Structure and embedding theorems for unique normal decomposition lattices

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In [2] we characterized those distributive Noether lattices which can be represented as the lattice of ideals of a Noetherian ring. Those lattices have the property that every element has a unique normal primary decomposition involving only powers of primes. In this paper we consider a broader class of multiplicative lattices, namely those which satisfy the less restrictive condition that every element has a unique normal decomposition. For such a lattice $\mathfrak{L}$ we obtain a structure theorem (Theorem 1), and a characterization in terms of a "dense" embedding of the lattice of ideals of a suitable Noetherian ring (Theorem 2). We also give a condition under which $\mathfrak{L}$ can be represented as the lattice of ideals of some Noetherian ring (Theorem 3).

A (commutative) *multiplicative lattice* is a complete lattice in which there is defined a commutative, associative and join-distributive multiplication for which the greatest element, denoted by I, is the multiplicative identity. An element M in a multiplicative lattice $\mathfrak{L}$ is a *join-principal element* if $(A\vee BM):M = A:M\vee B$ for all $A, B\in\mathfrak{L}$. Each principal ideal in a commutative ring R is a join-principal element in the lattice of ideals of R [1].

Throughout this paper, $\mathfrak{L}$ will denote a commutative multiplicative lattice in which each element is a join (not necessarily finite) of join-principal elements and in which I is compact.

An element Q *is primary for* a prime element P (or Q is P-primary) if $P^n\leqslant Q\leqslant P$ for some integer n and the relation $AB\leqslant Q$ implies that $A\leqslant Q$ or $B\leqslant P$. An irredundant decomposition $A = Q_1\wedge\ldots\wedge Q_k$ is a *normal decomposition* of A if Q_i is primary for, say, P_i and the P_i are distinct. $\mathfrak{L}$ is a *unique normal decomposition lattice* (or UND lattice) if each element of $\mathfrak{L}$ has a unique normal decomposition.

We will first prove the Krull Intersection Theorem for $\mathfrak{L}$.

LEMMA 1. *Let $\mathfrak{L}$ be a lattice in which each element has a normal decomposition. Let $\mathfrak{J}$ denote the greatest lower bound of the collection of maximal elements of $\mathfrak{L}$, and let B be an element of $\mathfrak{L}$ such that $B \leqslant \mathfrak{J}$. Then $\bigwedge_{k=1}^{\infty}(A \vee B^k) = A$ for each $A \in \mathfrak{L}$.*

Proof. It suffices to prove that if M is a join-principal element such that $M \leqslant \bigwedge_{k=1}^{\infty}(A \vee B^k)$, then $M \leqslant A$. Hence, let M be a join-principal element which satisfies the relation $M \leqslant \bigwedge_{k=1}^{\infty}(A \vee B^k)$. Let $A = Q_1 \wedge \ldots \wedge Q_n$ be a normal decomposition of A where Q_i is P_i-primary, and let $A \vee BM = T_1 \wedge \ldots \wedge T_m$ be a normal decomposition of $A \vee BM$ where T_i is S_i-primary. Since T_i is S_i-primary, we have that for each i $(1 \leqslant i \leqslant m)$, either $M \leqslant T_i$ or $B^{k_i} \leqslant T_i$ for some integer k_i. In the latter case the relations $T_i \geqslant A \vee B^{k_i} \geqslant \bigwedge_{k=1}^{\infty}(A \vee B^k) \geqslant M$ hold. Thus, in either case $M \leqslant T_i$ for each i, and so $M \leqslant A \vee BM$ and $A \vee M = A \vee BM$. Since M is join-principal, we have $A : M \vee B = (A \vee BM) : M = (A \vee M) : M = I$. From this and from the relation $B \leqslant \mathfrak{J}$, we conclude that $A : M \not\leqslant P_i$ for each $i = 1, \ldots, n$. Since $(A : M)M \leqslant A \leqslant Q_i$, we have $M \leqslant Q_i$ for each $i = 1, \ldots, n$. Therefore, $M \leqslant A$.

LEMMA 2. *If D and M are elements of $\mathfrak{L}$ such that $M^m \leqslant D \leqslant M$ for some integer m and if M is a maximal element in $\mathfrak{L}$, then D is M-primary.*

Proof. Suppose that $AB \leqslant D$ but $B \not\leqslant M$. Then $(M \vee B)^m = I$, hence $A = A(M \vee B)^m = AM^m \vee ABM^{m-1} \vee \ldots \vee AB^{m-1}M \vee AB^m$. Since $M^m \leqslant D$ and $AB \leqslant D$, it follows that $A \leqslant D$.

If $\mathfrak{L}$ has only one maximal element, $\mathfrak{L}$ is *quasi-local*. If $\mathfrak{L}$ has only one prime element, $\mathfrak{L}$ is *primary*. $\mathfrak{L}$ is said to be *one-dimensional* if there exists at least one pair of distinct primes which are comparable but no three distinct primes are pairwise comparable. The following lemma, which extends a known ring theoretic result to multiplicative lattices, classifies the quasi-local UND lattices.

LEMMA 3. *If $\mathfrak{L}$ is a quasi-local UND lattice, then $\mathfrak{L}$ is either primary or $\mathfrak{L}$ is a one-dimensional lattice in which 0 is prime.*

Proof. Assume that $\mathfrak{L}$ is not a primary lattice. Let M be the maximal element of $\mathfrak{L}$ and let P be a nonmaximal prime. Let $PM = Q_1 \wedge \ldots \wedge Q_k$ be the normal decomposition of PM where Q_i is P_i-primary. Then for each i, we have $PM \leqslant Q_i$, so either $P \leqslant Q_i$ or $M \leqslant P_i$. Suppose that $M \leqslant P_i$ for some i, say $M \leqslant P_1$. Then Q_1 is M-primary, so there is an integer n such that $M^n \leqslant Q_1$. It follows that $(M^r \vee PM) \wedge Q_2 \wedge \ldots \wedge Q_k$ is a normal decomposition of PM for all $r \geqslant n$, and hence that $Q_1 \leqslant \bigwedge_{r=1}^{\infty}(M^r \vee PM)$. But then, by Lemma 1, $M^n \leqslant Q_1 \leqslant PM \leqslant P$, which contradicts the fact that $M \neq P$. Hence $M \not\leqslant P_i$ for all i, so $P \leqslant Q_i$ for all i. It now follows that $P \leqslant PM$, and hence that $P \leqslant \bigwedge_{n=1}^{\infty} M^n = 0$. Since P was an arbitrary nonmaximal prime in $\mathfrak{L}$, this completes the proof of the lemma.

For an arbitrary element D in $\mathfrak{L}$, let $\mathfrak{L}/D$ denote the sublattice of $\mathfrak{L}$ which consists of all elements A in $\mathfrak{L}$ that satisfy the relation $A \geqslant D$. For $A, B \in \mathfrak{L}/D$, define $A \circ B = AB \vee D$. With this multiplication, $\mathfrak{L}/D$ is a commutative multiplicative lattice in which each element is a join of join-principal elements in $\mathfrak{L}/D$ [1].

Suppose that each element of $\mathfrak{L}$ has a normal decomposition and let $A = Q_1 \wedge \ldots \wedge Q_n$ be a normal decomposition of $A \in \mathfrak{L}$ where Q_i is P_i-primary. If D is an arbitrary element of $\mathfrak{L}$, then $\{P_i \mid P_i \vee D \neq I\}$ is an isolated set of primes of A. Let A_D denote the corresponding isolated component of A (i.e., $A_D = \bigwedge\{Q_i \mid P_i \vee D \neq I\}$), and define $A \equiv B(D)$ if and only if $A_D = B_D$. Dilworth [1] proved that the congruence mod D is a congruence relation on $\mathfrak{L}$ which preserves meet, join, multiplication, and residuation. Let $\mathfrak{L}_D$ denote the multiplicative lattice of congruence classes. For each element $A \in \mathfrak{L}$, we let $\{A\}$ denote the congruence class of A. Since join-principal elements are defined in terms of an equation involving join, multiplication, and residuation, it follows that the congruence class of a join-principal element in $\mathfrak{L}$ is join-principal in $\mathfrak{L}_D$. Therefore each element in $\mathfrak{L}_D$ is a join of join-principal elements. The primes and primaries of $\mathfrak{L}_D$ are precisely the congruence classes determined by the primes and primaries of $\mathfrak{L}$. We are now ready to prove the following structure theorem for UND lattices.

THEOREM 1. *$\mathfrak{L}$ is a UND lattice if and only if $\mathfrak{L}$ is a finite direct sum of primary lattices having nilpotent maximal elements and one-dimensional lattices in which 0 is prime and in which each nonzero element is greater than or equal to a product of nonzero prime elements.*

Proof. Assume that $\mathfrak{L}$ is a UND lattice. Let $0 = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_n \wedge \wedge P_1 \wedge \ldots \wedge P_m$ be the unique normal decomposition of 0 where each Q_i is primary for a maximal element, say M_i, and each P_j is primary for a nonmaximal prime element. Let M be any maximal element of $\mathfrak{L}$ such that $M \geqslant P_j$ for some j, and let P_j be P-primary. Since the UND property of $\mathfrak{L}$ is inherited by $\mathfrak{L}_M$, $\mathfrak{L}_M$ is a one-dimensional lattice in which $\{0\}$ is prime (Lemma 3). Therefore the primes $\{0\}$, $\{P\}$, and $\{M\}$ cannot be distinct and so $\{0\} = \{P\}$ (since $\{P\} \neq \{M\}$). Consequently $P = P_M = 0_M \leqslant P_j \leqslant P$ and so P_j is prime. Again by one-dimensionality of $\mathfrak{L}_M$ we also conclude that P_j is the only prime element in $\mathfrak{L}$ such that $P_j \leqslant M$. Thus if $i \neq j$, then P_i and P_j are comaximal primes. Since the Q_j are primary for maximal elements, they are pairwise comaximal. Furthermore, we claim that any pair P_i, Q_k is comaximal. For suppose $P_i \vee Q_k \neq I$, and let M be a maximal element such that $P_i \vee Q_k \leqslant M$. From $\{0\} = \{P_i\} \leqslant \{Q_k\}$ we conclude that $\{P_i\} = \{P_i\} \wedge \{Q_k\} = \{P_i \wedge Q_k\}$, hence $P_i = (P_i)_M = (P_i \wedge Q_k)_M = P_i \wedge Q_k$. This contradicts the irredundancy of the given normal decomposition of 0. Therefore the elements in $\{Q_1, \ldots, Q_n, P_1, \ldots, P_m\}$ are pairwise comaximal and $\mathfrak{L} \cong \mathfrak{L}/Q_1 \oplus \ldots \oplus \mathfrak{L}/Q_n \oplus \mathfrak{L}/P_1 \oplus \ldots \oplus \mathfrak{L}/P_m$.

For each Q_i, $\mathfrak{L}/Q_i$ is clearly a primary lattice in which the maximal element is nilpotent. For each j, $\mathfrak{L}/P_j$ is one-dimensional since if M is a maximal element of $\mathfrak{L}$ such that $M \geqslant P_j$, then $\mathfrak{L}_M$ is one-dimensional by an above argument. Since each element of $\mathfrak{L}/P_j$ has a normal decomposition, it is clear that each nonzero element of $\mathfrak{L}/P_j$ is greater than or equal to a product of nonzero prime elements. This completes the proof of the "only if" part of the theorem.

We will now prove that a finite direct sum of lattices which satisfy the conditions stated in the theorem is a UND lattice. Clearly a primary lattice having a nilpotent maximal elements is a UND lattice since each element of such a lattice is primary for the maximal element. Also, the direct sum of UND lattices is clearly a UND lattice, so we need only prove that a one-dimensional lattice in which zero is prime, and in which every nonzero element is greater than or equal to a product of nonzero prime elements, is a UND lattice. Let $\mathfrak{L}$ be such a lattice. Let A be a nonzero element of $\mathfrak{L}$, and let $P_1, \ldots, P_n$ be distinct nonzero primes such that $P_1^{k_1} \ldots P_n^{k_n} \leqslant A$ for some positive integers k_i. If M is a maximal element such that $M \geqslant A$, then $M \geqslant P_i$ for some i (since M is prime) and so $M = P_i$. Thus there are only a finite number of maximal elements which are greater than or equal to A and they are among the P_i. Let $\{M_1, \ldots, M_s\}$ be the collection of all distinct maximal elements such that $M_i \geqslant A$. For each M_i, define $F_i = \{B \in \mathfrak{L}|$ there exists an element $T \in \mathfrak{L}$ such that $T \nleqslant M_i$ and $BT \leqslant A\}$, and define $Q_i = \bigvee_{B \in F_i} B$. Clearly $A \leqslant Q_i$ (take $B = A$ and $T = I$). We will now prove that Q_i is M_i-primary. If B is an element such that $BT \leqslant A$ for some element $T \nleqslant M_i$, then, since M_i is prime and $BT \leqslant M_i$, we have $B \leqslant M_i$; consequently $Q_i \leqslant M_i$. Since M_i is one of the P_j, say $M_i = P_1$, and since the P_j are distinct, we conclude that $M_i^{k_1} T \leqslant A$ but $T \nleqslant M_i$ where $T = P_2^{k_2} \ldots P_n^{k_n}$. By definition of Q_i it follows that $M_i^{k_1} \leqslant Q_i$ and so Q_i is M_i-primary (Lemma 2). Let $D = Q_1 \wedge \ldots \wedge Q_s$ and observe that $A \leqslant D$. We will now show that $D \leqslant A$. Since the Q_i are primary for (distinct) maximal elements, they are pairwise comaximal, hence $D = Q_1 \wedge \ldots \wedge Q_s = Q_1 Q_2 \ldots Q_s$. Consequently

$$D = \Big(\bigvee_{B \in F_1} B\Big) \cdots \Big(\bigvee_{B \in F_s} B\Big) = \bigvee \{B_1 \ldots B_s |\ B_i \in F_i \text{ for each } i = 1, \ldots, s\}$$

where the last equality holds since multiplication distributes over arbitrary joins. Let $B = B_1 \ldots B_s$ be an arbitrary product where $B_i \in F_i$.

Then, for each i, there exists an element $T_i \nleqslant M_i$ such that $B_i T_i \leqslant A$. If M is a maximal element of $\mathfrak{L}$ such that M is not one of the M_j, then $A:B \nleqslant M$ (otherwise $A \leqslant A:B \leqslant M$). If $M = M_j$ for some j, then the relations $BT_j \leqslant B_j T_j \leqslant A$ and $T_j \nleqslant M_j$ imply that $A:B \nleqslant M_j$ (otherwise $T_j \leqslant A:B \leqslant M_j$). Thus $A:B$ is not less than or equal to any maximal element of $\mathfrak{L}$, so $A:B = I$ and $B \leqslant A$. Consequently $A = D$ and A has a normal decomposition. Since A has no embedded primes, this normal decomposition of A is unique. This completes the proof of the theorem.

Let $\mathfrak{L}$ and $\mathfrak{L}'$ be multiplicative lattices such that each element of $\mathfrak{L}'$ has a normal decomposition. A one-to-one function f mapping $\mathfrak{L}$ into $\mathfrak{L}'$ is a *multiplicative lattice embedding* if it preserves meets, joins, and products, and if it maps primes into primes, primaries into primaries, and 0 into 0. If f also has the property that for each element $A' \in \mathfrak{L}'$ which is not less than or equal to any isolated component of zero there exists a nonzero $A \in \mathfrak{L}$ such that $f(A) \leqslant A'$, then f is called a *dense multiplicative lattice embedding*.

THEOREM 2. *Let $\mathfrak{L}$ be a lattice in which each element has a normal decomposition. $\mathfrak{L}$ is a UND lattice if and only if there exists a dense embedding of a lattice of ideals of a finite direct sum of Dedekind domains and homomorphic images of regular local rings of altitude one into $\mathfrak{L}$ such that maximal ideals are mapped onto maximal elements of $\mathfrak{L}$.*

Proof. Let $\mathfrak{L}$ be a UND lattice. By Theorem 1, $\mathfrak{L}$ is a finite direct sum $\mathfrak{L} \cong \mathfrak{L}_1 \oplus \ldots \oplus \mathfrak{L}_n \oplus \mathfrak{L}_{n+1} \oplus \ldots \oplus \mathfrak{L}_m$ of lattices, where for each $i = 1, \ldots, n$, $\mathfrak{L}_i$ is a primary lattice having a nilpotent maximal element N_i, and for each $i = n+1, \ldots, m$, $\mathfrak{L}_i$ is a one-dimensional lattice in which 0 is prime and each nonzero element of $\mathfrak{L}_i$ is greater than or equal to a product of nonzero prime elements. Let R be a regular local ring of altitude one and let N denote its maximal ideal. Each non-zero ideal of R is a power of N [3]. For each $i = 1, \ldots, n$, let k_i be the least positive integer such that $N_i^{k_i} = 0$, let L_i denote the lattice of ideals of R/N^{k_i}, let M_i denote the maximal element of L_i, and define $f_i: L_i \to \mathfrak{L}_i$ by defining $f_i(M_i^k) = N_i^k$. Clearly f_i is a dense embedding.

Now fix an integer $i = n+1, \ldots, m$ and let $\mu\mathfrak{L}_i = \{M_1^{k_1} \ldots M_m^{k_m} |\ M_j$ is a maximal element of $\mathfrak{L}_i$ and $k_j \geqslant 0\} \cup \{0\}$. Since $\mathfrak{L}_i$ is one-dimensional, $\mu\mathfrak{L}_i$ is a distributive sublattice of $\mathfrak{L}_i$ in which each element is a unique product of prime elements and in which 0 is prime. Furthermore, for distinct nonzero primes (hence maximal) M_j of $\mu\mathfrak{L}_i$ and for integers $e_j, f_j \geqslant 0$, we have the following:

(a) $$\Big(\prod_1^n M_j^{e_j}\Big) \cdot \Big(\prod_1^n M_j^{f_j}\Big) = \prod_1^n M_j^{e_j+f_j}$$

(b) $$\Big(\prod_1^n M_j^{e_j}\Big)\wedge\Big(\prod_1^n M_j^{f_j}\Big) = \Big(\bigwedge_1^n M_j^{e_j}\Big)\wedge\Big(\bigwedge_1^n M_j^{f_j}\Big) = \bigwedge_1^n M_j^{\max(e_j,f_j)} = \prod_1^n M_j^{\max(e_j,f_j)},$$

and

(c) $$\Big(\prod_1^n M_j^{e_j}\Big)\vee\Big(\prod_1^n M_j^{f_j}\Big) = \Big(\bigwedge_1^n M_j^{e_j}\Big)\vee\Big(\bigwedge_1^n M_j^{f_j}\Big) = \bigwedge_1^n M_j^{\min(e_j,f_j)} = \prod_1^n M_j^{\min(e_j,f_j)}.$$

We will now construct a Dedekind domain whose lattice of ideals can be densely embedded in $\mathfrak{L}_i$. Let a be the cardinality of the collection of maximal primes in $\mathfrak{L}_i$, and let K be a field of cardinality $\beta \geqslant a$. Let A be a subset of K of cardinality a, and let S be the compliment in $K[x]$ of the union of the prime ideals $(a+x)$, $a \in A$. Then S is a multiplicatively closed subset of $K[x]$ which doesn't meet any of the prime ideals $(a+x)$, and which meets every other prime ideal. Hence $D_i = K[x]_S$ is a Dedekind domain with a maximal prime ideals [4]. Let L_i denote the lattice of ideals of D_i. Let f_i' be a one-to-one correspondence between the maximal primes of L_i and the maximal primes of $\mathfrak{L}_i$, and extend f_i' to a map f_i of L_i onto $\mu\mathfrak{L}_i$ by taking 0 to 0 and products to products. Since L_i also satisfies the above properties (a), (b) and (c), it follows that f_i is an isomorphism of L_i onto $\mu\mathfrak{L}_i$, and hence an embedding into $\mathfrak{L}_i$. Since each nonzero element of $\mathfrak{L}_i$ is greater than or equal to a product of maximal elements, f_i is dense. Therefore there is a dense embedding of $L_1\oplus\ldots\oplus L_m$ into $\mathfrak{L}_1\oplus\ldots\oplus\mathfrak{L}_m$. This completes the proof of the "only if" part of the theorem.

Conversely, let L be the lattice of ideals of a ring satisfying the conditions stated in the theorem, and let f be a dense embedding of L into $\mathfrak{L}$ such that f maps the maximals in L onto the maximals of $\mathfrak{L}$. The function f maps normal decomposition of 0 in L to a normal decomposition of 0 in $\mathfrak{L}$. Let $0 = R_1\wedge\ldots\wedge R_r$ be this normal decomposition of 0 in $\mathfrak{L}$ where R_i is P_i-primary. If $R_i \leqslant P_i$, then P_i is not less than or equal to any isolated component of 0, and so there exists a nonzero element $B \in L$ such that $f(B) \leqslant P_i$. Since B is a product of maximal elements in L, P_i is greater than or equal to a product of maximal elements in $\mathfrak{L}$ and thus P_i is maximal in $\mathfrak{L}$. Therefore, each R_i is either prime or is a primary for a maximal element. Consequently, the above normal decomposition of 0 is unique. From this and from the properties of f we conclude that if D is less than or equal to an isolated component of 0, then D also has a unique normal decomposition. If A is not less than or equal to any isolated component of zero, then A is greater than or equal to a product of maximal elements of $\mathfrak{L}$ (since f is a dense embedding), and so A is less than or equal to only finitely many distinct maximal elements M_i $(1 \leqslant i \leqslant n)$ in $\mathfrak{L}$. For each M_i, we construct a primary component Q_i of A as we did in the proof of Theorem 1. Then $A = Q_1\wedge\ldots\wedge Q_n$ is the unique normal decomposition of A. Therefore $\mathfrak{L}$ is a UND lattice.

THEOREM 3. *Let $\mathfrak{L}$ be a UND lattice. Each primary element of $\mathfrak{L}$ is a power of a prime if and only if $\mathfrak{L}$ is represented as a lattice of ideals of a direct sum of Dedekind domains and homomorphic images of regular local rings of altitude one.*

Proof. By Theorem 1, $\mathfrak{L}$ is a finite direct sum $\mathfrak{L} \cong \mathfrak{L}_1\oplus\ldots\oplus\mathfrak{L}_n\oplus\ldots\oplus\mathfrak{L}_m$ of lattices, where for each $i = 1, \ldots, n$, $\mathfrak{L}_i$ is a primary lattice having a nilpotent maximal element N_i, and for each $i = n+1, \ldots, m$, $\mathfrak{L}_i$ is a one-dimensional lattice in which 0 is prime and each nonzero element of $\mathfrak{L}_i$ is greater than or equal to a product of nonzero prime elements. If the primaries of $\mathfrak{L}$ are powers of primes, then $\mathfrak{L}_i = \{N_i^k \mid k = 1, \ldots, s_i$ where s_i is the least positive integer such that $N_i^{s_i} = 0\}$ for $i = 1, \ldots, n$, and $\mathfrak{L}_i = \mu\mathfrak{L}_i$ for $i = n+1, \ldots, m$. Consequently the embedding in the proof of Theorem 1 is an isomorphism onto $\mathfrak{L}$. The converse of the theorem is clear.

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Reçu par la Rédaction le 22. 8. 1969