd}(E_1)$, then $g^{-1}(x)$ is the union of the arcs I'_{ij} such that $h(x) = a'_{ij} \in \dot{I}'_{ij}$ and of the sets $g_i^{-1}(x)$, whenever $x \in A_i$. It follows from (9.2) and (9.3) that these arcs form a broom with $h(x)$ as a vertex, the sets $g_i^{-1}(x)$ being trees disjoint to one another and intersecting the broom at the respective points b'_{ij}. We conclude that $g^{-1}(x)$ is a tree, which completes the proof of (9.1) and therefore of Lemma A in the case under consideration.

10. The proof of Lemma A in the general case. Let X satisfy the assumptions of Lemma A. We can assume that X has infinitely many t.s.c.e.'s, $E_1, E_2, \ldots$ It follows from (4.8) and (4.10) that there is an $m_0 \geqslant 0$ such that $m > m_0$ implies $E_m \in \alpha'_0$ and that $\mathrm{Bd}(E_m)$ consists of exactly two points a_m and b_m. Let $I_m \subset E_m$ be an arc joining a_m with b_m and let

(10.1)
$$Y = (X - \bigcup_{m > m_0} E_m) \cup \bigcup_{m > m_0} I_m \,.$$

One can easily show that Y is a retract of X (cf. (4.6)), and therefore $Y \in \alpha$. Since X is a cyclic space, it is seen from (10.1) that Y is also a cyclic space. Evidently, the sets E_m, for $m \leqslant m_0$, are the t.s.c.e.'s of Y. It follows that the result (9.1) of Section 9 is applicable to Y, and therefore there are a set $Y' \subset E^n$ and a map $g_0\colon Y' \to Y$ which satisfy (9.1) with respect to Y. For each $m \leqslant m_0$, denote by E'_m the t.s.c.e.'s of Y' which corresponds to E_m (see Lemma A, 3°). Let $\hat{E}' = \bigcup_{m \leqslant m_0} E'_m$. Then, by (4.5) and (4.9), the set $G' = \overline{Y' - \hat{E}'}$ is a graph. It follows from (9.1) (with respect to Y, Y' and g_0) and from the definition of the arcs I_m that, for each $m > m_0$, $I'_m = g_0^{-1}(\mathring{I}_m) \subset G'$ and $g_0|g_0^{-1}(\mathring{I}_m)$ is one-to-one. We infer that I'_m is an arc and that the map $g_0|I'_m$ is a homeomorphism of I'_m onto I_m. Let

(10.2)
$$(a'_m) = I'_m \cap g_0^{-1}(a_m) \quad \text{and} \quad (b'_m) = I'_m \cap g_0^{-1}(b_m)\,.$$

Evidently, a'_m and b'_m are the end-points of I'_m. Since $\mathring{I}_m \subset \mathrm{Int}(E_m)$ is an open subset of Y, we infer that the sets $\mathring{I}'_m = g_0^{-1}(\mathring{I}_m)$ are open and disjoint subsets of Y'. Modifying the sets $\mathring{I}'_m$ if necessary, one can construct a sequence of (geometric) n-balls $Q'_m \subset E^n$ (where $m > m_0$) such that:

(10.3) $Q'_m \cap Y'$ *is a subarc of* I'_m *contained in* $\mathring{I}'_m$, *whose interior is contained in* $\mathring{Q}'_m$. *If* $m \neq p$, *then* $Q'_m \cap Q'_p = \emptyset$ *and* $\lim_{m=\infty} \delta(Q'_m) = 0$.

Now, consider a t.s.c.e. E_m of X, where $m > m_0$. By the definition of a_m and b_m, $\mathrm{Bd}(E_m) = (a_m) \cup (b_m)$. Since X is a cyclic space, there is an arc $J_m \subset X$ such that $J_m \cap E_m = (a_m) \cup (b_m) = \dot{J}_m$. By the assumption of Lemma A, there is an embedding h_m of $E_m \cup J_m$ into $\mathring{Q}'_m$. Let $E'_m = h_m(E_m)$, $\hat{a}'_m = h_m(a_m)$ and $\hat{b}'_m = h_m(b_m)$. Since $h_m(\mathring{J}_m) \subset E^n - E'_m$, we can assume that $\hat{a}'_m$ and $\hat{b}'_m$ belong to the closure of the unbounded component of $E^n - E'_m$ and, evidently, they are accessible from it. Denote the points belonging to $\dot{Q}'_m \cap I'_m$ by c'_m and d'_m in such a way that in the ordering of the arc I'_m from a'_m to b'_m the point c'_m precedes d'_m (cf. (10.2) and (10.3)). We infer that there are two arcs $K'_m, L'_m \subset Q'_m$ such that:

$$\text{(10.4)} \qquad \mathring{K}'_m \cup \mathring{L}'_m \subset \mathring{Q}'_m - E'_m\,, \qquad \dot{K}'_m = (c'_m) \cup (\hat{a}'_m)\,, \qquad \dot{L}'_m = (d'_m) \cup (\hat{b}'_m)\,, \qquad K'_m \cap L'_m = \emptyset\,.$$

Now, we can define the desired set $X' \subset E^n$ by the formula:

$$\text{(10.5)} \qquad X' = (Y' - \bigcup_{m>m_0} Q'_m) \cup \bigcup_{m>m_0} K'_m \cup E'_m \cup L'_m\,.$$

Each set E'_m, for $m > m_0$, belongs to α'_0, as a homeomorphic image of $E_m \in \alpha'_0$. Since $Y' \in \alpha'$, it is easily seen from the construction (cf. (10.3), (10.4) and (10.5)) that X' is a cyclic locally connected continuum. Moreover, each simple closed curve $S' \subset X'$ is contained either in a subset of X' homeomorphic with Y' or in a set E'_m, where $m > m_0$. Consequently, $X' \in \alpha'$. Evidently, the sets E'_m, $m = 1, 2, \ldots$ are the t.s.c.e.'s of X'.

Now, we can define the desired function $g\colon X' \to X$ by the formula:

$$g(x') = \begin{cases} g_0(x') & \text{if} \quad x' \in Y' - \bigcup_{m>m_0} I'_m\,, \\ h_m^{-1}(x') & \text{if} \quad x' \in E'_m \quad \text{for an } m > m_0\,, \\ a_m & \text{if} \quad x' \in K'_m \cup [a'_m c'_m]\,, \\ b_m & \text{if} \quad x' \in L'_m \cup [d'_m b'_m]\,, \end{cases}$$

where $[a'_m c'_m]$ and $[d'_m b'_m]$ denote the subarcs of the arc I'_m bounded by these points. It follows from (10.2), (10.4) and from the definition of the points $\hat{a}'_m, \hat{b}'_m, c'_m$ and d'_m that $g|E'_m \cup K'_m \cup L'_m \cup [a'_m c'_m] \cup [d'_m b'_m]$ is a map of this set onto E_m. Since the diameters of these sets converge to zero and since g_0 is a map, we infer that g is a map. Since g_0 maps Y' onto Y, it follows from (10.1) that g maps X' onto X. We shall prove that X' and g satisfy the conditions 1°–3° of Lemma A.

Considering the sets $g^{-1}(x)$, for $x \in X$, first notice that $x \in X - \bigcup_{m>m_0} E_m$ implies that $g^{-1}(x) = g_0^{-1}(x)$, because the points $x' \in X'$ such that $g(x') \neq g_0(x')$ are mapped by g into $\bigcup_{m>m_0} E_m$. In this case, by (9.1), I (with respect to Y, Y' and g_0), the sets $g^{-1}(x)$ are trees and, except of a finite number, they are points. If $x \in \mathrm{Int}(E_m)$ for an $m > m_0$, then evidently $g^{-1}(x) = h_m(x)$ is a point. Now, suppose that $x \in \mathrm{Bd}(E_m)$ for an $m > m_0$ and that $g_0^{-1}(x)$ is a point x'. Since Y, Y' and g_0 satisfy Lemma A, 3°, it follows that x' belongs to the graph $G' = \overline{Y' - \hat{E}'}$. If x' is not a ramification point of G', then x' belongs to (at most) two of the arcs I'_m and it is an end-point of either. It follows from (10.4) and from the definition of g that $g^{-1}(x)$ is the union of at most two arcs, either being of the form $[a'_m c'_m] \cup K'_m$, where $x' = a'_m$, or of the form $L'_m \cup [d'_m b'_m]$, where $b'_m = x'$. Thus $g^{-1}(x)$ is an arc. It follows from (10.3), (10.4) and from the fact that the sets $\mathring{I}'_m$ are open and disjoint subsets of the graph G' that the diameters of these arcs converge to zero. Now, suppose that x' is a ramification point of G'. Then $g^{-1}(x)$ is a broom which is the union of a finite number of arcs of the form described above, x' being a vertex of the broom. Evidently, this case can hold only for a finite number of points $x \in X$.

Finally, suppose that $g_0^{-1}(x)$ is a non-degenerate tree T'. Condition 3° of Lemma A (with respect to Y, Y' and g_0) implies that T' is contained in the graph G'. Thus, only a finite number of the arcs I'_m (which are contained in G' also) can intersect T', and if $T' \cap I'_m \neq \emptyset$, then either $T' \cap I'_m = (a'_m)$ or $T' \cap I'_m = (b'_m)$. It follows that $g^{-1}(x)$ is the union of T', of the arcs $[a'_m c'_m] \cup K'_m$ where $I'_m \cap T' = (a'_m)$ and of the arcs $L'_m \cup [d'_m b'_m]$ where $I'_m \cap T' = (b'_m)$. Since these arcs can intersect one another only at their common points with T' (i.e. in a'_m or b'_m), we conclude that $g^{-1}(x)$ is a tree. By (9.1), I, $g_0^{-1}(x)$ is non-degenerate only for a finite number of points $x \in X$. Thus, we see that X' and g satisfy the conditions 1° and 2° of Lemma A. Condition 3° is clear by the definition of g and h_m and by the respective property of g_0. Thus, the proof of Lemma A is concluded.

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A characterization of locally connected continua which are quasi-embeddable into E^2

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1. Introduction. We shall consider metrizable spaces only. A map f of a compactum X into a space Y is said to be an *ε-mapping* if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in f(X)$. A compact space X is said to be *quasi-embeddable* into Y if for every $\varepsilon > 0$ there is an ε-mapping $f\colon X \to Y$. The problem of finding a characterization of locally connected continua which can be quasi-embedded into S^2 (E^2) has been raised by Mardešić and Segal in [6] in connection with the following

THEOREM OF MARDEŠIĆ AND SEGAL. *If P is a connected polyhedron, hen the following statements are equivalent*:

(a) *P is embeddable into S^2*,

(b) *P is quasi-embeddable into S^2*,

(c) *P does not contain any homeomorphic images of the Kuratowski graphs K_1 and K_2 and any 2-umbrella.*

The graph K_1 is the 1-skelton of a 3-simplex with midpoints of a pair of non-adjacent edges joined by a segment and the graph K_2 is the 1-skelton of a 4-simplex. A *2-umbrella* is the one-point union of a disk and of an arc relative to an interior point of the disk and an end-point of the arc.

In [8] I have generalized that theorem, namely I have shown that the equivalence of (a), (b) and (c) holds for each locally connected continuum P satisfying the following condition: There is a number $\varepsilon > 0$ such that no simple closed curve $S \subset P$ with $\operatorname{diam} S < \varepsilon$ is a retract of P. Another similar generalization has been found by J. Segal (see [10]). He has shown the equivalence of (a) and (b) for locally connected continua which do not contain any homeomorphic images of the curves K_3 and K_4 (described by Kuratowski in [4]).

In this paper we shall prove the equivalence of (b) and (c) for arbitrary locally connected continua, i.e.