On ω_1-categorical theories of abelian groups

by

Angus Macintyre (Aberdeen, Scotland)

0. Introduction. In this paper we classify the totally transcendental complete theories of abelian groups, and the ω_1-categorical theories of abelian groups. The results were obtained while working on the corresponding, but more difficult, problem for theories of fields. Our results about fields will appear in a separate publication [6], where the results of this paper will be presupposed.

The work of Szmielew [11] gives a classification of complete theories of abelian groups. However, only at one point do we use a result from her paper, and the result in question can easily be proved using ultrapowers.

From model-theory we presuppose acquaintance with Morley's paper [7], as well as some results of Feferman and Vaught, and Mostowski, on products of structures [2, 8].

From group theory we presuppose some basic facts about the existence and uniqueness of certain direct sum decompositions of abelian groups. These facts can be found in Kaplansky's book [3].

If G is an abelian group, let $Th(G)$ be the set of all sentences, of first-order group theory, that are satisfied in G.

THEOREM 1. *If G is an abelian group, then $Th(G)$ is totally transcendental if and only if G is of the form $D \oplus H$, where D is divisible and H is of bounded order.*

THEOREM 2. *If G is an abelian group, then $Th(G)$ is ω_1-categorical if and only if G is of one of the following forms:*

(i) *$K \oplus H$, where H is finite and K is a direct sum of copies of a fixed finite cyclic group of prime-power order;*

(ii) *$D \oplus H$, where H is finite and D is a divisible group with the property that for each prime p there are only finitely many elements of D of order p.*

Using Szmielew's work, one can deduce from these theorems syntactic characterizations of complete totally transcendental theories of abelian groups, and ω_1-categorical theories of abelian groups.

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1. Model-theoretic preliminaries.

1.1. For the fundamental notions of model-theory, one should consult Tarski [12] or Robinson [9].

We will be working with first-order predicate logics $\mathcal{L}$, with connectives $\wedge$ and $\vee$, quantifiers $\exists$ and $\forall$, identity-symbol $=$, finitary relation-symbols and operation-symbols and variables $v_0, v_1, \ldots, v_n \ldots$ We assume the usual syntactic notions of term, formula, sentence, etc.

An $\mathcal{L}$-structure $\mathcal{M}$ is a relational structure which has a relation or operation for each relation-symbol or operation-symbol of $\mathcal{L}$. $|\mathcal{M}|$ is the underlying set of $\mathcal{M}$. We assume the usual semantic notions of satisfaction, model, consequence, and validity, and the various related uses of the symbol "$\models$".

If α is an ordinal, we form a logic $\mathcal{L}(\alpha)$ by adding to $\mathcal{L}$ distinct new individual constants c_η for $\eta < \alpha$. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $s \in |\mathcal{M}|^\alpha$, then $(\mathcal{M}, s)$ is the obvious $\mathcal{L}(\alpha)$-structure where $s(\eta)$ corresponds to c_η for each $\eta < \alpha$.

As is customary, cardinals are identified with initial ordinals. ω is the least infinite ordinal, and ω_1 is the least uncountable ordinal.

1.2. If Σ is an $\mathcal{L}$-theory, and $\varkappa$ is a cardinal, Σ is said to be $\varkappa$-categorical (or categorical in power $\varkappa$) if any two members of $\mathrm{Mod}(\Sigma)$ of cardinality $\varkappa$ are isomorphic. For the basic examples and background, one should consult Łoś [5] or Vaught [13].

The classical example of a theory that is categorical in every uncountable power is the theory of an algebraically closed field of specified characteristic. This follows from Steinitz's work [10]. In [6] we prove that no other theory of an infinite field is ω_1-categorical.

The classical example of a theory of abelian groups that is categorical in every uncountable power is the theory of a non-trivial torsion-free divisible abelian group. Such a group can be construed as a vector-space over the field of rational numbers, and the categoricity result follows easily from elementary facts about the dimension of vector-spaces.

The following theorem is very important.

THEOREM A [Morley, 7]. *Suppose $\mathcal{L}$ is countable and Σ is an $\mathcal{L}$-theory. If Σ is categorical in one uncountable power, Σ is categorical in all uncountable powers.*

It is because of Theorem A that we confine our attention to ω_1-categoricity.

In [7] Morley used the important idea of totally transcendental theory. We give a definition equivalent to Morley's. (The equivalence is proved in Theorem 2.8 of [7]. Our formulation follows [4].)

Suppose Σ is a theory in a countable logic $\mathcal{L}$. Σ is said to be totally transcendental if, for every $\mathcal{M}$ in $\mathrm{Mod}(\Sigma)$ and every $s \in |\mathcal{M}|^\omega$, $Th\big((\mathcal{M}, s)\big)$ has at most ω complete extensions in $\mathcal{L}(\omega+1)$.

The importance of the notion comes from:

THEOREM B [Morley, 7]. *Suppose $\mathcal{L}$ is countable and Σ is an $\mathcal{L}$-theory. If Σ is ω_1-categorical, Σ is totally transcendental.*

2. Conjugacy types.

The results of this section are too crude to have much general interest, but they are useful to us in dealing with certain abelian groups, because of the existence for these groups of nice direct-sum decompositions.

DEFINITION. Suppose $\mathcal{M}$ is a fixed $\mathcal{L}$-structure, and $A \subseteq |\mathcal{M}|$. If x and y are elements of $|\mathcal{M}|$, we say x is *A-conjugate to y*, and write $x \approx_A y$, if there is an automorphism f of $\mathcal{M}$, fixing every element of A, and such that $f(x) = y$.

It is clear that $\approx_A$ is an equivalence relation. We call the equivalence classes A-types.

The following lemma is well-known. It follows simply from Lemma 2.1 of Morley's paper, and his sufficient condition halfway down page 523 of the same paper.

LEMMA 0. *Suppose $\mathcal{L}$ is countable and $\mathcal{M}$ is an $\mathcal{L}$-structure. A sufficient condition for $Th(\mathcal{M})$ to be totally transcendental is the following:*

If $\mathcal{N} \equiv \mathcal{M}$ and A is a countable subset of $|\mathcal{N}|$, $\mathcal{N}$ has at most ω A-types.

The next lemma deals with conjugacy-types in direct sums of structures. We will apply the lemma only to abelian groups, but for the proof of the lemma there is no gain in confining ourselves to abelian groups.

In Feferman-Vaught, page 71, the notion of the weak direct product of an indexed family of similar relational systems is defined. Under their very general definition the next lemma would fail, but under a natural restriction (which they mention) the lemma holds. We refer the reader to their paper for the general definition of weak direct product, which we call direct sum.

Let $\Psi(v_0)$ be a fixed $\mathcal{L}$-formula with v_0 as its only free variable. We will define the direct sum $\bigoplus_{i \in I} \mathcal{M}_i$ of a family $(\mathcal{M}_i)_{i \in I}$ of $\mathcal{L}$-structures, but only under the following assumption:

For each $i \in I$ there is a unique e_i in $\mathcal{M}_i$ such that e_i satisfies $\Psi(v_0)$ in $\mathcal{M}_i$; in addition e_i satisfies $v_0 = \tau$ for all τ which are individual constants of $\mathcal{L}$; and, finally, the set $\{e_i\}$ is closed under the operations of $\mathcal{M}_i$, for each $i \in I$.

If this assumption is satisfied, then we define $\bigoplus_{i \in I} \mathcal{M}_i$ as the subsystem of $\prod_{i \in I} \mathcal{M}_i$ consisting of those f in $\prod_{i \in I} \mathcal{M}_i$ for which $f(i) = e_i$ for all but finitely many i in I. It follows directly from the assumption that we get a subsystem in this way.

The classical example is when the $\mathcal{M}_i$ are groups, and $\Psi(v_0)$ is a formula defining the identity element. Then the definition above obviously coincides with the group-theoretic notion of direct sum.

If $\mathcal{M}_i = \mathcal{M}$ for each $i \in I$, we write $\bigoplus_{i\in I} \mathcal{M}$ instead of $\bigoplus_{i\in I} \mathcal{M}_i$.

If $J \subseteq I$, we identify $\bigoplus_{i\in J} \mathcal{M}_j$ with the subsystem of $\bigoplus_{i\in I} \mathcal{M}_i$ consisting of those f for which $f(i) = e_i$ whenever $i \notin J$.

The next lemma originated in the observation that for some interesting classes K of abelian groups there are countably many groups G_n, $n < \omega$, each of cardinality $\leqslant \omega$, such that each member of K is a direct sum of copies of the groups G_n. Examples to be discussed later are the class of divisible groups, and the class of groups of bounded order.

LEMMA 1. (a) *If $\mathcal{M}$ has at most ω A-types for every countable subset A of $\mathcal{M}$, then $\bigoplus_{i\in I} \mathcal{M}$ has at most ω A-types for every countable subset A of $\bigoplus_{i\in I} \mathcal{M}$.*

(b) *If, for $n < \omega$, $\mathcal{M}_n$ has at most ω A_n-types for every countable subset A_n of $\mathcal{M}_n$, then $\bigoplus_{n<\omega} \mathcal{M}_n$ has at most ω A-types for every countable subset A of $\bigoplus_{n<\omega} \mathcal{M}_n$.*

We do not prove the lemma, but indicate a proof of (a). (b) is obvious, and implies the special case of (a) where I is countable. Suppose I is uncountable, and A is a countable subset of $\bigoplus_{i\in I} \mathcal{M}$. Then for some countable $J \subseteq I$, $A \subseteq \bigoplus_{i\in J} \mathcal{M}$. Select a countable $J_1 \subseteq I$ with $J \cap J_1 = \emptyset$. Then any element of $\bigoplus_{i\in I} \mathcal{M}$ is A-conjugate to a member of $\bigoplus_{i\in J\cup J_1} \mathcal{M}$, and the latter has only countably many A-types since $J \cup J_1$ is countable. Finally, observe that any automorphism of $\bigoplus_{i\in J\cup J_1} \mathcal{M}$ extends to an automorphism of $\bigoplus_{i\in I} \mathcal{M}$. This completes our sketch of the proof.

3. Abelian groups. We formalize the elementary theory of abelian groups in a logic $\mathcal{L}_{ab}$ having the 2-ary operation-symbol $+$, and no other operation-symbols or relation-symbols. We construe abelian groups as $\mathcal{L}_{ab}$-structures $\langle A, + \rangle$. By looking at any standard list of axioms for abelian groups, one sees easily that the class of abelian groups is an EC class of $\mathcal{L}_{ab}$-structures.

For the purposes of this paper, "group" will mean abelian group.

3.1. We follow the notation of [3], except when we indicate otherwise. We fix some notation for those groups that are the building-blocks of the theory.

3.1.1. $\boldsymbol{Z}$ is the additive group of integers.

3.1.2. $\boldsymbol{Q}$ is the additive group of rationals.

3.1.3. If n is an integer $\geqslant 1$, $\boldsymbol{Z}(n)$ is the cyclic group of order n.

3.1.4. If p is a prime, $\boldsymbol{Z}(p^\infty)$ is the multiplicative group of all roots of unity whose order is a power of p. (Alternatively, $\boldsymbol{Z}(p^\infty)$ is the direct limit of the groups $\boldsymbol{Z}(p^k)$, $k < \omega$, ordered by inclusion.)

3.2. If $n \geqslant 1$ and G is a group, we define nG as the subgroup $\{nx \mid x \in G\}$. If n divides m then $mG \subseteq nG$.

We say G is of bounded order if $nG = \{0\}$, for some n. We say G is divisible if $G = \bigcap_{n\geqslant 1} nG$.

If p is a prime, we define G_p as the subgroup of G consisting of all elements whose order is a power of p. If $G = G_p$, we say G is a p-group.

For $n \geqslant 1$ we define $t_n(G)$ as the subgroup $\{x \mid x \in G \wedge nx = 0\}$. We define $t(G)$ as $\bigcup_{n\geqslant 1} t_n(G)$. $t(G)$ is a subgroup of G. G is called a torsion-group if $G = t(G)$. G is called torsion-free if $t(G) = \{0\}$.

3.3. A basic fact about abelian groups is that if D is a divisible subgroup of G then D is a direct summand of G [3, page 8]. Furthermore G has a unique maximal divisible subgroup. Clearly, any divisible subgroup of G is a subgroup of $\bigcap_{n\geqslant 1} nG$. However, $\bigcap_{n\geqslant 1} nG$ need not be divisible. But if G is ω_1-saturated then $\bigcap_{n\geqslant 1} nG$ is divisible, and so is the maximal divisible subgroup of G. For suppose G is ω_1-saturated, $x \in \bigcap_{n\geqslant 1} nG$ and m is an integer. Then the infinitary condition $x = my \wedge y \in \bigcap_{n\geqslant 1} nG$ is finitely satisfiable in G and so satisfiable. Thus x is divisible in $\bigcap_{n\geqslant 1} nG$, and $\bigcap_{n\geqslant 1} nG$ is divisible.

The trivial group $\{0\}$ is divisible. Any non-trivial divisible group is infinite.

4. Theorem 1 (First Part). We can now prove that if $G = D \oplus H$, where D is divisible and H is of bounded order, then $Th(G)$ is totally transcendental.

LEMMA 2. *If $G = D \oplus H$ where D is divisible and H is of bounded order, and $G_1 \equiv G$, then G_1 is of the form $D_1 \oplus H_1$, where D_1 is divisible and H_1 is of bounded order.*

Proof. Suppose $G = D \oplus H$, where D is divisible and H is of bounded order. Select n such that $nH = \{0\}$. Then $nG = D$. Thus G has the property that nG is divisible. Suppose $G_1 \equiv G$. Then clearly nG_1 is divisible. nG_1 is a direct summand of G, say $G_1 = (nG_1) \oplus H_1$. Since H_1 is a subgroup of G_1 and $H_1 \cap nG_1 = \{0\}$, it follows that $nH_1 = \{0\}$. Let $D_1 = nG_1$. Then

$G_1 = D_1 \oplus H_1$, where D_1 is divisible and H_1 is of bounded order. This proves the lemma.

We now make use of two classical direct-sum decompositions. The uniqueness of these decompositions is not needed just now, but will be fundamental when we discuss ω_1-categoricity.

THEOREM C [3, page 10]. *A non-trivial divisible abelian group is a direct sum of groups each isomorphic to $\boldsymbol{Q}$ or to $\boldsymbol{Z}(p^{\infty})$ for various primes p.*

THEOREM D [3, page 17]. *An abelian group of bounded order is a direct sum of groups each isomorphic to $\boldsymbol{Z}(p^k)$, for various primes p and integers k.*

Since all the groups $\boldsymbol{Q}, \boldsymbol{Z}(p^{\infty})$ and $\boldsymbol{Z}(p^k)$ are countable or finite, and since the collection

$$\{\boldsymbol{Q}\} \cup \{\boldsymbol{Z}(p^{\infty}) \mid p \text{ prime}\} \cup \{\boldsymbol{Z}(p^k) \mid p \text{ prime},\ 0 \leqslant k < \omega\}$$

is countable, we can apply Lemma 1 to conclude the following:

If $G = D \oplus H$, where D is divisible and H is of bounded order, and if A is a countable subset of G, then G has at most ω A-types.

Using Lemma 2, we see that if $G_1 \equiv D \oplus H$, where D is divisible and H is of bounded order, and if A is a countable subset of G_1, then G_1 has at most ω A-types.

By Lemma 0, we conclude that if $G = D \oplus H$, where D is divisible and H is of bounded order, then $Th(G)$ is totally transcendental. We have proved the first half of Theorem 1.

5. Filtrations. In this section we give a sufficient condition for a theory not to be totally transcendental. This condition proved useful to us in [6], and we hope it may have other applications.

Suppose $\mathcal{L}$ has among its symbols a binary operation-symbol $+$. Let $\mathcal{M}$ be a fixed $\mathcal{L}$-structure, and let $+_{\mathcal{M}}$ be the operation on $\mathcal{M}$ corresponding to $+$. For convenience we drop the subscript $\mathcal{M}$, and write $+$ instead of $+_{\mathcal{M}}$.

DEFINITION. $\mathfrak{F}$ is a *filtration of* $\mathcal{M}$ if and only if $\mathfrak{F}$ is a sequence $\langle X_n \rangle_{n<\omega}$, where:

(i) Each X_n is a subset of $|\mathcal{M}|$, and if $m \geqslant n$ then $X_m \subseteq X_n$;

(ii) Each X_n is an abelian group under the operation $+$.

We will be mainly interested in filtrations $\langle X_n \rangle_{n<\omega}$ where each set X_n is definable.

A subset X of $|\mathcal{M}|$ is said to be definable if there is a formula $\Phi(v_0)$ of $\mathcal{L}$, with v_0 as its only free variable, such that X is the set of all elements of $\mathcal{M}$ which satisfy the formula $\Phi(v_0)$.

If $\mathfrak{F}$ is a filtration $\langle X_n \rangle_{n<\omega}$, where each X_n is definable, $\mathfrak{F}$ is called a definable filtration.

5.1. Suppose $\mathfrak{F}$ is a filtration $\langle X_n \rangle_{n<\omega}$ of $\mathfrak{m}$. Let 0 be the neutral element of the group $\langle X_0, + \rangle$. Then 0 is the neutral element of each group $\langle X_n, + \rangle$. For x in X_0, $-x$ is the inverse of x in $\langle X_0, + \rangle$, and, for x, y in X_0, $x - y$ is $x + (-y)$.

Define X_∞ as $\bigcap_{n<\omega} X_n$. Then $0 \in X_\infty$, and X_∞ is an abelian group under $+$.

We define a map $x \to \|x\|$ from X_0 to the set of real numbers as follows:

(a) if $x \in X_\infty$, then $\|x\| = 0$;

(b) if $x \notin X_\infty$, then $\|x\| = 2^{-n}$, where n is the least integer such that $x \notin X_n$.

The following are easily verified:

5.1.1. $\|x\| \geqslant 0$;

5.1.2. $\|x\| = 0$ if and only if $x \in X_\infty$;

5.1.3. $\|x\| = \|-x\|$;

5.1.4. $\|x+y\| \leqslant \max(\|x\|, \|y\|)$.

Now we define a map d from X_0^2 to the reals by:

$$d(x, y) = \|x - y\| .$$

Then d is a pseudo-metric on X_0, satisfying the ultrametric inequality:

5.1.5. $d(x, y) \leqslant \max\big(d(x, z), d(z, y)\big)$.

The following observation, familiar in valuation theory, follows from 5.1.5 and the fact that d is a pseudo-metric.

5.1.6. If $d(x, z) \neq d(z, y)$ then $d(x, y) = \max\big(d(x, z), d(z, y)\big)$.

d defines a pseudo-metric topology on X_0, and clearly $+$ is continuous for this topology. The topology is Hausdorff if and only if $X_\infty = \{0\}$.

DEFINITION. Suppose Γ is a subgroup of $\langle X_0, + \rangle$. We say Γ is *completely filtered* by $\mathfrak{F}$ if and only if

(a) $|\Gamma| \cap X_\infty = \{0\}$, and

(b) the chain $|\Gamma| \supset |\Gamma| \cap X_1 \supset \ldots \supset |\Gamma| \cap X_n \supset |\Gamma| \cap X_{n+1} \supset \ldots$ is strictly descending.

For us the importance of the notion comes from the following lemma.

LEMMA 3. *Suppose $\mathcal{L}$ is countable, and $\mathcal{M}$ is an $\mathcal{L}$-structure. Suppose $\mathfrak{F}$ is a definable filtration $\langle X_n \rangle_{n<\omega}$ of $\mathcal{M}$, and Γ is a subgroup of $\langle X_0, + \rangle$ which is completely filtered by $\mathfrak{F}$. Then $Th(\mathcal{M})$ is not totally transcendental.*

Proof. Assume the hypothesis of the lemma, and the notation of the preceding discussion. For $n < \omega$, let $\Phi_n(v_0)$ be a formula defining X_n.

We observe first that we can assume without loss of generality that Γ is countable. For if we start with an arbitrary Γ that is completely filtered, select elements x_n, for $n<\omega$, such that $x_n \in |\Gamma| \cap X_n$, $x_n \notin |\Gamma| \cap X_{n+1}$, and let Γ' be the group generated by the elements x_n. Then Γ' is countable, and is completely filtered by $\mathfrak{F}$.

So we assume Γ is countable. Let s $(\in |\mathcal{M}|^{\omega})$ be a fixed enumeration of $|\Gamma|$. We will show that $Th\big((\mathcal{M}, s)\big)$ has uncountably many complete extensions in $\mathcal{L}(\omega+1)$.

Since $|\Gamma| \cap X_\infty = \{0\}$, the pseudo-metric d is in fact a metric on Γ. We claim that no point of Γ is isolated. Since Γ is a metric group it suffices to prove that 0 is not isolated. Let ε be an arbitrary positive real number, and choose the integer n so that $2^{-n}<\varepsilon$. Since Γ is completely filtered by $\mathfrak{F}$, $|\Gamma| \cap X_{n-1} \neq \{0\}$. Select x in $|\Gamma| \cap X_{n-1}$ with $x \neq 0$. Then $\|x\| \leqslant 2^{-n}$, so $d(x,0) \leqslant 2^{-n} < \varepsilon$. Thus 0 is not isolated in Γ.

We now complete the metric group $\langle \Gamma, d\rangle$ to a metric group $\langle \Gamma^*, d^*\rangle$, where Γ^* is complete under d^*, and Γ is dense in Γ^*. Γ^* has no isolated points, and so is uncountable, by Baire Category.

A useful observation about d^* is that it satisfies the ultrametric inequality 5.1.5. This follows easily from the fact that d satisfies 5.1.5 and Γ is dense in Γ^*. Since d^* satisfies 5.1.5, it also satisfies 5.1.6.

We are going to define 1-1 map from Γ^* to the set of complete extensions of $Th\big((\mathcal{M}, s)\big)$ in $\mathcal{L}(\omega+1)$. This will prove the lemma. The basic idea is that a point of Γ^* can be specified by a Cauchy sequence of elements of Γ, and in turn this Cauchy sequence can be coded by a set of formulas of $\mathcal{L}(\omega+1)$.

Let x be a point of Γ^*. We define a set Σ_x of $\mathcal{L}(\omega+1)$-formulas, by specifying its members, which are of two kinds.

First kind. All formulas

$$\Phi_n(c_\omega + c_m)\,,$$

where $d^*\big(x, -s(m)\big) < 2^{-n}$.

Second kind. All formulas

$$\neg\Phi_k(c_\omega + c_m)$$

where $d^*\big(x, -s(m)\big) > 2^{-k}$.

We claim first that

$$Th\big((\mathcal{M}, s)\big) \cup \Sigma_x$$

is satisfiable. By the Compactness Theorem it suffices to prove that if Δ is a finite subset of Σ_x then

$$Th\big((\mathcal{M}, s)\big) \cup \Delta$$

is satisfiable. We enumerate Δ as

$$\Phi_{n_i}(c_\omega + c_{m_i}) \quad 0 \leqslant i \leqslant I \quad \text{and} \quad \neg\Phi_{k_j}(c_\omega + c_{l_j}) \quad 0 \leqslant j \leqslant J\,.$$

Consider the corresponding basic open sets of Γ^*:

$$\{t|\ d^*\big(t, -s(m_i)\big) < 2^{-n_i}\} \qquad 0 \leqslant i \leqslant I\,,$$

and

$$\{t|\ d^*\big(t, -s(l_j)\big) > 2^{-k_j}\} \qquad 0 \leqslant j \leqslant J\,.$$

Let U be the intersection of these open sets. Then U is open. Moreover, by the definition of Σ_x, $x \in U$. Since Γ is dense in Γ^*, we may select an element u from $U \cap |\Gamma|$.

Let s_u be the unique extension of s to $\omega+1$ such that $s_u(\omega) = u$. We claim that

$$(\mathcal{M}, s_u) \models Th\big((\mathcal{M}, s)\big) \cup \Delta\,.$$

It is obvious that it suffices to prove that

$$(\mathcal{M}, s_u) \models \Delta\,.$$

Consider first a member of Δ of the first kind, e.g. $\Phi_{n_i}(c_\omega + c_{m_i})$. Since u is chosen so that

$$d^*\big(u, -s(m_i)\big) < 2^{-n_i},$$

we have

$$d\big(u, -s(m_i)\big) < 2^{-n_i},$$

whence

$$\|u + s(m_i)\| < 2^{-n_i},$$

whence

$$u + s(m_i) \in X_{n_i}\,.$$

Since $\Phi_{n_i}(v_0)$ defines X_{n_i}, we conclude that

$$(\mathcal{M}, s_u) \models \Phi_{n_i}(c_\omega + c_{m_i})\,.$$

A completely analogous argument works for formulas of the second kind.

We conclude that

$$(\mathcal{M}, s_u) \models Th\big((\mathcal{M}, s)\big) \cup \Delta\,.$$

Since Δ was an arbitrary finite subset of Σ_x, we have proved that

$$Th\big((\mathcal{M}, s)\big) \cup \Sigma_x$$

is satisfiable. Thus Σ_x extends to a complete extension of $Th\big((\mathcal{M}, s)\big)$ in $\mathcal{L}(\omega+1)$. Select such an extension $\bar{\Sigma}_x$. We claim the map $x \to \bar{\Sigma}_x$ is 1-1. Clearly it suffices to prove that if $x \neq y$ then

$$Th\big((\mathcal{M}, s)\big) \cup \Sigma_x \cup \Sigma_y$$

is not satisfiable.

Suppose $x \neq y$. Select an integer n such that $2^{-n} < d^*(x, y)$. Using the density of Γ in Γ^*, select u in Γ so that $d^*(x, u) < 2^{-n}$. Then $d^*(x, u) < d^*(x, y)$, so by 5.1.6 for d^*,

$$d^*(y, u) = d^*(x, y).$$

Then $d^*(x, u) < 2^{-n}$, whereas $d^*(y, u) > 2^{-n}$.

Now $-u = s(m)$ for some m.

We conclude that

$$\Phi_n(c_\omega + c_m) \in \Sigma_x \quad \text{whereas} \quad \neg\Phi_n(c_\omega + c_m) \in \Sigma_y .$$

It follows that

$$Th\big((\mathcal{M}, s)\big) \cup \Sigma_x \cup \Sigma_y$$

is not satisfiable.

Therefore $x \to \bar{\Sigma}_x$ is 1-1, and since Γ^* is uncountable the proof is complete.

COROLLARY. *Suppose $\mathcal{L}$ is countable, and $\mathcal{M}$ is an $\mathcal{L}$-structure such that $Th(\mathcal{M})$ is totally transcendental. Suppose $\mathfrak{F}$ is a definable filtration $\langle X_n \rangle_{n<\omega}$ of $\mathcal{M}$, and suppose Γ is a subgroup of $\langle X_0, + \rangle$ such that $|\Gamma| \cap X_\infty = \{0\}$. Then there exists an integer n_0 such that for all $n \geqslant n_0$, $|\Gamma| \cap X_n = |\Gamma| \cap X_{n_0}$.*

Proof. Assume the hypotheses of the corollary, but suppose the conclusion fails. Then there exists an increasing sequence $\langle n_m \rangle_{m<\omega}$ such that the chain

$$|\Gamma| \cap X_{n_0} \supset |\Gamma| \cap X_{n_1} \supset \ldots \supset |\Gamma| \cap X_{n_m} \supset |\Gamma| \cap X_{n_{m+1}} \supset \ldots$$

is strictly descending. Then $\langle X_{n_m} \rangle_{m<\omega}$ is a definable filtration of $\mathcal{M}$, and Γ is completely filtered by $\langle X_{n_m} \rangle_{m<\omega}$. By Lemma 3, $Th(\mathcal{M})$ is not totally transcendental, contrary to hypothesis. This proves the corollary.

6. Theorem 1 (Second Part). We now prove that if G is an abelian group with $Th(G)$ totally transcendental, then G is of the form $D \oplus H$, where D is divisible and H is of bounded order. This will complete the proof of Theorem 1.

By Lemma 2, we may assume without loss of generality that G is ω_1-saturated, so $\bigcap_{n \geqslant 1} nG$ is divisible by 3.3. We define a filtration $\langle X_n \rangle_{n<\omega}$ of G by:

(a) $X_0 = G$;

(b) $X_{n+1} = (n+1)!\, G$, for $n > 0$.

Then $\langle X_n \rangle_{n<\omega}$ is obviously a definable filtration of G. X_∞, as defined in the last section, is clearly $\bigcap_{n \geqslant 1} nG$, the maximal divisible subgroup of G.

Select H so that $G = X_\infty \oplus H$. Then $H \cap X_\infty = \{0\}$.

Suppose $Th(G)$ is totally transcendental. Then by the corollary to

Lemma 3 there exists n_0 such that $H \cap X_n = H \cap X_{n_0}$ for all $n > n_0$. It follows that $H \cap X_{n_0} = H \cap X_\infty = \{0\}$. Thus $H \cap (n_0+1)!\, G = \{0\}$, and since H is a subgroup of G we conclude that $(n_0+1)!\, H = \{0\}$. Thus H is of bounded order. Since $G = X_\infty \oplus H$, and X_∞ is divisible, we conclude that G is a direct sum of a divisible group and a group of bounded order. Theorem 1 is now proved.

7. ω_1-categoricity. From Theorem B and Theorem 1, we now see that if $Th(G)$ is ω_1-categorical then G is of the form $D \oplus H$ where D is divisible and H is of bounded order. The converse is far from true.

LEMMA 4. *Suppose $G = D \oplus H$, where D is divisible, H is of bounded order, and D and H are infinite. Then $Th(G)$ is not ω_1-categorical.*

We sketch the (simple) proof. By the Löwenheim–Skolem Theorems and the fact that $\oplus$ preserves $\equiv$, we get, for $i = 1, 2$, groups $G_i \equiv G$, where G_i is of cardinality ω_1, and $G_i = D_i \oplus H_i$ where $D_i \equiv D$, $H_i \equiv H$, and D_1 is countable and D_2 uncountable. Clearly G_1 and G_2 are not isomorphic, so $Th(G)$ is not ω_1-categorical.

7.1. Because of the fact that a finite divisible group is trivial, we now see that if $Th(G)$ is ω_1-categorical then either G is of bounded order, or G is $D \oplus H$ where D is divisible and H is finite. In analysing these subcases we shall use a technique very similar to that in the preceding proof, making use of theorems about the uniqueness of the decompositions given in Theorems C and D.

Before looking at the separate cases of divisible groups and groups of bounded order, we get the following lemma out of the way.

LEMMA 5. *Suppose D is divisible and $Th(D)$ is ω_1-categorical, and suppose H is finite. Then $Th(D \oplus H)$ is ω_1-categorical.*

Proof. Assume the hypotheses of the lemma. Let n be the cardinality of H. Let $G = D \oplus H$. Then $nH = \{0\}$, and $nG = D$. It is clear that we can express by a set of first-order conditions that nG is elementarily equivalent to D, and that G/nG is isomorphic to the finite group H.

Suppose $G^{(1)}$ and $G^{(2)}$ are of cardinality ω_1, and are elementarily equivalent to G. Then $nG^{(1)}$ and $nG^{(2)}$ are elementarily equivalent to D, and $G^{(1)}/nG^{(1)}$ and $G^{(2)}/nG^{(2)}$ are both isomorphic to H. Since D is divisible, $nG^{(1)}$ and $nG^{(2)}$ are divisible. Thus $G^{(1)} = nG^{(1)} \oplus H_1$, and $G^{(2)} = nG^{(2)} \oplus H_2$, where H_1 and H_2 are isomorphic to H. Therefore H_1 and H_2 are isomorphic finite groups, and $nG^{(1)}$ and $nG^{(2)}$ have cardinality ω_1. Since $Th(D)$ is ω_1-categorical, $nG^{(1)} \cong nG^{(2)}$. It follows that $nG^{(1)} \oplus H_1 \cong nG^{(2)} \oplus H_2$, whence $G^{(1)} \cong G^{(2)}$.

Since $G^{(1)}$ and $G^{(2)}$ were arbitrary members of $\mathrm{Mod}\big(Th(D \oplus H)\big)$ of cardinality ω_1, we conclude that $Th(D \oplus H)$ is ω_1-categorical.

7.2. In this subsection we prove that if D is a divisible group then $Th(D)$ is ω_1-categorical if and only if for each prime p there are only finitely many elements of D of order p.

We look at the uniqueness statement corresponding to Theorem C.

THEOREM C$^+$. *Suppose D is divisible. Then there are index sets I_0, and I_p for prime p, such that*

$$D = [\bigoplus_{i\in I_0} \boldsymbol{Q}]\oplus[\bigoplus_{i\in I_2} \boldsymbol{Z}(2^\infty)]\oplus \ldots\oplus[\bigoplus_{i\in I_p} \boldsymbol{Z}(p^\infty)]\oplus \ldots$$

The cardinalities of these index sets are uniquely determined.

For a proof, one should consult [3, page 11]. We indicate how the cardinalities of these index sets may be characterized.

(a) For a prime p, I_p is infinite if and only if $t_p(D)$ is infinite. If $t_p(D)$ is finite, the cardinality of $t_p(D)$ is of the form p^n, where n is the cardinality of I_p.

(b) The cardinality of I_0 is the dimension of $D/t(D)$ construed as a vector-space over the rationals.

In particular, one sees that the conditions on D that I_p, for a fixed prime p, has a fixed finite cardinality, or is infinite, are first-order conditions. The situation is quite different for I_0, as we will soon see.

Suppose D is divisible. We define a function f_D from the set of primes to $\omega+1$, as follows:

(i) $f_D(p) = n$ if $t_p(D)$ has cardinality p^n;

(ii) $f_D(p) = \omega$ if $t_p(D)$ is infinite.

The next theorem follows directly from Szmielew's criterion [11] for the elementary equivalence of abelian groups.

THEOREM E. *Suppose $D^{(1)}$ and $D^{(2)}$ are non-trivial divisible abelian groups. Then $D^{(1)} \equiv D^{(2)}$ if and only if $f_{D^{(1)}} = f_{D^{(2)}}$.*

(Remark. One can avoid appeal to [11], as follows. By the Löwenheim–Skolem Theorem, it suffices to prove the result when $D^{(1)}$ and $D^{(2)}$ are countable. Let E be a non-principal ultrafilter on ω, and let $D^{(3)}$, $D^{(4)}$ be respectively $(D^{(1)})^\omega/E$, $(D^{(2)})^\omega/E$. Then $D^{(3)} \equiv D^{(1)}$, $D^{(4)} \equiv D^{(2)}$. $D^{(3)}$ and $D^{(4)}$ have the cardinality of the continuum. Assuming $f_{D^{(1)}} = f_{D^{(2)}}$ one readily proves that $t_p(D^{(3)})$ and $t_p(D^{(4)})$ have the same cardinality for all primes p. Finally one shows that $D^{(3)}/t(D^{(3)})$ and $D^{(4)}/t(D^{(4)})$ have the same dimension as vector-spaces over the rationals. It follows that $D^{(3)} \cong D^{(4)}$, whence $D^{(1)} \equiv D^{(2)}$.)

With reference to the decomposition of D given by Theorem C$^+$, one sees immediately from Theorem E that the elementary type of D is independent of the cardinality of I_0. We exploit this idea in the following lemma.

LEMMA 6. *Suppose D is a divisible abelian group. Then $Th(D)$ is ω_1-categorical if and only if $f_D(p) < \omega$ for all primes p.*

Proof. The result is clearly true for the divisible group $\{0\}$. Henceforward we assume D is an infinite divisible abelian group. Because of Theorem E and the Löwenheim–Skolem Theorems, we may assume D has cardinality ω_1.

Necessity. Suppose $Th(D)$ is ω_1-categorical.

Using Theorem C$^+$, decompose D as

$$[\bigoplus_{i\in I_0} \boldsymbol{Q}]\oplus[\bigoplus_{i\in I_2} \boldsymbol{Z}(2^\infty)]\oplus \ldots\oplus[\bigoplus_{i\in I_p} \boldsymbol{Z}(p^\infty)]\oplus \ldots$$

Suppose $f_D(q) = \omega$ for some prime q. Then I_q is infinite.

By Theorem E, we can assume without loss of generality that I_q has cardinality ω_1.

Let J_0 be an arbitrary extension of I_0, of cardinality ω_1. Let J_q be an arbitrary subset of I_q of cardinality ω. For a prime $p \neq q$, let J_p be I_p. Let $D^{(1)}$ be

$$[\bigoplus_{j\in J_0} \boldsymbol{Q}]\oplus[\bigoplus_{j\in J_2} \boldsymbol{Z}(2^\infty)]\oplus \ldots\oplus[\bigoplus_{j\in J_p} \boldsymbol{Z}(p^\infty)]\oplus \ldots$$

Then clearly $f_{D^{(1)}} = f_D$, so $D^{(1)} \equiv D$ by Theorem E. Clearly $D^{(1)}$ has cardinality ω_1, since J_0 has cardinality ω_1. But $D^{(1)}$ is not isomorphic to D, since $t_q(D^{(1)})$ is countable, while $t_q(D)$ is uncountable.

Thus we contradict the ω_1-categoricity of $Th(D)$. It follows that if $Th(D)$ is ω_1-categorical then $f_D(p) < \omega$ for all primes p.

Sufficiency. Suppose $f_D(p) < \omega$ for all primes p. Decompose D as

$$[\bigoplus_{i\in I_0} \boldsymbol{Q}]\oplus[\bigoplus_{i\in I_2} \boldsymbol{Z}(2^\infty)]\oplus \ldots\oplus[\bigoplus_{i\in I_p} \boldsymbol{Z}(p^\infty)]\oplus \ldots$$

Then I_p is finite, for each prime p. Since D has cardinality ω_1, it follows that I_0 has cardinality ω_1.

Suppose $D^{(1)} \equiv D$, and $D^{(1)}$ has cardinality ω_1. Then $f_{D^{(1)}} = f_D$, so $f_{D^{(1)}}(p) < \omega$ for all primes p. Decompose $D^{(1)}$ as

$$[\bigoplus_{j\in J_0} \boldsymbol{Q}]\oplus[\bigoplus_{j\in J_2} \boldsymbol{Z}(2^\infty)]\oplus \ldots\oplus[\bigoplus_{j\in J_p} \boldsymbol{Z}(p^\infty)]\oplus \ldots$$

Then J_p has the same cardinality as I_p, for each prime p, since $f_{D^{(1)}}(p) = f_D(p)$. Also, J_0 has cardinality ω_1, since each J_p is finite and $D^{(1)}$ has cardinality ω_1. Now it is obvious that $D^{(1)} \cong D$.

Since $D^{(1)}$ was arbitrary, $Th(D)$ is ω_1-categorical.

This completes the proof of the lemma.

7.2′. The condition that $f_D(p) < \omega$ for all primes p is equivalent to the condition that, for each prime p, D has only finitely many elements of order p.

We can now characterize these abelian groups G such that G is not of bounded order and $Th(G)$ is ω_1-categorical.

LEMMA 7. *Suppose G is not of bounded order. Then $Th(G)$ is ω_1-categorical if and only if G is of the form $D \oplus H$, where H is finite and D is a divisible group with the property that for each prime p D has only finitely many elements of order p.*

Proof. Suppose G is not of bounded order.

Necessity. Suppose $Th(G)$ is ω_1-categorical. Then, by Theorem B, $Th(G)$ is totally transcendental. By Theorem 1, G is of the form $D \oplus H$, where D is divisible and H is of bounded order. By Lemma 4, since $Th(G)$ is ω_1-categorical, either D or H is finite. Since G is not of bounded order we conclude that H is finite.

We leave the rest of the proof to the reader. It is just like the corresponding part of the proof of Lemma 6.

Sufficiency. Suppose $G = D \oplus H$, where H is finite and D is divisible with the property that, for each prime p, D has only finitely many elements of order p. Then $f_D(p) < \omega$ for all primes p, so by Lemma 6 $Th(D)$ is ω_1-categorical. By Lemma 5, $Th(D \oplus H)$ is ω_1-categorical, i.e. $Th(G)$ is ω_1-categorical.

This proves the lemma.

7.3. We now have to characterize those abelian groups G such that G is of bounded order and $Th(G)$ is ω_1-categorical.

We first look at the uniqueness statement corresponding to Theorem D.

The notation $\bigoplus_{p^m}$ indicates a direct sum taken over all integers p^m where p is a prime and m is positive.

THEOREM D$^+$. *Suppose G is an abelian group of bounded order. Then, for each prime p and positive integer m, there is an index set I_{p^m} such that*

$$G = \bigoplus_{p^m}[\bigoplus_{i \in I_{p^m}} \boldsymbol{Z}(p^m)] .$$

Moreover, the cardinalities of the index sets I_{p^m} are uniquely determined.

For a proof one should consult Kaplansky, pages 17 and 27.

As with Theorem C$^+$, we indicate how the cardinalities of the above index sets can be characterized.

Let p be prime, and m a positive integer. We define $U_{m,p}(G)$ as $t_p(G_p) \cap p^m G_p$. Then

$$U_{m+1,p}(G) \subseteq U_{m,p}(G) .$$

Also, $pU_{m,p}(G) = \{0\}$, so $U_{m,p}(G)$ can be construed as a vector-space over GF_p, the prime field of characteristic p.

Then it turns out [Kaplansky, page 27] that the cardinality of I_{p^m} is the dimension, as a vector-space over GF_p, of the quotient-space

$$U_{m-1,p}/U_{m,p} .$$

Let k be a fixed finite cardinal. It is easily verified that the following condition on G is expressible by a first-order sentence of $\mathfrak{L}_{ab}$: the dimension of $U_{m-1,p}/U_{m,p}$ is k.

From this one sees that the following condition is expressed by an infinite set of first-order sentences: the dimension of $U_{m-1,p}/U_{m,p}$ is infinite.

This leads us to define a map μ_G from the set of prime powers to $\omega+1$, thus:

(a) $\mu_G(p^m) =$ the dimension of $U_{m-1,p}(G)/U_{m,p}(G)$, if this dimension is finite;

(b) $\mu_G(p^m) = \omega$, if the above dimension is infinite.

It turns out that μ_G characterizes the elementary type of G, for G of bounded order, just as f_G characterizes the type of G, for divisible G. This follows easily from Szmielew's work, but it is convenient for us to give a proof.

THEOREM F. *Suppose $G^{(1)}$ and $G^{(2)}$ are abelian groups of bounded order. Then $G^{(1)} \equiv G^{(2)}$ if and only if $\mu_{G^{(1)}} = \mu_{G^{(2)}}$.*

Proof. Necessity is clear by the preceding remarks.

Sufficiency. Suppose $G^{(1)}$ and $G^{(2)}$ are of bounded order and $\mu_{G^{(1)}} = \mu_{G^{(2)}}$. Using Theorem D$^+$ we decompose $G^{(1)}$ and $G^{(2)}$ thus:

$$G^{(1)} = \bigoplus_{p^m}[\bigoplus_{i \in I_{p^m}} \boldsymbol{Z}(p^m)] \quad \text{and} \quad G^{(2)} = \bigoplus_{p^m}[\bigoplus_{j \in J_{p^m}} \boldsymbol{Z}(p^m)] .$$

Since $\mu_{G^{(1)}} = \mu_{G^{(2)}}$, it follows that, for each prime power p^m, either I_{p^m} and J_{p^m} have the same finite cardinality, or both I_{p^m} and J_{p^m} are infinite.

If I_{p^m} and J_{p^m} have the same finite cardinality, then clearly

$$\bigoplus_{i \in I_{p^m}} \boldsymbol{Z}(p^m) \equiv \bigoplus_{j \in J_{p^m}} \boldsymbol{Z}(p^m) .$$

We will prove that if I_{p^m} and J_{p^m} are infinite then

$$\bigoplus_{i \in I_{p^m}} \boldsymbol{Z}(p^m) \equiv \bigoplus_{j \in J_{p^m}} \boldsymbol{Z}(p^m) .$$

From this it will follow, from the fact [2, 8] that the direct sum operation preserves elementary equivalence, that $G^{(1)} \equiv G^{(2)}$. This will prove the theorem.

Suppose then that I_{p^m} and J_{p^m} are infinite. Let $I = I_{p^m}$ and $J = J_{p^m}$. Let $H^{(1)}$ be $\bigoplus_{i \in I} Z(p^m)$, and $H^{(2)}$ be $\bigoplus_{j \in J} Z(p^m)$. We have to prove that $H^{(1)} \equiv H^{(2)}$.

We observe that $\mu_{H^{(1)}} = \mu_{H^{(2)}}$. Furthermore,

(a) $\mu_{H^{(1)}}(p^m) = \omega$, and

(b) $\mu_{H^{(1)}}(q^n) = 0$ if $q^n \neq p^m$.

Since $H^{(1)}$ and $H^{(2)}$ are infinite, there exist, by the Löwenheim–Skolem Theorems, $H^{(3)}$ and $H^{(4)}$ of cardinality ω_1 such that $H^{(3)} \equiv H^{(1)}$ and $H^{(4)} \equiv H^{(2)}$. Then $H^{(3)}$ and $H^{(4)}$ are of bounded order, and $\mu_{H^{(3)}} = \mu_{H^{(1)}} = \mu_{H^{(2)}} = \mu_{H^{(4)}}$.

Using Theorem D^+, and (a) and (b) above, we deduce that there exist index sets I' and J' such that

$$H^{(3)} = \bigoplus_{i \in I'} \boldsymbol{Z}(p^m) \quad \text{and} \quad H^{(4)} = \bigoplus_{j \in J'} \boldsymbol{Z}(p^m).$$

Since $H^{(3)}$ and $H^{(4)}$ have cardinality ω_1, it follows that I' and J' have cardinality ω_1, whence $H^{(3)}$ and $H^{(4)}$ are isomorphic. Thus $H^{(3)} \equiv H^{(4)}$, whence $H^{(1)} \equiv H^{(2)}$.

This concludes the proof.

We now prove the analogue of Lemma 6.

LEMMA 8. *Suppose G is an abelian group of bounded order. Then $Th(G)$ is ω_1-categorical if and only if there is at most one prime power p^m such that $\mu_G(p^m) = \omega$.*

Proof. By Theorem F and the Löwenheim–Skolem Theorems, it suffices to prove the lemma when G is countable or finite. So we suppose G has cardinality $\leqslant \omega$.

Necessity. If there are two distinct prime powers q^n and r^k with $\mu_G(q^n) = \mu_G(r^k) = \omega$, then we can use the same technique as in the necessity part of Lemma 6, to get non-isomorphic G_1 and G_2 of cardinality ω_1, with $G_1 \equiv G \equiv G_2$. G_1 will have ω copies of $Z(q^n)$ in its decomposition relative to Theorem D, while G_2 will have ω_1 copies of $Z(q^n)$. By Theorem D^+, G_1 and G_2 are not isomorphic. We leave the details to the reader.

Sufficiency. Suppose G is of bounded order, and there exists at most one prime power p^m such that $\mu_G(p^m) = \omega$. Obviously if G is finite $Th(G)$ is ω_1-categorical. If G is infinite, $Th(G)$ has models of cardinality ω_1, and by looking at the direct-sum decomposition of such a model we see that there exists a prime power p^m such that $\mu_G(p^m) = \omega$.

So we suppose G is infinite and q^n is the unique prime power such that $\mu_G(q^n) = \omega$. Then $\mu_G(p^m) < \omega$ if $p^m \neq q^n$. Let $G^{(1)}$ and $G^{(2)}$ be two models of $Th(G)$ of cardinality ω_1, and decompose $G^{(1)}$ and $G^{(2)}$ as:

$$G^{(1)} = \bigoplus_{p^m} [\bigoplus_{i \in I_{p^m}} \boldsymbol{Z}(p^m)], \quad G^{(2)} = \bigoplus_{p^m} [\bigoplus_{j \in J_{p^m}} \boldsymbol{Z}(p^m)].$$

Since $G^{(1)} \equiv G \equiv G^{(2)}$, it follows that $\mu_{G^{(1)}}(p^m) = \mu_{G^{(2)}}(p^m) < \omega$ if $p^m \neq q^n$.

Therefore I_{p^m} and J_{p^m} have the same finite cardinality if $p^m \neq q^n$. Since $G^{(1)}$ and $G^{(2)}$ have cardinality ω_1, it follows that I_{q^n} and J_{q^n} have cardinality ω_1. We conclude that $G^{(1)} \cong G^{(2)}$.

We conclude that $Th(G)$ is ω_1-categorical. This proves the lemma.

7.4. We can now give a complete classification of those abelian groups G such that $Th(G)$ is ω_1-categorical.

Firstly, Theorem B, Theorem 1 and Lemma 4 tell us that we can confine our attention to groups of bounded order, and groups $D \oplus H$ where D is divisible and H is finite.

Case 1. G is of bounded order. We decompose G as

$$\bigoplus_{p^m} [\bigoplus_{i \in I_{p^m}} \boldsymbol{Z}(p^m)].$$

Lemma 8 tells us that $Th(G)$ is ω_1-categorical if and only if $\mu_G(p^m) = \omega$ for at most one prime power p^m. By the remarks following Theorem D^+, it follows that $Th(G)$ is ω_1-categorical if and only if there is at most one p^m such that I_{p^m} is infinite.

Since G is of bounded order it is clear that there are only finitely many prime powers q^n such that I_{q^n} is non-empty. If I_{q^n} is finite for each q^n, then G is finite. If there is exactly one p^m such that I_{p^m} is infinite, then

$$G = [\bigoplus_{i \in I_{p^m}} \boldsymbol{Z}(p^m)] \oplus H,$$

where H is finite.

We deduce that if $Th(G)$ is ω_1-categorical then $G = K \oplus H$, where H is finite and K is a direct sum of copies of a fixed finite cyclic group of prime power order.

Conversely, suppose G is of this form. Then $\mu_G(p^m) = \omega$ for at most one p^m, so by Lemma 8 $Th(G)$ is ω_1-categorical.

This proves that if G is of bounded order then $Th(G)$ is ω_1-categorical if and only if G is of the form $K \oplus H$ where H is finite and K is a direct sum of copies of a fixed finite cyclic group of prime power order.

Case 2. G is not of bounded order. Then, by Lemma 7 $Th(G)$ is ω_1-categorical if and only if G is of the form $D \oplus H$ where H is finite and D is divisible with the property that for each prime p D has only finitely many elements of order p.

This completes our classification of ω_1-categorical theories of abelian groups, and proves Theorem 2.

8. Concluding remarks. We would like to extend our classification to theories $Th(G)$ where G is a non-abelian group. Of course, we have no classification of complete theories of groups, but this need not prevent us classifying ω_1-categorical theories of groups. (We have no classification of complete theories of fields, but in [6] we classify the ω_1-categorical theories of fields.) It seems likely that in order to make an advance on the problem one will have to use techniques like Ehrenfeucht's condition, or Keisler's finite cover property [1, 4].

When working on this paper we proved the following result, which may be useful.

THEOREM 3. *Suppose $\mathcal{L}$ is countable, and $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{L}$-structures such that $Th(\mathcal{M}_1)$ and $Th(\mathcal{M}_2)$ are totally transcendental. Then $Th(\mathcal{M}_1 \oplus \mathcal{M}_2)$ is totally transcendental.*

This result fails if we replace "totally transcendental" by "ω_1-categorical". To see this, take $\mathcal{M}_1$ as $\boldsymbol{Q}$, $\mathcal{M}_2$ as $\bigoplus_{i \in I} \boldsymbol{Z}(p)$ where I is infinite and p is prime, and use Lemma 4.

The result also fails for infinite direct sums and products. Thus, $Th\big(Z(p^n)\big)$ is totally transcendental, but, by Theorem 1, neither

$$Th\big(\bigoplus_n Z(p^n)\big),$$

nor

$$Th\big(\prod_n Z(p^n)\big)$$

is totally transcendental.

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Some theorems about the embeddability of ANR-sets into decomposition spaces of E^n

by

H. Patkowska (Warszawa)

1. Introduction. This paper is a continuation of my earlier paper [18], in which the following general theorem has been proved:

THEOREM A ([18], p. 290). *If X is a connected ANR containing no n-umbrella and if the cyclic elements of X are embeddable into E^n, then X is embeddable into an n-dimensional Cartesian divisor of E^{n+1}.*

As a corollary to this theorem and to Claytor's results ([6] and [7]) the following theorem has been deduced:

THEOREM B ([18], p. 291). *If X is a connected ANR which does not contain any 2-umbrellas and any homeomorphic images of the graphs of Kuratowski, then X is embeddable into S^2.*

This theorem gives a positive answer to a problem of Mardešić and Segal ([13], p. 637). In [18] some historical remarks concerning Theorems A and B have been given, which we do not repeat here. The following remarks concern the terminology. Only metrizable separable spaces are considered. The ANR-spaces are always assumed to be compact. We base our considerations on the definition and the propositions concerning cyclic elements given in [12], § 47, which have been recalled in [18]. Therefore, we do not repeat them here, although, in general we give references to respective propositions proved in [12], § 47. By an n-umbrella we mean a one-point union of a (topological) n-ball Q and of an arc I relative to a point $p \in \mathring{Q}$ and a point $q \in \dot{I}$. By a graph we mean any space which is a homeomorphic image of a compact, at most 1-dimensional polyhedron. A connected, acyclic graph (i.e. a graph which is an AR-set) is called a tree. The graphs of Kuratowski (which are called *primitive skew curves* by Mardešić and Segal) are the following polyhedra K_1 and K_2 (cf. [11]): K_1 is the 1-skelton of a 3-simplex in which the midpoints of a pair of non-adjacent edges are joined by a segment, K_2 is the 1-skelton of a 4-simplex. Given a space X, any space Y is called a *Cartesian divisor* of X if there is a space Z such that the product $Y \times Z$ is homeomorphic with X.