On ring theoretic lattice modules

by

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The lattice of ideals of a commutative Noetherian ring with identity has been abstracted by the notion of a Noether lattice. This note generalizes this by presenting an abstraction of the lattice of submodules of a unitary module over a commutative ring with identity. The basic idea is to extend R. P. Dilworth's formulation of a principal element in a multiplicative lattice into a module setting.

Section 1 gives basic definitions and some examples. Section 2 presents an abstract version of Nagata's principle of idealization thereby enabling much of the theory of Noether lattices to be extended immediately to Noetherian ring theoretic modules. In Section 3 form modules and localization are investigated.

§ 1. Ring theoretic lattice modules. Let L be a complete modular lattice with maximum element I and minimum element O and with a multiplication satisfying:

$$(AB)C = A(BC)\,, \tag{1.1}$$

$$AB = BA\,, \tag{1.2}$$

$$(\bigvee_{\alpha} A_\alpha)(\bigvee_{\beta} B_\beta) = \bigvee_{\alpha,\beta} A_\alpha B_\beta\,, \tag{1.3}$$

$$IA = A\,. \tag{1.4}$$

Under these conditions L is said to be *multiplicative*. An L-*module* M is a complete modular lattice with maximum element $\mathfrak{m}$ and minimum element O_M, and which admits a (left) multiplication from L satisfying:

$$(AB)N = A(BN)\,, \tag{1.5}$$

$$(\bigvee_{\alpha} A_\alpha)(\bigvee_{\beta} N_\beta) = \bigvee_{\alpha,\beta} A_\alpha N_\beta\,, \tag{1.6}$$

$$IN = N \quad \text{and} \quad ON = O_M\,. \tag{1.7}$$

Associated with an L-module M are operations called residual division. If $A \in L$ and $N \in M$, then $N:A$ is the join of all $N' \in M$ such that $AN' \leqslant N$. Similarly if N and N' belong to M, then $N:N'$ is the join of all $A \in L$ such that $AN' \leqslant N$.

Now in order to abstract the notion of a module over a commutative ring it is necessary to formulate an abstraction of a principal element. The following definition is due to E. W. Johnson [3]. An element N of an L-module M is said to be *principal* if for arbitrary $A \in L$ and $N' \in M$ the following holds:

(1.8) $$\big(A \wedge (N':N)\big)N = AN \wedge N',$$

(1.9) $$(N' \vee AN):N = (N':N) \vee A .$$

An L-module M is said to be *principally generated* if every element is a join of principal elements. A principally generated multiplicative lattice satisfying the ascending chain condition is called a *Noether lattice*.

For the remainder of this paper L will denote a principally generated multiplicative lattice. A principally generated L-module is said to be *ring theoretic* provided every element of L is a join of principal elements A which satisfy the following for all $N_1, N_2 \in M$:

(1.10) $$(N_1 \vee AN_2):A = N_1:A \vee N_2$$

(1.11) $$A(N_1 \wedge N_2:A) = AN_1 \wedge N_2$$

Such elements of L will be called *M-principal*. A ring theoretic module is said to be *Noetherian* if it satisfies the ascending chain condition. For the remainder of this paper, M will denote a ring theoretic L-module. Arbitrary elements of L will be denoted $A, B, C, \ldots$ and principal elements $a, b, c, \ldots$ Arbitrary elements of M will be denoted $N, N_1, N_2, \ldots$ and principal elements $n, n_1, n_2, \ldots$

Examples of principally generated multiplicative lattices and ring theoretic lattice modules are abundant. The lattice $L(R)$ of graded ideals of a graded commutative ring R with identity is a principally generated multiplicative lattice, and the set of graded R-submodules of a graded unitary R-module is a ring theoretic $L(R)$-module. Furthermore, an arbitrary (possibly non-commutative) ring with identity which satisfies the ascending chain condition on ideals has a principally generated multiplicative lattice for an ideal system provided:

(1.12) $AB = BA$, for ideals A, B of R,

(1.13) Every ideal is a join of principal ideals RaR such that $RaR = aR = Ra$.

Verification of the above statements is straightforward.

The following lemma cites some elementary facts about M.

LEMMA 1.1. *The following hold for L and M:*

(1) $\bigwedge_\alpha (N_\alpha:N) = (\bigwedge_\alpha N_\alpha):N$, $\bigwedge_\alpha (N_\alpha:A) = (\bigwedge_\alpha N_\alpha):A$.

(2) $N:\bigvee_\alpha N_\alpha = \bigwedge_\alpha (N:N_\alpha)$, $N:\bigvee_\alpha A_\alpha = \bigwedge_\alpha (N:A_\alpha)$.

(3) $N:AB = (N:A):B$, $N:AN_1 = (N:A):N_1 = (N:N_1):A$.

(4) *If* $A \leqslant B$, *then* $AN \leqslant BN$ *and* $AN \leqslant N$; *and if* $N_1 \leqslant N_2$, *then* $AN_1 \leqslant AN_2$.

(5) $N \geqslant A(N:A)$, $N \geqslant (N:N_1)N_1$, *and* $N:A \geqslant N$.

(6) $I = N_1:N_2$ *if and only if* $N_2 \leqslant N_1$.

(7) *If* $A \leqslant B$, *then* $N:B \leqslant N:A$; *if* $N_1 \leqslant N_2$, *then* $N_1:A \leqslant N_2:A$ *and* $N:N_1 \geqslant N:N_2$.

(8) $N_1:N_2 = N_1:(N_1 \vee N_2)$, $N_1:N_2 = (N_1 \wedge N_2):N_2$.

(9) $(A:B)N \leqslant AN:B$, $A:B \leqslant AN:BN$.

(10) *If $n \in M$ is principal, then* $n \wedge N = (N:n)n$ *and* $An:n = A \vee O_M:n$.

(11) *If $a \in L$ is M-principal and $n \in M$ is principal, then an is principal in M.*

Proof. The proofs of (1)–(9) are quite simple and may be omitted. (10) may be proved as follows: $n \wedge N = In \wedge N = (I \wedge N:n)n = (N:n)n$, since n is principal. Also since n is principal, $An:n = (O_M \vee An):n = O_M:n \vee A$. (11) may be proved in a way very similar to the proof of Corollary 3.3 in [2], q.e.d.

The following theorem generalizes the notions of factor module and submodule.

THEOREM 1. *Let N_1/N_2 be an interval of M, let $A \leqslant N_2:N_1$, and let L/A denote the set of $B \in L$ such that $B \geqslant A$. Then L/A is a principally generated multiplicative lattice under the multiplication $B \cdot C = BC \vee A$ and N_1/N_2 is a ring theoretic module over L/A when multiplication is defined by $B \cdot N = BN \vee N_2$, for $B \in L/A$ and $N \in N_1/N_2$.*

Proof. It can be easily verified that L/A is a multiplicative lattice. In [1, p. 488] Dilworth proved that, when L is Noetherian, if a is principal in L, then $a \vee A$ is principal in L/A. As his proof does not rely upon the ascending chain condition, it also suffices for the general case. This clearly implies that L/A is principally generated.

It is easily verified that N_1/N_2 is complete, modular, and satisfies (1.5)–(1.7). Every element of L/A is a join of N_1/N_2-principal elements because elements of the form $a \vee A$ where a is M-principal are also N_1/N_2-principal in L/A. (1.11) is shown as follows: Let $N, N' \in N_1/N_2$. Now $\big((a \vee A)\cdot N\big) \wedge N' = \big((a \vee A)N \vee N_2\big) \wedge N' = (aN \vee N_2) \wedge N'$ since $AN \leqslant N_2$. By modularity $(aN\vee N_2)\wedge N'=(aN\vee N')\wedge N_2$. Since a is M-principal $(aN\wedge N')\vee N_2=a(N\wedge N':a)\vee N_2=(a\vee A)\cdot(N\wedge(N':a))$. Letting $:'$ denote residuation with respect to L/A and N_1/N_2, it is easy to compute that $N':'(a\vee A)=N':(a\vee A)\wedge N_1=(N':a)\wedge N_1$. Therefore $(a\vee A)\cdot N\wedge N'=(a\vee A)\cdot N\wedge(N':'(a\vee A))$ and (1.11) is satisfied. (1.10) can be verified similarly for $a\vee A$. Also it can be shown similarly that, if $n\leqslant N_1$, then $n\vee N_2$ is principal in N_1/N_2 with respect to L/A, q.e.d.

At the end of this section it seems appropriate to pose the following problem: Classify the non-commutative rings which have a Noether lattice for an ideal system.

§ 2. The principle of idealization. Again in this section L will denote a principally generated multiplicative lattice and M a ring theoretic L-module. The essential result in this section is the following theorem which is called the principle of idealization after its ring theory counterpart (see [4, p. 2]).

THEOREM 2. *Let $L\oplus M$ denote the set of all ordered pairs (A,N) where $A\in L$ and $N\in M$ are such that $A\mathfrak{m}\leqslant N$. Under the operations below, $L\oplus M$ is a principally generated multiplicative lattice. Furthermore if L and M are Noetherian, then $L\oplus M$ is a Noether lattice.*

$$(A_1,N_1)\vee(A_2,N_2)=(A_1\vee A_2,N_1\vee N_2)\,,$$

$$(A_1,N_1)\wedge(A_2,N_2)=(A_1\wedge A_2,N_1\wedge N_2)\,,$$

$$(A_1,N_1)(A_2,N_2)=(A_1A_2,A_1N_2\vee A_2N_1)\,.$$

Proof. The details of proving that $L\oplus M$ is a multiplicative lattice are quite straightforward and will be omitted. It remains to show that $L\oplus M$ is principally generated. To show this it suffices to show that elements of the form $(a,a\mathfrak{m})$ and (O,n) are principal, where a is M-principal, since every element of $L\oplus M$ is clearly a join of such elements. The following calculation shows that $(a,a\mathfrak{m})$ satisfies (1.8). Let (A,N) and (B,N') be arbitrary elements of $L\oplus M$. Now it is easy to compute that $(B,N'):(a,a\mathfrak{m})=(B:a,N':a)$. Hence $\big((A,N)\wedge\big((B,N'):(a,a\mathfrak{m})\big)\big)(a,a\mathfrak{m})=\big((A,N)\wedge(B:a,N':a)\big)(a,a\mathfrak{m})$. By definition of meet and product in $L\oplus M$ the latter reduces to $\big((A\wedge B:a)a,a(N\wedge N':a)\big)$. Since a is M-principal, this equals $(aA\wedge B,aN\wedge N')$. Now $\big((a,a\mathfrak{m})(A,N)\big)\wedge(B,N')=(aA,aA\mathfrak{m}\vee aN)\wedge(B,N')=(aA\wedge B,aA\mathfrak{m}\vee aN\wedge N'$. Since $A\mathfrak{m}\leqslant N$, $(a,a\mathfrak{m})(A,N)\wedge(B,N')=(aA\wedge B,aN\wedge N')$. Thus $(a,a\mathfrak{m})$ satisfies (1.8). The other identities for (O,n) and $(a,a\mathfrak{m})$ can be verified similarly.

If L and M are Noetherian, then any ascending chain $\{(A_i;N_i)\}$ will terminate when both chain $\{A_i\}$ and $\{N_i\}$ terminate, and hence $L\oplus M$ is a Noether lattice, q.e.d.

The principle of idealization provides a useful technique in developing the theory of Noetherian ring theoretic L-modules, where L is a Noether lattice. *For the remainder of this paper L and M will be assumed Noetherian.* The following corollaries indicate the usefulness of Theorem 2. A Noether lattice is said to be *local* if it has a unique maximal prime element.

COROLLARY 1. *Let L be local with unique maximal prime element P, and let $N\neq O\in M$. Then every minimal set of principal elements generating N has the same cardinality which is equal to $l(N/PN)$.*

Proof. This result is known already in the case $M=L$ (see [2]). Now $(O,N)\in L\oplus M$ which is local with maximal prime element $(P,\mathfrak{m})$. Let $n_1,\ldots,n_k$ be a minimal set of principal generators for N. Hence $(O,N)=(O,n_1)\vee\ldots\vee(O,n_k)$ where each (O,n_i) is principal in $L\oplus M$. So by the already known result about Noether lattices, $k=l\big((O,N)/(P,\mathfrak{m})(O,N)\big)=l\big((O,N)/(O,PN)\big)=l(N/PN)$, q.e.d.

COROLLARY 2. *Let $A\in L$, and let $N,N'\in M$. Then there exists a positive integer r such that for $n>r$ $A^nN\wedge N'=A^{n-r}(A^rN\wedge N')$.*

Proof. In [2] this result is proved for the case $L=M$.

Passing to $L\oplus M$, it follows that there exists an r such that for $n>r$,

$$\big((A,A\mathfrak{m})^n(O,N)\big)\wedge(O,N')=(A,A\mathfrak{m})^{n-r}\big((A,A\mathfrak{m})^r(O,N)\wedge(O,N')\big)\,.$$

Simplifying both sides yields $(O,A^nN\wedge N')=\big(O,A^{n-r}(A^rN\wedge N')\big)$.

Thus for $n>r$, $A^nN\wedge N'=A^{n-r}(A^rN\wedge N')$, q.e.d.

§ 3. Form lattices and localization. In this section two of the standard constructions of ring theory, form modules and localization, are abstracted to ring theoretic lattice modules. Throughout this section L will be a Noether lattice and M a Noetherian ring theoretic L-module. The convention $A^i=I$, for $i\leqslant 0$, will also be adopted.

Let $A\neq I\in L$. Then the *form lattice of L with respect to A*, denoted $F(A)$, is the set of all formal sums $\sum_{i=0}^{\infty}B_i$, where $A^i\geqslant B_i\geqslant B_{i+1}\geqslant AB_i$ and $B_i\geqslant A^{i+1}$, together with the following operations:

$$\Big(\sum_i B_i\Big)\vee\Big(\sum_i C_i\Big)=\sum_i(B_i\vee C_i)\,,$$

$$\Big(\sum_i B_i\Big)\wedge\Big(\sum_i C_i\Big)=\sum_i(B_i\wedge C_i)\,,$$

$$\Big(\sum_i B_i\Big)\Big(\sum_i C_i\Big)=\sum_i\bigvee_{k+j=i}(B_kC_j\vee A^{i+1})\,,\quad\text{and}$$

$$\sum_i B_i\leqslant\sum_i C_i\quad\text{if and only if } B_i\leqslant C_i,\quad\text{for all } i\,.$$

The *form lattice of M with respect to A*, denoted $G(A, M)$, is the set of all formal sums $\sum_{i=0}^{\infty} N_i$, where $A^i\mathfrak{m} \geq N_i \geq N_{i+1} \geq AN_i$ and $N_i \geq A^{i+1}\mathfrak{m}$, together with the operations

$$\Big(\sum_i N_i\Big)\vee\Big(\sum_i N_i'\Big) = \sum_i (N_i \vee N_i'),$$

$$\Big(\sum_i N_i\Big)\wedge\Big(\sum_i N_i'\Big) = \sum_i (N_i \wedge N_i'), \quad \text{and}$$

$$\sum_i N_i \leq \sum_i N_i' \quad \text{if and only if } N_i \leq N_i', \quad \text{for all } i.$$

If n is principal in M and $n \leq A^k\mathfrak{m}$, $n \nleq A^{k+1}\mathfrak{m}$, then the *leading form* of n, denoted n', is $\sum_{i=0}^{\infty}(A^{i-k}n \vee A^{i+1}\mathfrak{m})$.

As might be supposed, the following is true.

THEOREM 3. *$F(A)$ is a Noether lattice and $G(A, M)$ is a Noetherian ring theoretic $F(A)$-module under multiplication given by*

$$\Big(\sum_i B_i\Big)\Big(\sum_i N_i\Big) = \sum_i \bigvee_{k+j=i} B_k N_j \vee A^{i+1}\mathfrak{m}.$$

Furthermore if L is local, then $F(A)$ is local.

Proof. The easiest way to obtain this result is to apply the theory A-transforms for Noether lattices as developed by E. W. Johnson in [2]. Let $\mathfrak{R}(L, A)$ denote the A-transform of L which Johnson proves to be a Noether lattice which is local when L is local. In $\mathfrak{R}(L, A)$ consider the element $I^{(-1)} = \sum_{-\infty}^{\infty} B_i$, where $B_i = A^{i+1}$. Upon examination it is obvious that $\mathfrak{R}(L, A)/I^{(-1)}$ and $F(A)$ are isomorphic as multiplicative lattices under the correspondence $\sum_{i=-1}^{-\infty} I + \sum_{i=0}^{\infty} C_i \Leftrightarrow \sum_{i=0}^{\infty} C_i$. Thus $F(A)$ is a Noether lattice since $\mathfrak{R}(L, A)/I^{(-1)}$ is a Noether lattice.

To get the module structure of $G(A, M)$ consider the Noether lattice $L' = L \oplus M$ and then consider the form lattice $F(A') = F'$ of L' where $A' = (A, A\mathfrak{m})$. Now the interval $\sum_{i=0}^{\infty}(A^{i+1}, A^i\mathfrak{m})/\sum_{i=0}^{\infty}(A^{i+1}, A^{i+1}\mathfrak{m})$ is a ring theoretic $F'/\sum_{i=0}^{\infty}(A^{i+1}, A^i\mathfrak{m})$-module by Theorem 1. Under the obvious correspondences $F'/\sum_{i=0}^{\infty}(A^{i+1}, A^i\mathfrak{m})$ is isomorphic to $F(A)$ and $\sum_{i=0}^{\infty}(A^{i+1}, A^i\mathfrak{m})/\sum_{i=0}^{\infty}(A^{i+1}, A^{i+1}\mathfrak{m})$ is isomorphic to $G(A, M)$. This completes the proof, q.e.d.

It should be pointed out that the previous theorem could also be proved by long and tedious computation. Indeed, it can be verified that leading forms of principal elements in L and M are principal in $F(A)$ and $G(A, M)$. Also straightforward computation shows that the leading forms of M-principal elements are G-principal. Furthermore an application of the Artin–Rees lemma (Corollary 2 above) can be made to show F and G Noetherian.

The basic idea for abstracting localization was given by Dilworth in [1]. Here Dilworth's results will be extended via the principle of idealization. The first step in the development of localization for M is to note that the standard primary decomposition theorems for Noetherian modules over Noetherian commutative rings with identity also hold for M. The details of proof for this can be lifted easily from the standard proofs in ring theory (see for example [6]). Also this could be shown by using the principle of idealization and the well known results about primary decomposition in Noether lattices. For the remainder of this paper it will be assumed that the reader is conversant with this decomposition theory as well as the usual terminology of P-primary, P-component, and so forth.

Now let $D \in L$ and $N, N'(\neq \mathfrak{m}) \in M$ and define $N \sim N'(D)$, if $N_D = N'_D$, where N_D denotes the isolated component of N with respect to the sets of primes less than or equal to D; also set $\mathfrak{m}_D = \mathfrak{m}$. Also let $[N]$ denote the equivalence class of N and let M_D be the set of such equivalence classes. The following result describes the relation between L_D and M_D.

THEOREM 4. *M_D is a Noetherian ring theoretic L_D-module under the operations*

$$[N_1]\vee[N_2] = [N_1 \vee N_2],$$

$$[N_1]\wedge[N_2] = [N_1 \wedge N_2],$$

$$[A][N] = [AN].$$

Proof. The basic idea is to show that $\sim$ is a congruence relation with respect to join, meet, product, and residuation. The result then is immediate because then the above operations are well defined and satisfy the required properties of a ring theoretic L_D-module. In particular, equivalence classes containing principal (M-principal) elements are principal (M-principal) since principal (M-principal) elements are defined by identities.

The congruence properties of $\sim$ will be shown by passing to $L\oplus M$ and using the fact that $\sim$ is a congruence relation in a Noether lattice (this is shown in [1, p. 489]). The following is true and will be proved in a lemma below: $N\sim N'(D)$ and $A\sim A'(D)$ if and only $(A,N)\sim(A',N')\big((D,\mathfrak{m})\big)$. Now assume $N_1\sim N_1'$, $N_2\sim N_2'$, and $A\sim A'$. The congruence with respect to $\vee$ is shown in the following way: $(O,N_1\vee N_2)=(O,N_1)\vee(O,N_2)\sim(O,N_1')\vee(O,N_2')=(O,N_1'\vee N_2')$ by the above remark and the congruence property of $\sim$ in $L\oplus M$. Also by the above remark it follows that $N_1\vee N_2\sim N_1'\vee N_2'$. Similarly $N_1\wedge N_2\sim N_1'\wedge N_2'$. By the congruence of $\sim$ with respect to product in a Noether lattice $(O,AN_1)=(A,\mathfrak{m})(O,N_1)\sim(A',\mathfrak{m})(O,N_1')=(O,AN_1')$. Thus $AN_1\sim AN_1'$. It can also be shown similarly that $N_1:N_2\sim N_1':N_2'$ and $N_1:A\sim N_1':A'$. Hence $\sim$ satisfies the desired congruence properties, q.e.d.

LEMMA. $(A,N)\sim(B,N')\big((D,\mathfrak{m})\big)$ *if and only if* $A\sim B(D)$ *and* $N\sim N'(D)$.

Proof. The following statements can be proved easily from the definitions of prime and primary elements. The prime elements of $L\oplus M$ are exactly those elements $(P,\mathfrak{m})$, where P is prime in L. Furthermore primary elements in $L\oplus M$ are of the form $(N:\mathfrak{m},N)$ where N is P-primary or $(Q,\mathfrak{m})$ where Q is P-primary.

Now to prove the lemma it clearly suffices to show that $(A,N)_{(D,\mathfrak{m})}=(A_D,N_D)$. Let $N=\bigwedge_{i=1}^{k}N_i$ and $A=\bigwedge_{i=k+1}^{n}Q_i$ be normal decompositions for N and A, where N_i is P_i-primary and Q_j is P_j-primary. Now $(A,N)=\bigwedge_j(Q_j,\mathfrak{m})\wedge\bigwedge_i(N_i:\mathfrak{m},N_i)$ since $A\mathfrak{m}\leqslant N$ implies that $N_i:\mathfrak{m}\geqslant A$, for $i=1,\ldots,k$. By the remarks in the previous paragraph it follows that $\bigwedge_j(Q_j,\mathfrak{m})$ is a normal decomposition for $(A,\mathfrak{m})$ and that $\bigwedge_i(N_i:\mathfrak{m},N_i)$ is a normal decomposition for $(N:\mathfrak{m},N)$. Now it can be easily proved from Lemma 5.2 of [2] that $(E\wedge F)_D=E_D\wedge F_D$, where E, F, and D belong to an arbitrary Noether lattice. In particular, $(A,N)_{((D,\mathfrak{m}))}=\bigwedge_j(Q_j,\mathfrak{m})_{(D,\mathfrak{m})}\wedge\bigwedge_i(N_i:\mathfrak{m},N_i)_{((D,\mathfrak{m}))}=\big(A_D\wedge(N:\mathfrak{m})_D,\mathfrak{m}\wedge N_D\big)=\big((A\wedge N:\mathfrak{m})_D,N_D\big)$. The last element equals (A_D,N_D) because $A\leqslant N:\mathfrak{m}$, q.e.d.

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