Proof. Same as Theorem 2.9.

THEOREM 4.9. *Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be a locally finite Lebesgue cover of a uniform space $(X, \mathcal{U})$. Then $\mathcal{G}$ has a Δ-refinement which is locally finite and Lebesgue.*

Proof. By the proof of Theorem 3.2 above, it suffices to show that if $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ and $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ are Lebesgue covers of X and $\mathcal{F}$ is a uniform shrink of $\mathcal{G}$, then $\mathcal{H} = \bigwedge_{\alpha \in A} \{G_\alpha, X - F_\alpha\}$ is Lebesgue in the uniform sense. As before we may assume that there exists $U \in \mathcal{U}$ such that $F_\alpha = \{x: U(x) \subset G_\alpha\}$ for all $\alpha \in A$. Choose $V \in \mathcal{U}$ such that V is symmetric and $V^2 \subset U$. Let $x \in X$ and define $A_x = \{\alpha \in A: V(x) \subset G_\alpha\}$. Note that $\beta \notin A_x$ implies that $V(x) \cap (X - G_\beta) \neq \emptyset$, so let $z \in V(x) \cap (X - G_\beta)$. Then for $y \in V(x)$ we have $(x, y) \in V$ and $(x, z) \in V$, so that $(y, z) \in V^2 \subset U$. Thus $z \in U(y)$, and hence $y \notin F_\beta$. Therefore $V(x) \cap F_\beta = \emptyset$ for all $\beta \in A - A_x$. Finally we have $V(x) \subset [\bigcap_{\alpha \in A_x} G_\alpha] \cap [\bigcap_{\beta \in A - A_x} (X - F_\beta)]$, so that $\mathcal{H}$ is Lebesgue.

THEOREM 4.10. *Every locally finite Lebesgue cover of a uniform space $(X, \mathcal{U})$ is Lebesgue normal.*

5. Concluding remarks. It is still unknown whether an arbitrary Lebesgue cover of a metric space (X, ϱ) has a locally finite Lebesgue refinement. This problems seems very difficult. An affirmative answer to this question would answer a number of unsolved problems in Dimension Theory as well as give the extremely strong property that every Lebesgue cover is Lebesgue normal.

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On compactifications with continua as remainders

by

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1. Introduction. A *compactification* of a space X is a compact Hausdorff space $\hat{X}$ with a dense subspace X' homeomorphic to X. The set $\hat{X} - X'$ is called a remainder of X in $\hat{X}$. We are concerned here with spaces that have every continuum (compact connected metric space) as a remainder in some compactification. Aarts and van Emde Boas have shown [1] that every locally compact, non-compact, separable metric space is such a space. Earlier, in [4], K. D. Magill had given an argument that, as observed in [2], shows that every Peano continuum is a remainder in some compactification of any locally compact, non-pseudocompact Hausdorff space. (A space is *pseudocompact* if and only if there is no unbounded real-valued continuous function on it.) More recently, Steiner and Steiner have observed [5] that the methods of Aarts and van Emde Boas are also applicable to Magill's theorem. We show here that their methods can in fact be used to generalize both their theorem and Magill's, i.e. we show in Theorem 2 that non-pseudocompactness is a necessary and sufficient condition on a locally compact Hausdorff space X in order that every continuum be a remainder of a *certain type* of X in some compactification of X.

It would be of interest to characterize the spaces which have every continuum as a remainder in some compactification, without any added conditions on the remainder. We give in Section 3 an example to show that there is a pseudocompact space with this property.

2. Theorems.

DEFINITION. A collection G of subsets of a space X is *discrete* (in X) if and only if each point of X lies in an open subset of X which does not intersect two elements of G.

THEOREM 1. *A completely regular space is pseudocompact if and only if there is no infinite discrete collection of open subsets of it.*

Proof. If a space X is not pseudocompact, it is not difficult to get a map f from X into the non-negative real numbers such that $f(X)$ contains every non-negative integer. For each positive integer n, let O_n denote the set of all points of X whose images lie in the segment $(n-\frac{1}{4}, n+\frac{1}{4})$. Clearly $O_1, O_2, \ldots$ is an infinite discrete collection of open sets in X.

Suppose that X is pseudocompact, but there is an infinite discrete collection $O_1, O_2, \ldots$, of distinct open subsets of X. It follows that the closures of O_i and O_j are mutually exclusive if $i \neq j$. For each n, let P_n denote a point of O_n, and let f_n denote a continuous function from X into $[0, n]$ such that $f_n(P_n) = n$ and $f_n(X - O_n) = 0$. Let $f(A) = f_n(A)$ if A is a point of O_n, and $f(A) = 0$ if A is a point of $X - \bigcup_{i=1}^{\infty} O_i$. Then f is an unbounded real-valued continuous function on X, which is impossible.

Definition. Suppose $\hat{X}$ is a compactification of a space X. Then the remainder R of X in $\hat{X}$ is *sequentially accessible* if and only if it is true that if P is a point of R, then there is a sequence $\{O_i\}$ of distinct open subsets of X which is discrete in $\hat{X} - P$.

Theorem 2. *If X is a locally compact Hausdorff space, then every continuum is a sequentially accessible remainder of X in some compactification of X if and only if X is not pseudocompact.*

Proof. We first show that the first condition implies non-pseudocompactness by considering the one-point compactification $\hat{X} = X \cup \omega$ of X. Since ω is sequentially accessible, there is a sequence $\{O_i\}$ of distinct open subsets of X which is discrete in $\hat{X} - \omega = X$. So from Theorem 1, X is not pseudocompact.

Conversely, suppose that X is not pseudocompact. Then from Theorem 1 and the local compactness of X there is an infinite discrete collection $O_1, O_2, \ldots$ of distinct open subsets of X whose closures are compact. For each i, let P_i denote a point of O_i. Then since X is completely regular, there is a map f_i: $X \to [0, 1]$ such that $f_i(P_i) = 1$ and $f_i(X - O_i) = 0$.

Now suppose that K is any non-degenerate continuum (if K is degenerate, the one-point compactification has the desired properties). As in [1], we consider K to be a subset of the Hilbert Cube I^{∞}, and take a countable dense subset $A_0, A_1, \ldots$ of distinct points of K. Let d be a metric for I^{∞}. For each positive integer j, there is a sequence C_0^j, $C_1^j, \ldots, C_{n_j}^j$ of distinct points of K such that $C_0^j = A_0$, $C_{n_j}^j = A_j$, and if $0 \leqslant i < n_j$, then $d(C_i^j, C_{i+1}^j) < 1/j$. If $0 < i \leqslant n_1$, let $N(C_i^1) = i$, and if $j > 1$ and $0 < i \leqslant n_j$, let $N(C_i^j) = i + \sum_{e=1}^{j-1} n_e$. For each positive integer i, let $a_i = \bigcup_{e=1}^{n} C_{e-1}^j C_e^j$, where n and j are the positive integers such that $N(C_n^j) = i$, and $C_{e-1}^j C_e^j$ is the straight-line interval in I^{∞} from C_{e-1}^j to C_e^j. Clearly, for each i, there is a map g_i: $[0, 1] \to a_i$ such that $g_i(1) = C_n^j$ and $g_i(0) = C_0^j = A_0$.

For each point P of X, let $h(P) = A_0$ if P lies in $X - \bigcup_{j=1}^{\infty} O_j$ and let $h(P) = g_i f_i(P)$ if P lies in O_i for some positive integer i. Using the fact that the collection $O_1, O_2, \ldots$ is discrete it is easy to show that h is a continuous transformation from X into I^{∞}. We observe that

(1) $h(\bigcup_{i=1}^{\infty} P_i)$ contains $\bigcup_{i=1}^{\infty} A_i$ (If $i = N(C_{n_j}^j)$ for any positive integer j, then $h(P_i) = g_i f_i(P_i) = g_i(1) = C_{n_j}^j = A_j$.),

(2) if $\varepsilon > 0$, there is a positive integer I such that each point of $h(X - \bigcup_{i=1}^{I} O_i)$ lies at a distance less than ε from some point of K (Pick $k > 1/\varepsilon$ and $I > \sum_{e=1}^{k} n_e$. Then if $i > I$ and $i = N(C_n^j)$, then $j > k$ and $1/j < \varepsilon$. But $h(O_i)$ lies in a_i, each point of which lies at a distance less than $1/j$ from one of the points $C_0^j, \ldots, C_{n_j}^j$. Also, $h(X - \bigcup_{i=1}^{\infty} O_i) = A_0$, a point of K.), and

(3) if P is a point of K, there is an infinite discrete (in X) collection $V_1, V_2, \ldots$ of open sets in X such that the collection $h(V_1), h(V_2), \ldots$ is discrete in $I^{\infty} - P$ (Pick inductively an increasing sequence $n_1, n_2, \ldots$ of positive integers such that for each i, $h(P_{n_i}) = A_{m_i}$ for some positive integer m_i, and the sequence $\{A_{m_i}\}$ of points, all distinct from one another and from P, converges to P. Then for each positive integer j, let V_j denote the set of all points Q of O_{n_j} such that $d(A_{m_j}, h(Q))$ is less than $\frac{1}{3}$ the distance from A_{m_j} to any other point of the sequence $\{A_{m_i}\}$.).

Now, as in [1], we consider the graph H of h in $a(X) \times I^{\infty}$, where $a(X)$ denotes the one-point compactification $X \cup \omega$ of X. Since h is continuous, H is a closed subset of $X \times I^{\infty}$ which is homeomorphic to X. Thus $\bar{H}$ is a compactification of X with remainder $\bar{H} - H$ in $\omega \times I^{\infty}$.

$\bar{H} - H$ contains $\omega \times K$, for suppose P is a point of K, and U and V are open sets in $a(X)$ and I^{∞} containing ω and P, respectively. Then $X - U$ is compact and can contain at most finitely many points of $\bigcup_{i=1}^{\infty} P_i$. So by observation (1) above, $h(U)$ contains all but at most finitely many of the points of $\bigcup_{i=1}^{\infty} A_i$, which is dense in K. There is a point E of U, then, such that $h(E)$ lies in V, so that $(E, h(E))$ is a point of H in $U \times V$ distinct from (ω, P). Thus each point of $\omega \times K$ is a limit point of H.

Also, $\omega \times K$ contains $\bar{H} - H$, for suppose (ω, Q) is a point of $\omega \times I^{\infty}$, and $\frac{1}{2} d(Q, K) = \varepsilon > 0$. Let V denote an open subset of I^{∞} containing Q of diameter $< \varepsilon$, and I denote a positive integer as in observation (2) above. Then no point of $h(X - \bigcup_{i=1}^{I} O_i)$ is in V. Let U denote the open set in $\alpha(X)$ containing ω such that $U-\omega = X-\bigcup_{i=1}^{I}\bar{O}_i$. Then $U\times V$ is an open set in $\alpha(X)\times I^\infty$ containing (ω, Q) but no point of H.

So the remainder of X in $\bar{H}$ is $\omega\times K$, which is homeomorphic to K. We need only show that $\omega\times K$ is sequentially accessible. Suppose P is a point of K, and $V_1, V_2, \ldots$ is a sequence of open sets in X as given in observation (3) above. For each n, let W_n denote the set of all points $(E, h(E))$ of H such that E is in V_1. That the sequence $\{W_i\}$ of open subsets of H is discrete in $\bar{H}-(\omega, P)$ easily follows from the properties of observation (3).

3. An example. It is well known that the only compactification of the first uncountable ordinal $[0, \Omega)$, with the order topology, is the one-point compactification, and that $[0, \Omega)$ is pseudocompact. So not every non-compact locally compact Hausdorff space can be compactified by the addition of any continuum. We now give an example to show that there is a pseudocompact, locally compact Hausdorff space X which has every continuum as a remainder in some compactification, which shows the need for the requirement of sequential accessibility in Theorem 2.

There are only c topologically different continua. Let C denote a transformation from the number interval $[0, 1]$ into the class of all continua such that (1) if K is a continuum, then K is homeomorphic to $C(t)$ for some t in $[0, 1]$, and (2) if $0 \leqslant t_1 < t_2 \leqslant 1$, then $C(t_1)$ and $C(t_2)$ are not homeomorphic.

Let $M = \prod_{t\in[0,1]} C(t)$. By the Tychonov theorem, M is compact. Let $X = [0, \Omega)\times M$. Then X is pseudocompact ([3], Theorem 9.14, p. 134). Also, $\hat{X} = [0, \Omega]\times M$ is a compactification of X.

Now, each continuum K is homeomorphic to $C(a)$ for some a in $[0, 1]$. We use the projection map p_a: $M\to C(a)$ to attach $C(a)$ to X. Let $X' = X\cup C(a)$ and define f: $\hat{X}\to X'$ such that f is the identity on X, and if (Ω, P) is in $\Omega\times M$, then $f(\Omega, P) = p_a(P)$. Topologize X' by taking as open sets those subsets O of X' such that $f^{-1}(O)$ is open in $\hat{X}$. Then f is a continuous transformation from the compact space $\hat{X}$ onto the Hausdorff space X', so that X' is a compactification of X with remainder homeomorphic to K.

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