On orderings of the system of subsets of ordered sets

by

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In many considerations it is necessary to introduce some relation in the system of all substructures of a given mathematical structure. Usually one takes inclusion as this relation, but in some cases other types of relations should also be considered. In this note, some observations on this subject made in [1] and [2] will be completed.

Let us first introduce some basic notions and notations. If $\mathcal{K}$ is a category, then the class of all morphisms of $\mathcal{K}$ will be denoted by $M(\mathcal{K})$, the class of all objects of $\mathcal{K}$ by $O(\mathcal{K})$. $\mathcal{S}$ will denote the category of all sets with all mappings as morphisms, $\mathcal{K}_2$ the category of all partially ordered sets with isotone mappings as morphisms (the notation taken from [2]). $\mathcal{S}^f$, $\mathcal{K}_2^f$ are the corresponding categories containing only finite objects. If X is a set, $\operatorname{Exp} X$ denotes the system of all subsets of X. If X is a partially ordered set, then the elements of $\operatorname{Exp} X$ are taken as the full relational subobjects (the ordering on subset is the restriction of the ordering of the whole set).

Definition 1. Let $\mathcal{K}$ denote one of the categories $\mathcal{S}$, $\mathcal{S}^f$, $\mathcal{K}_2$, $\mathcal{K}_2^f$. For each $X \in O(\mathcal{K})$, let ϱ_X be an ordering of $\operatorname{Exp} X$. We say that the system $\{\varrho_X: X \in O(\mathcal{K})\}$ is *stable* (*locally stable*) if for each $f \in \operatorname{Hom}(X, Y)$ ($f \in \operatorname{Hom}(X, X)$) $X_1, X_2 \in \operatorname{Exp} X$, $X_1 \varrho_X X_2 \Rightarrow f(X_1) \varrho_Y f(X_2)$ (respectively $f(X_1) \varrho_X f(X_2)$).

Locally stable orderings for the category with sets as objects and permutations of sets as morphisms have been described in [1]. In [2] stable orderings for the category $\mathcal{K}_2^*$ of partially ordered sets as objects with one-to-one isotone mappings as morphisms have been studied. Especially two results have been derived.

Proposition 1. *If $\{\varrho_X: X \in O(\mathcal{K}_2^*)\}$ is stable, $(\operatorname{Exp} X, \varrho_X)$ is a lower semi-lattice for every X and the conditions $X_1, X_2 \in \operatorname{Exp} X$, $X_1 \subset X_2$ imply $X_1 \varrho_X X_2$, then the converse is valid, i.e. $X_1 \varrho_X X_2 \Rightarrow X_1 \subset X_2$, and so ϱ_X is the set inclusion.*

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PROPOSITION 2. *Let $\mathcal{K}_2^{*f}$ be the full subcategory of $\mathcal{K}_2^*$ having finite partially ordered sets as objects. For $X_1, X_2 \in \operatorname{Exp} X$, let*

$X_1 \nu_X X_2 \equiv$ {*if $x \in X_1 - X_2$, then there exists a $y \in X_2 - X_1$ such that $x < y$*}.

Then $\{\nu_X: X \in O(\mathcal{K}_2^{*f})\}$ is maximal stable and $(\operatorname{Exp} X, \nu_X)$ is a lattice for every $X \in O(\mathcal{K}_2^{*f})$.

Maximality is considered in the sense of the following obvious definition.

DEFINITION 2. The stable (locally stable) system $\{\varrho_X: X \in O(\mathcal{K})\}$ is called *maximal* if for every stable (locally stable) system $\{\sigma_X: X \in O(\mathcal{K})\}$ with $\varrho_X \subset \sigma_X$ one has $\varrho_X = \sigma_X$ for each $X \in O(\mathcal{K})$.

Now, we shall be interested in stable systems in $\mathcal{K}_2$, especially in those for which $X_1 \subset X_2 \Rightarrow X_1 \varrho_X X_2$, i.e. which contain inclusion. Let us start with a proposition concerning $\mathcal{S}$.

PROPOSITION 3. 1. *Let $\{\varrho_X: X \in O(\mathcal{S})\}$ be a locally stable system. Then either*

$$\{X_1, X_2 \subset X, X_1 \neq \emptyset \neq X_2\} \Rightarrow \{X_1 \varrho_X X_2 \Rightarrow X_1 \subset X_2\}$$

or

$$\{X_1, X_2 \subset X_1, X_1 \neq \emptyset \neq X_2\} \Rightarrow \{X_1 \varrho_X X_2 \Rightarrow X_2 \subset X_1\}.$$

2. *Let $\{\varrho_X: X \in O(\mathcal{S})\}$ be maximal. Then*

$$\{X_1 \subset X \Rightarrow \emptyset \varrho_X X_1\} \quad \text{or} \quad \{X_1 \subset X \Rightarrow X_1 \varrho_X \emptyset\}.$$

Proof. Ad 1. Let $X_1, X_2 \subset X$, $X_1 \not\subset X_2$, $X_2 \not\subset X_1$. Let $x_0 \in X_1 - X_2$, $y_0 \in X_2 - X_1$ and, if $X_1 \cap X_2 \neq \emptyset$, $z_0 \in X_1 \cap X_2$. Let f_1, f_2 be functions defined as follows: $f_i(x) = x_0$ for $x \in X - X_1 \cup X_2$ and $i = 1, 2$. $f_1(x) = x_0$ for $x \in X_1 - X_2$. $f_1(x) = y_0$ for $x \in X_2 - X_1$. $f_2(x) = y_0$ for $x \in X_1 - X_2$. $f_2(x) = x_0$ for $x \in X_2 - X_1$. $f_i(x) = z_0$ for $x \in X_1 \cap X_2$ and $i = 1, 2$.

Then $f_1(X_1) = \{x_0, z_0\}$, $f_1(X_2) = \{y_0, z_0\}$, $f_2(X_1) = \{y_0, z_0\}$, $f_2(X_2) = \{x_0, z_0\}$ (if z_0 does not exist, then the symbol for it is to be omitted). Thus we cannot have $X_1 \varrho_X X_2$, since then $\{x_0, z_0\} \varrho_X \{y_0, z_0\}$ and simultaneously $\{y_0, z_0\} \varrho_X \{x_0, z_0\}$.

Suppose that for certain two subsets X_1 and X_2 we have simultaneously $X_1 \varrho_X X_2$, $\emptyset \neq X_1 \neq X_2$, $X_1 \subset X_2$. Suppose that there exist $Y_1, Y_2 \subset X$ such that $Y_1 \varrho_X Y_2$, $\emptyset \neq Y_2 \neq Y_1$, $Y_2 \subset Y_1$. Choose $x_2 \in Y_2$, $y_2 \in Y_1 - Y_2$. Let $f: X \to X$ be defined as follows: $f(x) = x_2$ for $x \in Y_2$, $f(x) = y_2$ otherwise.

Then $\{x_2, y_2\} \varrho_X \{x_2\}$. An analogous construction for X_1, X_2 yields $\{y_1\} \varrho_X \{x_1, y_1\}$, where $y_1 \in X_1$, $x_1 \in X_2 - X_1$. Let $g: X \to X$ be such a map that $g(x_2) = y_1$, $g(y_2) = x_1$. Then $\{x_2, y_2\} \varrho_X \{x_2\} \Rightarrow \{x_1, y_1\} \varrho_X \{y_1\}$, which is a contradiction.

Ad 2. As ϱ_X is maximal, we must have $\emptyset \varrho_X X_1$ or $X_1 \varrho_X \emptyset$ at least for one $X_1 \subset X$, $X_1 \neq \emptyset$. Let the first case occur (the second case can be treated dually). Thus $\emptyset \varrho_X \{x\}$ for all $x \in X$. Assume $X_2 \varrho_X \emptyset$ for some X_2. Then $\{x\} \varrho \emptyset$ for all $x \in X$, which is a contradiction. As ϱ_X is maximal, we get 2.

COROLLARY. *All maximal locally stable systems for $\mathcal{S}$ are described as follows*

$$\varrho_X^1: X_1 \neq \emptyset \neq X_2 \Rightarrow X_1 \subset X_2 \equiv X_1 \varrho_X^1 X_2 \quad \text{and} \quad \emptyset \varrho_X^1 X_1 \text{ for all } X_1.$$

$$\varrho_X^2: X_1 \neq \emptyset \neq X_2 \Rightarrow X_1 \subset X_2 \equiv X_1 \varrho_X^2 X_2 \quad \text{and} \quad X_1 \varrho_X^2 \emptyset \text{ for all } X_1.$$

$$\varrho_X^3: X_1 \neq \emptyset \neq X_2 \Rightarrow X_1 \subset X_2 \equiv X_2 \varrho_X^3 X_1 \quad \text{and} \quad \emptyset \varrho_X^3 X_1 \text{ for all } X_1.$$

$$\varrho_X^4: X_1 \neq \emptyset \neq X_2 \Rightarrow X_1 \subset X_2 \equiv X_2 \varrho_X^4 X_1 \quad \text{and} \quad X_1 \varrho_X^4 \emptyset \text{ for all } X_1.$$

Remarks. For $X = \emptyset$, all orderings coincide. For X with $\operatorname{card} X = 1$, $\varrho_X^1 = \varrho_X^3$ and $\varrho_X^2 = \varrho_X^4$. For X with $\operatorname{card} X > 1$ the orderings are mutually different. Moreover, ϱ_X^1, ϱ_X^4 are lattice orderings; ϱ_X^2, ϱ_X^3 are lattice orderings only for X with $\operatorname{card} X \leqslant 1$.

Let $i = 1, 2, 3, 4$. Then $\{\varrho_X^i: X \in O(\mathcal{S})\}$ is a maximal stable system and there are exactly these four maximal stable systems for $\mathcal{S}$.

Consider now the category $\mathcal{K}_2$. The symbol $x \| y$ will mean that neither $x \leqslant y$ nor $x \geqslant y$.

DEFINITION 3. Define for $X \in O(\mathcal{K}_2)$ the relation μ_X on $\operatorname{Exp} X$ in the following way:

If $X_1, X_2 \in \operatorname{Exp} X$, then $X_1 \mu_X X_2$ iff the following assertions are valid.

(1) $x \in X_1 \Rightarrow$ there exist $y_1, y_2 \in X_2$, $y_1 \leqslant x \leqslant y_2$.

(2) If $\{x, y, z\} \subset X_1$, $x \neq y \neq z \neq x$ and $x \| y$, $x \| z$, then $x, y, z \in X_2$.

(3) If $\{x, y, z\} \subset X_1$, $x \neq y \neq z \neq x$ and $x > y$, $x > z$, $y \| z$, then $y, z \in X_2$.

(4) If $\{x, y, z\} \subset X_1$, $x \neq y \neq z \neq x$ and $x < y$, $x < z$, $y \| z$, then $y, z \in X_2$.

(5) If $\{x, y, z\} \subset X_1$, $x < y < z$, then $y \in X_2$.

Remarks. 1. If X contains at most two-element chains, then $X_1 \mu_X X_2 \equiv X_1 \subset X_2$.

2. If X is a chain, then clearly only the validity of (1) and (5) is needed.

3. Let $\check{X}$ be a partially ordered set with the order dual to that on X. Then $\mu_X = \mu_{\check{X}}$ ($\operatorname{Exp} X$ and $\operatorname{Exp} \check{X}$ are identified in a natural way).

4. $X_1 \subset X_2 \Rightarrow X_1 \mu_X X_2$.

LEMMA 1. *μ_X is an order on $\operatorname{Exp} X$.*

Proof. Reflexivity is clear.

Antisymmetry. Let $X_1 \mu_X X_2$, $X_2 \mu_X X_1$. Let $x \in X_1$. By (1) there exist $y_1, y_2 \in X_2$, $y_1 \leqslant x \leqslant y_2$. If somewhere in this relation equality holds, we have $x_1 \in X_2$. Let $y_1 < x < y_2$. Then by (1) for $X_2 \mu_X X_1$ we have some $x_1' \in X_1$, $x_2' \in X_2$, $x_1' \leqslant y_1$, $y_2 \leqslant x_2'$. Then $x_1' < x < x_2'$ and by (5) $x \in X_2$. So $X_1 \subset X_2$. Similarly $X_2 \supset X_1$ and so $X_2 = X_1$.

Transitivity. Let $X_1 \mu_X X_2$, $X_2 \mu_X X_3$.

Ad (1). Let $x \in X_1$. Then there exist $y_1, y_2 \in X_2$, $y_1 \leqslant x \leqslant y_2$. Further, z_1, z_2 exist in X_3 so that $z_1 \leqslant y_1$, $y_2 \leqslant z_2$, so $z_1 \leqslant x \leqslant z_2$.

Ad (2). Let $\{x, y, z\} \subset X_1$, $x \| y$, $x \| z$, $z \neq y$. Then $x, y, z \in X_2$ and so $x, y, z \in X_3$.

Ad (3), (4), (5). Similar arguments work as for (2) using (1) for $X_2 \mu_X X_3$.

Lemma 2. *$\{\mu_X\colon X \in O(\mathcal{K}_2)\}$ is stable.*

Proof. Let $f\colon X \to Y$, and $X_1 \mu_X X_2$.

Ad (1). Let $x \in f(X_1)$. Let $y \in X_1$, $f(y) = x$. There exists by (1) for $X_1 \mu_X X_2$ $y_1, y_2 \in X_2$ with $y_1 \leqslant y \leqslant y_2$. Then $f(y_1), f(y_2) \in f(X_2)$ and $f(y_1) \leqslant x \leqslant f(y_2)$.

Ad (2). Let $\{x, y, z\} \subset f(X)$, $x \neq y \neq z \neq x$ and $x \| y$, $x \| z$. x', y', z' exist in X_1 such that $f(x') = x$, $f(y') = y$, $f(z') = z$. Then $x' \| y'$, $x' \| z'$, $x' \neq y' \neq z' \neq x'$. By (2) for $X_1 \mu_X X_2$ $x', y', z' \in X_2$ and $x, y, z \in f(X_2)$.

Ad (3). Let $\{x, y, z\} \subset f(X_1)$, $x \neq y \neq z \neq x$, and $x > y$, $x > z$, $y \| z$. Let $f(x') = x$, $f(y') = y$, $f(z') = z$. We have $y' \| z'$, for x', y' we have $x' \| y'$ or $x' > y'$, for x', z', $x' \| z'$ or $x' > z'$. If $x' \| y'$ then, as $y' \| z'$, by (2), $x', y', z' \in X_2$ and so $x, y, z \in X_2$. The same holds for $x' \| z'$. If $x' > y'$, $x' > z'$, then by (3) for X_1 $y', z' \in X_2$. So $y, z \in f(X_2)$.

Ad (4). Dually to Ad (3).

Ad (5). Let $\{x, y, z\} \subset f(X_1)$, $x < y < z$, $f(x') = z$, $f(y') = y$, $f(z') = z$. Then $x' \| y'$ or $x' < y'$, $y' \| z'$ or $y' < z'$, $x' \| z'$ or $x' < z$.

If at least in two cases incomparability holds, then by (2) $x', y', z' \in X_2$, so $x, y, z \in f(X_2)$.

Let us check the remaining cases.

If $x' \| y'$, $y' < z'$ and $x' < z'$, then $x', y' \in X_2$ (by (3)) and $x, y \in f(X_2)$.

If $x' < y'$, $y' \| z'$, $x' < z'$, then $y', z' \in X_2$ (by (4)) and $y, z \in f(X_2)$.

If $x' < y'$, $y' < z'$, then $y' \in X_2$ (by (5) for X_1) and $y \in f(X_2)$.

In the next lemma let $\{\varrho_X\colon X \in O(\mathcal{K}_2)\}$ be some locally stable system, for which $X_1 \subset X_2 \Rightarrow X_1 \varrho_X X_2$.

Lemma 3. *If $X_1, X_2 \subset X$, $\operatorname{card} X_1 \geqslant 2$, $\operatorname{card} X_2 \geqslant 2$ and $X_1 \varrho_X X_2$, then $X_1 \mu_X X_2$.*

Proof. If X is an antichain (does not contain distinct comparable elements), then the assertion is true by Proposition 3 and Remark 4 after Definition 3. So we shall assume that X is not an antichain.

Suppose that there exist $X_1, X_2 \subset X$, $\operatorname{card} X_1 \geqslant 2$, $\operatorname{card} X_2 \geqslant 2$, $X_1 \varrho_X X_2$ and X_1 non $\mu_X X_2$. Thus at least one of the following cases occurs.

1° There exists an $x \in X_1$, for which either there exists no $y_1 \in X_2$ with $y_1 \leqslant x$ or there exists no $y_2 \in X_2$ with $y_2 \geqslant x$.

2° There exist $\{x, y, z\} \subset X_1$, $x \| y$, $x \| z$, $y \neq z$ and $\{x, y, z\} \not\subset X_2$.

3° There exist $\{x, y, z\} \subset X_1$, $x > y$, $x > z$, $y \| z$ and $\{y, z\} \not\subset X_2$.

4° There exist $\{x, y, z\} \subset X_1$, $x < y$, $x < z$, $y \| z$ and $\{y, z\} \not\subset X_2$.

5° There exist $\{x, y, z\} \subset X_1$, $x < y < z$ and $y \notin X_2$.

Ad 1°. Suppose that no $y_1 \in X_2$ satisfies $y_1 \leqslant x$. Let $a, b \in X$, $a < b$. Let us consider two cases.

(a) $X_1 \subset (x]$ where $(x] = \{u \in X\colon u \leqslant x\}$; then define $f\colon X \to X$ as follows: $y < x \Rightarrow f(y) = a$, $f(y) = b$ otherwise. Then $f(X_1) = \{a, b\}$, $f(X_2) = \{b\}$. Simultaneously $X_1 \varrho_X X_2 \Rightarrow \{a, b\} \varrho_X \{b\}$, which is a contradiction.

(b) $X_1 \not\subset (x]$; then define $f\colon X \to X$ as follows: $y \leqslant x \Rightarrow f(y) = a$, $f(y) = b$ otherwise. The conclusion is the same.

For the case concerning y_2 the reasoning is the same.

Ad 2°. (i) Let a, b, c exist in X such that $a < b < c$. If y, z from 2° are comparable, suppose that $y > z$.

(a) Assume $x \notin X_2$. Define $f\colon X \to X$ as follows: $v > x$ or $v > y \Rightarrow f(v) = c$, $f(x) = b$, $f(v) = a$ otherwise. $f \in M(\mathcal{K}_2)$ and $f(X_1) = \{a, b, c\}$, $f(X_2) \subset \{a, c\}$. So $\{a, b, c\} \varrho_X \{a, c\}$, which is a contradiction.

(b) Assume $y \notin X_2$. Define $f\colon X \to X$ as follows: $v \geqslant x$ or $v > y \Rightarrow f(v) = c$, $f(y) = b$, $f(v) = a$ otherwise. Again $f(X_1) = \{a, b, c\}$, $f(X_2) \subset \{a, c\}$.

(c) For $z \notin X_2$, the construction is dual to that in the previous case.

(ii) Let only at most two-element chains exist in X. Let a be such a point from $\{x, y, z\}$, for which $a \notin X_1$. If there exist elements comparable to a and distinct from a, let b be one of them. In the opposite case let b be some element of X distinct from a. Define $f\colon X \to X$ as follows: $f(a) = a$, $f(v) = b$ otherwise. Then $f \in M(\mathcal{K}_2)$, $f(X_1) = \{a, b\}$, $f(X_2) = \{b\}$, which leads to the same sort of contradiction as above.

Ad 3°. (i) Assume the situation as in Ad 2° (i). Assume $y \notin X_2$. Define $f\colon X \to X$ as follows: $v < y$ or $v \leqslant z \Rightarrow f(v) = a$, $f(y) = b$, $f(v) = c$ otherwise. Again $f(X_1) = \{a, b, c\}$, $f(X_2) \subset \{a, c\}$. For $z \notin X_2$ the construction is the same.

(ii) If X is as in Ad 2° (ii), the same constructions work as there.

Ad 4°. The case is dual to case 3°.

Ad 5°. Let us construct $f\colon X \to X$ as follows: $v < y \Rightarrow f(v) = x$, $f(y) = y$, $f(v) = z$ otherwise.

So $f(X_1) = \{x, y, z\}$, $f(X_2) = \{x, z\}$.

Lemma 4. *Let $\operatorname{card} X_1 \geqslant 2$, $X_1 \varrho_X X_2$. Then $\operatorname{card} X_2 \geqslant 2$.*

Proof. (i) Let X be an antichain. Then the assertion is valid by Proposition 3.

(ii) Let $a, b \in X$, $a < b$. Assume $X_2 = \{x\}$. We have $x \notin X_1$. Let $y, z \in X_1$, $y \neq z$. If y, z are comparable, let $y > z$.

Define $f\colon X \to X$ as follows $v \leqslant z \Rightarrow f(v) = a$, $f(v) = b$ otherwise. Then $f(X_2) \subset f(X_1)$, $f(X_2) \neq f(X_1)$, which implies a contradiction.

LEMMA 5. *Let $\{x\}\varrho_X\{y\}$. Then x and y are comparable. If $x_0 < y_0$ are two points in X with $x_0\varrho_X y_0$, then $x < y \Rightarrow x\varrho_X y$, if $y_0 < x_0$ and $x_0\varrho_X y_0$, then $y < x \Rightarrow x\varrho_X y$.*

Proof. (i) For an antichain the assertion is clear by Proposition 3.

(ii) Let $a, b \in X$, $a < b$. Let $\{x\}\varrho_X\{y\}$; assume $x \| y$. Define f and g on X as follows:

$$f(v) = a \text{ for } v \leqslant x\,, \quad f(v) = b \text{ otherwise.}$$
$$g(v) = a \text{ for } v \leqslant y\,, \quad g(v) = b \text{ otherwise.}$$
$$f(\{x\}) = g(\{y\}) = \{a\}\,, \quad f(\{y\}) = g(\{x\}) = \{b\}.$$

So $\{a\}\varrho_X\{b\}$, $\{b\}\varrho_X\{a\}$, which is a contradiction.

The proof of the second assertion is evident.

Let $X \in O(\mathcal{K}_2)$. Let $\operatorname{Exp}^* X$ be the system of all one-element subsets of X, $\operatorname{Exp}^{**} X$ the system of all at least two-element subsets. Let λ_X^1 be the ordering on $\operatorname{Exp}^* X$ defined by $\{x\}\lambda_X^1\{y\} \equiv x \leqslant y$. Let λ_X^2 be the ordering of $\operatorname{Exp}^* X$ dual to λ_X^1.

Let π_X^1 be the relation on $\operatorname{Exp} X$ defined as follows.

(α) $\emptyset\pi_X^1 X_1$ for each $X_1 \subset X$.

(β) On $\operatorname{Exp}^* X$ π_X^1 coincides with λ_X^1.

(γ) On $\operatorname{Exp}^{**} X$ π_X^1 coincides with μ_X.

(δ) If $x \in X$, $\operatorname{card} X_1 \geqslant 2$ $\{x\}\pi_X^1 X_1$ iff there exists a $y_1 \in X_1$ such that $x \leqslant y_1$.

π_X^2 is defined similarly, in (β) is λ_X^2 instead of λ_X^1 and $y_1 \leqslant x$ in (δ).

LEMMA 6. *π_X^1, π_X^2 are orderings and $\{\pi_X^1\colon X \in O(\mathcal{K}_2)\}$, $\{\pi_X^2\colon X \in O(\mathcal{K}_2)\}$ are stable.*

Proof. Let us prove it for π_X^1. Reflexivity is clear. Let $X_1\pi_X^1 Y_1$, $Y_1\pi_X^1 X_1$. If $X_1 = \emptyset$, then $Y_1 = \emptyset$. If $X_1 \in \operatorname{Exp}^* X$, then $Y_1 \in \operatorname{Exp}^* X$ and so $X_1 = Y_1$. The same for $X_1 \in \operatorname{Exp}^{**} X$.

Transitivity. The only case to be checked is $x \in X$, $X_1, X_2 \in \operatorname{Exp}^{**} X$ $\{x\}\pi_X^1 X_1$, $X_1\pi_X^1 X_2$. But $\{x\}\pi_X^1 X_2$ follows by (1) in Definition 3 and by (δ).

Stability. It is immediately clear that the relations given by (α), (β) or (δ) are preserved by isotone mappings. Let $X_1, X_2 \in \operatorname{Exp}^{**} X$, $X_1\pi_X^1 X_2$, $f\colon X \to Y$. By Lemma 4 applied for μ_Y we have $\operatorname{card} f(X_2) = 1 \Rightarrow \operatorname{card} f(X_1) = 1$ and then $f(X_1) = f(X_2)$ by (1) of the Definition 3. If $\operatorname{card} f(X_2) \geqslant 2$, then clearly $f(X_1)\pi_X^1 f(X_2)$.

THEOREM 1. *$\{\pi_X^1\colon X \in O(\mathcal{K}_2)\}$, $\{\pi_X^2\colon X \in O(\mathcal{K}_2)\}$ are the only maximal stable systems containing inclusion. If $\{\varrho_X\colon X \in O(\mathcal{K}_2)\}$ is a locally stable system containing inclusion, then $\varrho_X \subset \pi_X^1$ or $\varrho_X \subset \pi_X^2$.*

Proof. In view of Lemmas 3 and 5 it is sufficient to prove the following assertions (a) and (b).

(a) If $\{x_0\}\varrho_X X_0$ for certain x_0 and X_0, $(x_0] \cap X_0 = \emptyset$, $X_0 \cap [x_0) \neq \emptyset$, then ϱ_X contains λ_X^1 (similarly a dual assertion can be formulated). Namely, let $y_0 \in X_0$, $x_0 < y_0$. Define $f\colon X \to X$ as follows: $x \leqslant x_0 \Rightarrow f(x) = x_0$ and $f(x) = y_0$ otherwise. Then $\{x_0\}\varrho_X\{y_0\}$ and by Lemma 5 $\lambda_X^1 \subset \varrho_X$.

(b) If $\{x_0\}\varrho_X X_0$, then in X_0 there exists a y_0 comparable with x_0.

Suppose this is not the case. Take $y_0 \in X_0$ (y_0 exists, X_0 is not empty) and construct f as above. Then $\{x_0\}\varrho_X\{y_0\}$, which contradicts Lemma 5.

Now, we shall be interested in the cases where the orderings introduced above are lattice orderings.

PROPOSITION 4. *μ_X is a lattice order if and only if X is an ordinal sum* (see [3], p. 198) *$M \oplus N \oplus P$ where M, P are antichains and N a finite chain or if X contains only at most two-element chains.*

Proof. Let X contain an infinite chain. Then it contains a sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $x_1 < x_2 < \ldots$, or $x_1 > x_2 > \ldots$ (it follows e.g. from the characterization of well-ordered sets by the non-existence of chain of the type ω^*—see e.g. [4], p. 214). We shall deal with the first case; the second one is dual.

Put $X_1 = \{x_1, x_3, x_5, \ldots\}$, $X_2 = \{x_1, x_2, x_4, x_6, \ldots\}$. We have $\{x_i\}\mu_X X_1$, $\{x_i\}\mu_X X_2$ for all i. Let X_3 be the infimum of X_1, X_2 for the order μ_X. For all i, there exist $y_1^i, y_2^i \in X_3$ such that $y_1^i \leqslant x_i \leqslant y_2^i$. Further for all $y \in X_3$ there exists an $x', x'' \in X_1$ such that $x' \leqslant y \leqslant x''$. Suppose that $x' < y < x''$. Thus we must have $y', y'' \in X_3$ such that $y' \leqslant x' < y < x'' \leqslant y''$. Hence $y \in X_1$ in any case. The same is valid for X_2. But $X_1 \cap X_2 = \{x_1\}$ and neither $\{x_1\}$ nor $\emptyset$ can be X_3.

Let X contain no infinite chain. Suppose that X is not a chain and let there exist $a, b, c \in X$, $a < b < c$. Complete $\{a, b, c\}$ to some maximal chain Y in X. There exist $d \in X$ such that $d \notin Y$. Thus, let e be an element of Y for which $e \| d$. The following cases may occur:

(a) There exist $f, g \in Y$, $f < e < g$, $f < d < g$.

(b) There exist $f, g \in Y$, $f < e < g$, $f < d$, $g \| d$.

(c) There exist $f, g \in Y$, $f < e < g$, $g > d$, $f \| d$.

(d) There exist $f, g \in Y$, $f < e < g$, $d \| g$, $d \| f$.

(e) There exist $f, g \in Y$, $e < f < g$, $f > d$ or dual.

(f) There exist $f, g \in Y$, $e < f < g$, $g > d$ or dual.

(g) There exist $f, g \in Y$, $e < f < g$, $d \| f$, $d \| g$ or dual.

Thus if X does not contain an infinite chain, one can consider the following four cases.

(1) There exist a, b, c, d, $a < b < d$, $a < c < d$, $b \| c$.

(2) There exist a, b, c, d such that $a < b < c$ and $a \| d$, $b \| d$ or $b \| d$, $c \| d$.

(3) $X = M \oplus N \oplus P$, where M, P are antichains, and N is a finite chain.

(4) X contains only at most two-element chains.

We shall prove that in cases (1), (2) μ_X is not a lattice ordering.

Ad (1). Let X_1 be a supremum of $\{a, b\}$ and $\{a, c\}$. Then $X_1 \mu_X \{a, d\}$, $X_1 \mu_X \{a, b, c\}$, $\{a, b\} \mu_X X_1$, $\{a, c\} \mu_X X_1$. So for $x \in X_1$ we have $a \leqslant x$, $x \leqslant c$ or $x \leqslant b$, at the same time there exist $y_1 \in X_1$, $y_1 \geqslant c$, $y_2 \in X_1$, $y_2 \leqslant a$, $y_3 \in X_1$, $y_3 \geqslant b$. So $\{a, b, c\} \subset X_1$ and this implies $X_1 = \{a, b, c\}$. But $\{a, b, c\}$ non μ_X $\{a, d\}$ by (4) of Definition 3.

Ad (2). Let X_1 be a supremum of $\{a, d\}$, $\{b, d\}$ in μ_X, $a \| d$, $b \| d$. Again $X_1 = \{a, b, d\}$, but $\{a, b, d\}$ non μ_X $\{a, c, d\}$. Similarly for $b \| d$, $c \| d$.

Thus, let X have the form from (3). We shall prove that μ_X is a lattice order. Let $N = \{n_1, n_2, \ldots, n_k\}$, $n_1 < n_2 < \ldots < n_k$. Let $X_1 = M_1 \oplus \oplus \{a_1, \ldots, a_m\} \oplus P_1$, $X_2 = M_2 \oplus \{b_1, \ldots, b_p\} \oplus P_2$, $a_1 < \ldots < a_m$, $b_1 < \ldots < b_p$, $M_1, M_2 \subset M$, $P_1, P_2 \subset P$, $N_1 = \{a_1, \ldots, a_m\} \subset N$, $N_2 = \{b_1, \ldots, b_p\} \subset N$.

Put $V^* = \{a_1, b_1\}$ if $M_1 \neq \emptyset \neq M_2$, $N_1 \neq \emptyset \neq N_2$, $P_1 \neq \emptyset$ or $m > 1$, $P_2 \neq \emptyset$ or $p > 1$.

$V^* = \{a_1\}$ in the following cases:

$$M_1 \neq \emptyset \neq M_2,\ N_1 \neq \emptyset = N_2,\ P_1 \neq \emptyset \text{ or } m > 1.$$

$$M_1 \neq \emptyset = M_2,\ N_1 \neq \emptyset,\ P_1 \neq \emptyset \text{ or } m > 1.$$

$$M_1 = \emptyset = M_2,\ N_1 \neq \emptyset = N_2.$$

$$M_1 \neq \emptyset \neq M_2,\ N_1 \neq \emptyset \neq N_2,\ P_1 \neq \emptyset \text{ or } m > 1,\ P_2 = \emptyset,\ p = 1.$$

Put $V^* = \{b_1\}$ in cases dual to those in the previous case.

Put $V^* = \{\min(a_1, b_1)\}$ if $M_1 = \emptyset = M_2$, $N_1 \neq \emptyset \neq N_2$.

Put $V^* = \emptyset$ otherwise. Define V^{**} in a dual way, dealing with P_1, P_2 instead of M_1, M_2.

It is routine to check that $V = V^* \cup V^{**} \cup M_1 \cup M_2 \cup \{a_2, \ldots, a_{m-1}, b_2, \ldots, b_{p-1}\} \cup P_1 \cup P_2$ is the supremum of X_1 and X_2 in μ_X.

Put

$$W^* = M_1 \cap M_2 \text{ if } M_1 \cap M_2 \neq \emptyset.$$

$$W^* = \{n_1\} \text{ if } M_1 \cap M_2 = \emptyset,\ M_1 \neq \emptyset \neq M_2.$$

$$W^* = \{a_1\} \text{ if } M_1 = \emptyset \neq M_2,\ N_1 \neq \emptyset.$$

$$W^* = \{b_1\} \text{ if } M_1 \neq \emptyset = M_2,\ N_2 \neq \emptyset.$$

$$W^* = \{\max(a_1, b_1)\} \text{ if } M_1 = \emptyset = M_2,\ N_1 \neq \emptyset \neq N_2.$$

$$W^* = \emptyset \text{ if } M_1 = \emptyset,\ N_1 = \emptyset \text{ or } M_2 = \emptyset,\ N_2 = \emptyset.$$

Dually dealing with P_1, P_2 we define W^{**}.

Put $W = W^* \cup W^{**} \cup (N_1 \cap N_2)$ if for every $x \in W^* - X_1 \cap X_2$ there exists a $y \in W^{**}$ for which $x \leqslant y$ and, dually, for every $y \in W^{**} - X_1 \cap X_2$ there exists an $x \in W^*$ with $x \leqslant y$. Put $W = \emptyset$ in the other cases. It can be checked that W is the infimum of X_1 and X_2 in μ_X.

If X contains only at most two-point chains, then μ_X is identical with the inclusion. (Remark 1 after Definition 3.)

PROPOSITION 5. *π_X^1 is a lattice order iff $X = N \oplus P$ where N is a finite chain and P is an antichain.*

π_X^2 is a lattice order iff $X = M \oplus N$ where M is an antichain and N is a finite chain.

Proof. Let us accomplish the proof for π_X^1. Let π_X^1 be a lattice order. Let X contain an infinite chain. Then it contains

$$(\mathrm{a}_1) \qquad x_1 < x_2 < x_3 < \ldots$$

or

$$(\mathrm{a}_2) \qquad x_1 > x_2 > \ldots$$

Case (a_1) is as in the proof for μ_X. Let (a_2) occur. Let $X_1 = \{x_1, x_3, \ldots\}$, $X_2 = \{x_1, x_2, x_4, \ldots\}$. Then $\{x_1, x_i\} \pi_X^1 X_1$, $\{x_1, x_i\} \pi_X^1 X_2$ for all $i \geqslant 2$. Thus if V is the infimum of X_1 and X_2 in π_X^1, then for each x_i we have $y_i, y'_i \in V$ such that $y_i \geqslant x_i \geqslant y'_i$. V is not a one-element set. Namely, assume $V = \{v\}$. Then $v \geqslant x_i$ for all i and $v \leqslant x_k$ for some $x_k \in X_1$ and $v \leqslant x_h$ for some $x_h \in X_2$. So $v \geqslant x_1$, $v \leqslant x_2$, which is a contradiction. Thus, for $y \in V$ we have $x', x'' \in X_i$ such that $x' \leqslant y \leqslant x''$. So $V \subset X_1 \cap X_2 = \{x_1\}$.

Let us now consider cases (1), (2), (3), (4) from the previous proof. (1) and (2) cannot occur by the same arguments. Further, we can now prove the following two statements.

(5) In X there is no triple $\{a, b, c\}$ with $a < b$, $a \| c$, $b \| c$.

Indeed, assume that we have such triple. Then $\{a\} \pi_X^1 \{b, c\}$, $\{c\} \pi_X^1 \{b, c\}$. Let V be the supremum of $\{a\}$ and $\{c\}$. As $V \pi_X^1 \{a, c\}$, $\{a\} \pi_X^1 V$, $\{c\} \pi^1 V$, we have $V = \{a, c\}$. But $\{a, c\}$ non $\pi_X^1 \{b, c\}$.

(6) In X there is no triple $\{a, b, c\}$ with $a < b$, $c < b$, $a \| c$.

Indeed, assume that we have such triple. Then $\{a\} \pi_X^1 \{b\}$, $\{c\} \pi_X^1 \{b\}$. Let V be the supremum of $\{a\}, \{c\}$. By the same reasons as above $V = \{a, c\}$, but $\{a, c\}$ non $\pi_X^1 \{b\}$.

Thus, if π_X^1 is a lattice order, then X has the prescribed form and that will be assumed in the sequel. Let, again, $N = \{n_1, \ldots, n_k\}$, $n_1 < n_2 < \ldots < n_k$. Let $X_1, X_2 \subset X$, $X_1 = \{a_1, \ldots, a_m\} \cup P_1$, $X_2 = \{b_1, \ldots, b_p\} \cup P_2$, $a_i, b_i \in N$, $a_1 < \ldots < a_m$, $b_1 < \ldots < b_p$, $P_i \subset P$ for $i = 1, 2$.

Suppose that X_1 and X_2 are incomparable in π_X^1. In the following formulas put $\min(a_1, b_1) = a_1$ if $N_1 \neq \emptyset = N_2$, $\min(a_1, b_1) = b_1$ if $N_1 = \emptyset \neq N_2$, $\{\min(a_1, b_1)\} = \emptyset$ if $N_1 = \emptyset = N_2$. Similarly for $\max(a_m, b_p)$.

(1) If $P_1 \neq \emptyset \neq P_2$ put $V = \{\min(a_1, b_1)\} \cup \{a_2, \ldots, a_{m-1}, b_2, \ldots, b_{p-1}\} \cup \cup P_1 \cup P_2$.

(2) If $P_1 \cup P_2 = \emptyset$ put $V = \{\min(a_1, b_1)\} \cup \{a_2, \ldots, a_{m-1}, b_2, \ldots, b_{p-1}\} \cup \cup \{\max(a_m, b_p)\}$.

(3) If $P_1 = \emptyset \neq P_2$ put $V = \{\min(a_1, b_1)\} \cup \{a_2, \ldots, a_{m-1}, b_2, \ldots, b_p\} \cup P_2$.

(4) If $P_1 \neq \emptyset = P_2$ put $V = \{\min(a_1, b_1)\} \cup \{a_2, \ldots, a_m, b_2, \ldots, b_{p-1}\} \cup P_1$.

Then V is the supremum of X_1 and X_2 in π_X^1.

Suppose again that X_1 and X_2 are incomparable in π_X^1.

In the following formulas put $\{\max(a_1, b_1)\} = \emptyset$ if $N_1 = \emptyset$ or $N_2 = \emptyset$.

(1′) If $P_1 \cap P_2 \neq \emptyset$ put $W = (P_1 \cap P_2) \cup (N_1 \cap N_2) \cup \{\max(a_1, b_1)\}$.

(2′) If $P_1 \cap P_2 = \emptyset$, $P_1 \neq \emptyset \neq P_2$ put $W = \{\max(a_1, b_1)\} \cup \{n_k\} \cup \cup (N_1 \cap N_2)$.

(3′) If $P_1 = \emptyset = P_2$ and $b_1 \leqslant a_m$, $a_1 \leqslant b_p$, put $W = \{\min(a_m, b_p)\} \cup \cup \{\max(a_1, b_1)\} \cup (N_1 \cap N_2)$.

(4′) If $P_1 = \emptyset \neq P_2$, $N_2 \neq \emptyset$, $b_1 \leqslant a_m$ put $W = \{a_m\} \cup \{\max(a_1, b_1)\} \cup \cup (N_1 \cap N_2)$.

(5′) If $P_1 \neq \emptyset = P_2$, $N_1 \neq \emptyset$, $a_1 \leqslant b_p$, put $W = \{b_p\} \cup \{\max(a_1, b_1)\} \cup \cup (N_1 \cap N_2)$.

(6′) If $P_1 = \emptyset$ and $N_2 = \emptyset$ or $a_m < b_1$ put $W = \{a_m\}$.

(7′) If $P_1 = \emptyset \neq P_2$, $N_2 \neq \emptyset$ and $b_p < a_1$, put $W = \{a_1, a_m\}$.

(8′) If $P_1 \neq \emptyset = P_2$, $N_1 \neq \emptyset$ and $a_m < b_1$, put $W = \{b_1, b_p\}$.

(9′) If $P_2 = \emptyset$, $N_1 = \emptyset$ or $b_p < a_1$ put $W = \{b_p\}$.

It can be checked that W is infimum of X_1 and X_2. Thus the proposition is proved.

PROPOSITION 6. *Let $\mathcal{K}$ be a full subcategory of the category $\mathcal{K}_2$. Let $\mathcal{K}$ contain a four-element Boolean lattice as object. Let $\{\varrho_X: X \in O(\mathcal{K})\}$ be stable and let ϱ_X contain inclusion for all $X \in O(\mathcal{K})$. Then ϱ_X is equal to an inclusion for all $X \in O(\mathcal{K})$ provided ϱ_X is a lattice order for all X.*

Proof. Let B be a four-element Boolean lattice, $B = \{a, b, c, d\}$, $a < b < d$, $a < c < d$, $c \| b$. Assume that there exists an $X \in O(\mathcal{K})$ such that $X_1 \varrho_X X_2$, $X_1 \not\subset X_2$ for certain $X_1, X_2 \subset X$. Suppose that $\operatorname{card} X_1 \geqslant 2$, $\operatorname{card} X_2 \geqslant 2$. Let $x \in X_1 - X_2$. Define f_1, f_2: $X \to B$ as follows

$$u > x \Rightarrow f_1(u) = f_2(u) = d\,,$$

$$f_1(x) = b\,, \quad f_2(x) = c\,, \quad f_1(u) = f_2(u) = a \text{ otherwise}\,.$$

We have $f_1(X_2) = f_2(X_2) = \{a, d\}$ as, by Lemma 3, $X_1 \mu_X X_2$ and so y_1, y_2 exist in X_2 such that $y_1 < x < y_2$. Further, $b \in f_1(X_1)$,

$$c \in f_2(X_1),\ f_1(X_1) \subset \{a, b, d\},\ f_2(X_1) \subset \{a, c, d\}\,.$$

As $f_1(X_1) \varrho_B f_1(X_2)$, we have $f_1(X_1) \neq \{a, b, d\}$, and so we have the following possibilities:

(1) $f_1(X_1) = \{a, b\}$, $f_2(X_1) = \{a, c\}$.

(2) $f_1(X_1) = \{b, d\}$, $f_2(X_1) = \{c, d\}$.

(3) $f_1(X_1) = \{b\}$, $f_2(X_1) = \{c\}$.

Denote in the sequel the supremum of $f_1(X_1)$, $f_2(X_1)$ in ϱ_B by V.

Ad (1). As $\{a, b\} \mu_B V$, $\{a, c\} \mu_B V$ (by Lemma 3 and Lemma 4) and $V \mu_B \{a, b, c\}$ we get $V = \{a, b, c\}$. But $\{a, b, c\}$ non $\mu_B \{a, d\}$, which is contradiction of $f_1(X_1) \varrho_B \{a, d\}$, $f_2(X_1) \varrho_B \{a, d\}$.

Ad (2). The considerations are similar to that in the previous case.

Ad (3). As the mapping f: $B \to B$, where $f(a) = a$, $f(d) = d$, $f(b) = c$, $(c) = d$, preserves ϱ_B, V is preserved by f also, and so $V = f(V)$. Hence V is of one of the forms $\{b, c\}$, $\{a, d\}$, $\{a\}$, $\{d\}$. Let W be the infimum of $\{b\}$, $\{c\}$. By Lemmas 4 and 5, W is $\{a\}$ or $\{d\}$.

(a) Let $W = \{a\}$. Then $\{c\} \varrho_B \{d\}$, $\{b\} \varrho_B \{d\}$, so $V \neq \{b, c\}$, $\{a\}$, $\{a, d\}$. The last possibility $V = \{d\}$ is contradictory to the relation $\{d\}$ non $\varrho_B \{b, c\}$.

(b) If $W = \{d\}$ the consideration is dual.

Suppose that $\operatorname{card} X_1 = 1$, $\operatorname{card} X_2 \geqslant 2$. By constructing analogical mappings, we get one of the following cases;

(1) $\{b\} \varrho_B \{a, d\}$, $\{c\} \varrho_B \{a, d\}$.

(2) $\{b\} \varrho_B \{a\}$, $\{c\} \varrho_B \{a\}$.

(3) $\{b\} \varrho_B \{d\}$, $\{c\} \varrho_B \{d\}$.

Case (1) was dealt with in case (3) for $\operatorname{card} X_1 \geqslant 2$, $\operatorname{card} X_2 \geqslant 2$.

Ad (2). Let V be again the supremum of $\{b\}$ and $\{c\}$ in ϱ_B. Then V is of the form $\{a\}$, $\{b, c\}$. We have $\{b\} \varrho_B \{b, c\}$, $\{c\} \varrho_B \{b, c\}$ but $\{a\}$ and $\{b, c\}$ are incomparable in ϱ_B.

Ad (3). The case is dual to the previous one.

Suppose $\operatorname{card} X_1 = 1$, $\operatorname{card} X_2 = 1$. Then we get (2) or (3) as for $\operatorname{card} X_1 = 1$, $\operatorname{card} X_2 \geqslant 2$.

Remark. There exist full subcategories in $\mathcal{K}_2$, not containing a four-element Boolean lattice in which the conclusion is also true, for instance the category of all chains (proof follows immediately, e.g., from the proof of Proposition 1). Among the categories dealt in Proposition 6 there is, of course, a full subcategory in $\mathcal{K}_2$ containing all finite objects, i.e. $\mathcal{K}_2^f$. It may be of certain interest to compare this special case with Proposition 2.

The previous questions can be asked for different categories and in many of them they can have a reasonable sense. Let us conclude this paper with a description of the case of complete completely distributive Boolean algebras, i.e. in fact the case of the systems of all subsets of sets. As morphisms we take complete homomorphisms of Boolean algebras, i.e. mappings preserving all meets and joins, the least element 0, the greatest element 1 and complements (the complement of x is denoted by x'). Let $\mathcal{B}_C$ stand for this category. $\operatorname{Ext} X$ now denotes the system of all subalgebras A of X, which are complete in the sense that all meets and joins in A coincide with the corresponding meets and joins in X, $0, 1 \in A$ and $x \in A \Rightarrow x' \in A$. So a subalgebra means in the sequel a subalgebra with the properties just prescribed.

Let $X \in O(\mathcal{B}_C)$ and A_X the set of all atoms of X. Then any decomposition R of A_X determines a subalgebra of X (with the atoms of the form $\bigvee Y, Y \in R$) and any subalgebra of X is obtainable in this way by means of exactly one decomposition of A_X.

Let X_1 and X_2 be two subalgebras of X. Then $X_1 \subset X_2$ iff the decomposition R_2 belonging to X_2 is finer than R_1 belonging to X_1.

If f is a complete homomorphism between X and Y, $X, Y \in O(\mathcal{B}_C)$ and X_1 is a subalgebra in X, then $f(X_1)$ is a subalgebra in Y.

PROPOSITION 7. *The only maximal locally stable systems $\{\varrho_X : X \in O(\mathcal{B}_C)\}$ are the set inclusion and the order dual to the set inclusion. Every locally stable ordering is included in one of them.*

Proof. It is clear that the above-mentioned systems are both locally stable. Suppose we have an algebra B and a locally stable system $\{\varrho_X : X \in O(\mathcal{B}_C)\}$ such that $X_1 \not\subset X_2$, $X_2 \not\subset X_1$ and $X_1 \varrho_B X_2$. Let R_1 and R_2 be the corresponding decompositions of A_B. Then $Z_1 \in R_2$ $Z_2, Z_3 \in R_1$ exist such that $Z_2 \neq Z_3$, $Z_1 \cap Z_2 \neq \emptyset \neq Z_1 \cap Z_3$. Let $a \in Z_1 \cap Z_2$, $b \in Z_1 \cap Z_3$.

Define $f\colon B \to B$ as follows: $f(a) = a$, $f(b) = a'$, $f(c) = 0$ for $c \in A_B$, $a \neq c \neq b$, $f(x) = \bigvee_{c \in C} f(c)$, where $x = \bigvee_{c \in C} c$, $C \subset A_B$.

It can be checked that $f \in M(\mathcal{B}_C)$, $f(X_1) = \{0, a, a', 1\}$, $f(X_2) = \{0, 1\}$. Similarly one gets $g(X_2) = \{0, a, a', 1\}$, $g(X_1) = \{0, 1\}$ for a suitable $g \in M(\mathcal{B}_C)$, which is a contradiction.

Thus $X_1 \varrho_B X_2 \Rightarrow R_2$ is finer or coarser than R_1. Let the first possibility occur for certain X_1, X_2 (B being fixed), $X_1 \neq X_2$. Then $\{0, 1\} \varrho_B \{0, a, a', 1\}$ by the same argument as above. Assume that $X_1' \varrho_B X_2'$ and R_2 is finer than R_1 for certain distinct X_1', X_2'. Then we can deduce $\{0, a, a', 1\} \varrho_B \{0, 1\}$, which is a contradiction.

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