Accumulation functions on the ordinals

by

Arthur L. Rubin and **Jean E. Rubin** (Lafayette, Ind.)

§ 1. Introduction. For each ordinal α, we calculate $T(\alpha) = \sum_{\xi<\alpha} \xi$ and $\Gamma(\alpha) = \prod_{0<\xi<\alpha} \xi$, and then extend these functions to the higher operations of Doner and Tarski [1]. The notation used is that given in [1]. For convenience we shall repeat several of the definitions and theorems given in the Doner–Tarski paper. When referring to a theorem, lemma, etc. in their paper we shall prefix the numeral by the symbol "D–T".

We shall use the following notation: $\alpha, \beta, \gamma, \xi, \eta, \ldots$ are variables whose ranges are all ordinal numbers; $n, m, p, q, \ldots$ are variables whose ranges are all finite ordinal numbers; ω is the smallest infinite ordinal; Ω is the class of all ordinals; and $\emptyset$ is the empty set. The reader is referred to Sierpinski [4], Chapter 14, and Rubin [3], Chapters 8 and 9, for the traditional properties of ordinal arithmetic used in this paper.

DEFINITION 1.1. HIGHER OPERATIONS. [D–T 1]

(i) $\alpha O_0 \beta = \alpha + \beta$.

(ii) If $\gamma > 0$, $\alpha O_\gamma \beta = \bigcup_{\eta<\beta, \zeta<\gamma} (\alpha O_\gamma \eta) O_\zeta \alpha$.

COROLLARY 1.2. [D–T 2]

(i) $\alpha O_\gamma 0 = 0 O_\gamma \alpha = 0$, *if* $\gamma > 0$.

(ii) $\alpha O_\gamma 1 = 1 O_\gamma \alpha = \alpha$, *if* $\gamma > 0$.

(iii) $2 O_\gamma 2 = 4$.

THEOREM 1.3. [D–T 3]

(i) $\alpha O_1 \beta = \alpha \cdot \beta$.

(ii) *If* $\alpha \neq 1$ *and* $\beta \neq 0$ *then* $\alpha O_2 \beta = \alpha^\beta$.

(iii) *If* $\alpha \neq 1$ *then* $\alpha O_3 (1+\beta) = \alpha^{\alpha^\beta}$.

THEOREM 1.4. MONOTONICITY LAWS.

(i) *If* $\beta \geqslant \beta'$ *then* $\alpha O_\gamma \beta \geqslant \alpha O_\gamma \beta'$. [D–T 4(i)]

(ii) *If* $\alpha \geqslant 1$ *and* $\beta > \beta'$ *then* $\alpha O_\gamma \beta > \alpha O_\gamma \beta'$. [D–T 4(ii)]

(iii) *If* $\alpha \geqslant \alpha'$ *then* $\alpha O_\gamma \beta \geqslant \alpha' O_\gamma \beta$. [D–T 6]

(iv) *If* $\gamma \geqslant \gamma'$ *and either* $\alpha, \beta \geqslant 2$ *or* $\gamma' \geqslant 1$ *then* $\alpha O_\gamma \beta \geqslant \alpha O_{\gamma'} \beta$. [D–T 8]

(v) *If* $\beta \geqslant 1$ *then* $\alpha O_\gamma \beta \geqslant \alpha$. [D–T 5(i)]

(vi) *If* $\alpha \geqslant 1$ *and* $\beta \geqslant 2$ *then* $\alpha O_\gamma \beta > \alpha$. [D–T 5(ii)]

THEOREM 1.5. [D–T 9]

If $\alpha \geqslant 2$ *and* $\beta \geqslant 1$ *then*

$$\alpha O_{\gamma+1}(\beta+1) = (\alpha O_{\gamma+1}\beta) O_\gamma \alpha .$$

THEOREM 1.6. *$\alpha O_\gamma \beta$ is a limit ordinal if any one of the following conditions hold.*

(i) $\gamma \geqslant \omega$, $\alpha, \beta \geqslant 2$, *and* $\alpha = \beta = 2$ *does not hold.*

(ii) $2 \leqslant \gamma < \omega$, $\alpha \geqslant 2$ *and* $\beta \geqslant \omega$.

(iii) $3 \leqslant \gamma < \omega$, $\alpha \geqslant \omega$ *and* $\beta \geqslant 2$.

Proof. [2], Lemma 7.

THEOREM 1.7. [D–T 27 (i)]

If $\alpha \geqslant 2$, $\beta \geqslant 1$, *and* $\gamma = \bigcup \gamma \neq 0$ *then*

$$\alpha O_\gamma(\beta+\beta') = (\alpha O_\gamma \beta) O_\gamma (1+\beta') .$$

THEOREM 1.8. [D–T 32]

If $\beta, \beta' \geqslant 1$ *and* $\gamma \neq 1$ *then*

(i) *if* $\beta' = \bigcup \beta' \neq 0$ *and* $\alpha \geqslant 2$, *then*

$$\alpha O_{2\gamma}(\beta+\beta') = (\alpha O_{2\gamma}\beta) O_{2\gamma} \beta' .$$

(ii) *if* $\alpha = \bigcup \alpha$ *or if* $\alpha \geqslant 2$ *and* $\beta' = \bigcup \beta' \neq 0$ *then*

$$\alpha O_{2\gamma+1}(\beta+\beta') = (\alpha O_{2\gamma+1}\beta) O_{2\gamma} (\alpha \cdot \beta') .$$

DEFINITION 1.9. MAIN NUMBERS.

(i) If O is a binary operation from $\Omega \times \Omega$ to Ω then $\delta \geqslant \omega$ is a *main number of* O iff for all $\alpha, \beta < \delta$, $\alpha O \beta < \delta$. [D–T 38]

(ii) $M(O)$ denotes the class of all main numbers of O. [D–T 38]

(iii) $\mu(\eta, O)$ is the ηth successive main number of O. ($\mu(0, O)$ is the smallest main number of O.) [D–T 40]

(iv) If $\gamma = \bigcup \gamma \neq 0$ then $M_\gamma = \bigcap_{\eta<\gamma} M(O_\eta)$. ([2], 4(iii))

(v) If $\gamma = \bigcup \gamma \neq 0$ then $\mu_\gamma(\eta)$ is the ηth successive element of M_γ.

The main numbers of O_γ are its fixed points. That is,

THEOREM 1.10.

(i) *δ is a main number of O_γ iff $\delta \geqslant 3$ and $\alpha O_\gamma \delta = \delta$ for all α, $2 \leqslant \alpha < \delta$.* [D–T 46]

(ii) *If* $\gamma \geqslant 2$, *then for all* α, $2 \leqslant \alpha < \delta$, $\delta \in M(O_\gamma)$ *iff* $\alpha O_\gamma \delta = \delta$. [D–T 47]

Thus, for example, the main numbers of O_0 (addition) are positive powers of ω; the main numbers of O_1 (multiplication) are all ordinals in the form ω^{ω^η}; and the main numbers of O_2 (exponentiation) are ω and the epsilon numbers.

THEOREM 1.11. *If* $\beta = \bigcup \beta \neq 0$ *then*

(i) $\alpha O_\gamma \beta = \bigcup_{\eta<\beta} \alpha O_\gamma \eta$.

(ii) $\mu(\beta, O_\gamma) = \bigcup_{\eta<\beta} \mu(\eta, O_\gamma)$.

(iii) $\mu_\gamma(\beta) = \bigcup_{\eta<\beta} \mu_\gamma(\eta)$.

Proof. Part (i) is D–T 15(iii); (ii) is D–T 41(i); and (iii) follows from (i) and D–T 55.

THEOREM 1.12. [D–T 48(ii)]

If $\alpha \geqslant 2$, $\gamma \geqslant 1$, *and* $\mu(\lambda, O_{2\gamma})$ *is the least main number of* $O_{2\gamma}$ *exceeding* α *then*

$$\mu(\lambda+\eta, O_{2\gamma}) = \alpha O_{2\gamma+2}[\omega(1+\eta)] .$$

Thus, the function $\psi_\gamma(\eta) = 2 O_{2\gamma+2}[\omega(1+\eta)]$ enumerates the elements of $M(O_{2\gamma})$.

THEOREM 1.13. [D–T 54]

If $\alpha \geqslant 3$ *and* $\gamma = \bigcup \gamma \neq 0$ *then* $\xi \in M_\gamma$ *and* $\xi > \alpha$ *iff there is an* η *so that* $\xi = \alpha O_\gamma(2+\eta)$.

THEOREM 1.14. [D–T 37]

If $\gamma = \bigcup \gamma \neq 0$ *then* $2 O_\gamma(3+\eta) = 3 O_\gamma(2+\eta)$.

It follows from 1.12 and 1.13 that if $\gamma = \bigcup \gamma \neq 0$, $2 O_\gamma 3 = \mu_\gamma(0)$ is the smallest element of M_γ, and if $\alpha \geqslant 3$, $\alpha O_\gamma(2+\eta)$ is the ηth element of M_γ exceeding α. Moreover, the function $\psi(\eta) = 3 O_\gamma(2+\eta)$ enumerates the elements of M_γ.

THEOREM 1.15.

(i) *If* $\gamma \geqslant 1$ *then* $M(O_{2\gamma}) = M(O_{2\gamma+1}) \supsetneq M(O_{2\gamma+2})$. [D–T 52(i)]

(ii) *If* $\gamma = \bigcup \gamma \neq 0$ *then* $M(O_\gamma) \subsetneq M_\gamma$. [D–T 57]

(iii) *If* $\lambda \in M(O_\gamma)$ *then* $\lambda = \bigcup \lambda \neq 0$. [D–T 42(i)]

DEFINITION 1.16. ACCUMULATION FUNCTIONS.

(i) $\Xi_0(0) = \Xi_0(1) = 0$.

(ii) If $\gamma > 0$, $\Xi_\gamma(0) = \Xi_\gamma(1) = 1$.

(iii) If $\alpha > 0$, $\Xi_\gamma(\alpha+1) = \Xi_\gamma(\alpha) O_\gamma \alpha$.

(iv) If $\alpha = \bigcup \alpha \neq 0$, $\Xi_\gamma(\alpha) = \bigcup_{\beta<\alpha} \Xi_\gamma(\beta)$.

Thus, for example, $\Xi_0(\alpha) = \sum_{\xi<\alpha} \xi$ and $\Xi_1(\alpha) = \prod_{0<\xi<\alpha} \xi$.

DEFINITION 1.17. CONTINUOUS.

If F is a function from Ω to Ω then F is said to be *continuous* iff for all a such that $a = \bigcup a \neq 0$, $F(a) = \bigcup_{\beta<a} F(\beta)$.

It follows from 1.16 and 1.4, the monotonicity laws for O_γ, that for every γ, Ξ_γ is a continuous non-decreasing function and

LEMMA 1.18. *If* $2 \leqslant a < \beta$ *then* $\Xi_\gamma(a) < \Xi_\gamma(\beta)$.

THEOREM 1.19. *If* $a \geqslant 4$ *then* $a \leqslant \Xi_\gamma(a) \leqslant aO_{\gamma+1}a$.

Proof. The proof is by transfinite induction on a. First suppose $a = 4$ and $\gamma = 0$. Then $\Xi_0(4) = 6$ and $4\cdot 4 = 16$ so $a \leqslant \Xi_\gamma(a) \leqslant aO_{\gamma+1}a$ in this case. Next suppose $a = 4$ and $\gamma > 0$ then

$$\begin{aligned} a = 4 &= 2O_\gamma 2 && [1.2\ (\text{iii})]\\ &< 2O_\gamma 3 && [1.4\ (\text{ii})]\\ &= \Xi_\gamma(4) && [1.16\ (\text{iii})]\\ &< 4O_\gamma 4 && [1.4\ (\text{ii}),\ (\text{iii})]\\ &\leqslant 4O_{\gamma+1}4 && [1.4\ (\text{iv})]. \end{aligned}$$

Suppose the theorem is true for all $\beta < a$ and $a = \beta+1 > 4$. Then

$$\begin{aligned} a = \beta+1 &\leqslant \Xi_\gamma(\beta)+1 && [\text{Ind. Hyp.}]\\ &< \Xi_\gamma(\beta)+\beta && \\ &\leqslant \Xi_\gamma(\beta)O_\gamma\beta && [1.4\ (\text{iv})]\\ &= \Xi_\gamma(\beta+1) && [1.16\ (\text{iii})]\\ &\leqslant (\beta O_{\gamma+1}\beta)O_\gamma\beta && [\text{Ind. Hyp. } 1.4\ (\text{iii})]\\ &= \beta O_{\gamma+1}(\beta+1) && [1.5]\\ &\leqslant (\beta+1)O_{\gamma+1}(\beta+1) && [1.4\ (\text{iii})] \end{aligned}$$

Finally, if $a = \bigcup a \neq 0$, the theorem follows from 1.16 (iv) and the monotonicity law, 1.4.

THEOREM 1.20. *If* $a \geqslant 4$ *and* $\Xi_\gamma(a) = a$ *then* $a = \bigcup a \neq 0$.

Proof. Suppose $a = \beta+1 > 4$. Then

$$\begin{aligned} \Xi_\gamma(\beta+1) &= \Xi_\gamma(\beta)O_\gamma\beta && [1.16\ (\text{iii})]\\ &\geqslant \Xi_\gamma(\beta)+\beta && [1.4\ (\text{iv})]\\ &> \Xi_\gamma(\beta)+1 && \\ &\geqslant \beta+1. && [1.19] \end{aligned}$$

If $a = 4$, $\Xi_\gamma(a) = \Xi_\gamma(3)O_\gamma 3$, and it follows from 1.1 and 1.16 that $a < \Xi_\gamma(a)$. Thus, we have shown that if $a \geqslant 4$ is not a limit ordinal, then $a < \Xi_\gamma(a)$.

In what follows we frequently use the following well-known result.

THEOREM 1.21. *For each* $a \in \Omega$, $a \neq 0$, *there is a unique* $n \in \omega$, $n \neq 0$, *unique ordinal numbers* $a_0, a_1, \ldots, a_n$ *such that* $a_0 > a_1 > \ldots > a_n$, *and unique natural numbers* $a_i \neq 0$, $i = 0, 1, \ldots, n$, *such that*

$$(*)\qquad a = \omega^{a_0}a_0 + \omega^{a_1}a_1 + \ldots + \omega^{a_n}a_n\,.$$

The form $(*)$ is called the *normal form* of a. For a proof of 1.21 see, for example, Sierpinski [4], pp. 319–323.

In the next section we discuss the function Ξ_0. In section 3, we consider Ξ_1, Ξ_2, and Ξ_3. Then, in the last section, we consider Ξ_γ with $\gamma > 3$.

§ 2. The function $T(a) = \sum_{\xi<a} \xi$.

An ordinal number of the form $\sum_{\xi<a}\xi$ is called *triangular*. (This terminology is used by Sierpinski [4], p. 289. Sierpinski calculates all infinite triangular numbers $\leqslant \omega^3$.) Clearly, if $0 < a < \omega$, then $\sum_{\xi<a}\xi = \frac{1}{2}a(a-1)$. In this section we shall calculate all infinite triangular numbers.

DEFINITION 2.1. $T(a) = \sum_{\xi<a}\xi = \Xi_0(a)$.

THEOREM 2.2. *If* $a \geqslant 4$, $a \leqslant T(a) \leqslant a^2$.

Proof. 1.19 and 1.3 (i).

THEOREM 2.3. $T(a+\beta) = T(a) + \sum_{\xi<\beta}(a+\xi)$.

Proof. 2.1.

THEOREM 2.4. *If* $0 < n < \omega$ *and* $a \geqslant \omega$ *then*

$$T(a+n) = T(a) + a\cdot n + (n-1)\,.$$

Proof. 2.3 and traditional properties of ordinal arithmetic.

THEOREM 2.5. *If* $a > 0$, $\beta = \bigcup\beta \leqslant \omega^{a+1}$, *and* $0 < m < \omega$ *then* $\sum_{\xi<\beta}(\omega^a m + \xi) = \omega^a\beta$.

Proof. First we have

$$\omega^a\beta = \sum_{\xi<\beta}\omega^a \leqslant \sum_{\xi<\beta}(\omega^a m + \xi)\,.$$

Since the theorem is clearly true if $\beta = 0$ we can assume β is a limit ordinal. Then,

$$\sum_{\xi<\beta}(\omega^a m + \xi) = \bigcup_{\eta<\beta}\sum_{\xi<\eta}(\omega^a m + \xi)\,.$$

Therefore, if $\gamma \in \sum_{\xi<\beta}(\omega^a m + \xi)$ then there is an $\eta < \beta$ such that $\gamma \in \sum_{\xi<\eta}(\omega^a m + \xi)$.

Moreover,

$$\sum_{\xi<\eta}(\omega^{\alpha}m+\xi)\leqslant\sum_{\xi<\eta}(\omega^{\alpha}m+\eta)=(\omega^{\alpha}m+\eta)\,\eta\,.$$

If $\eta=\bigcup\eta$ then by [3], Theorem 9.1.6, $(\omega^{\alpha}m+\eta)\,\eta=\omega^{\alpha}\eta<\omega^{\alpha}\beta$. On the other hand, if $\eta\neq\bigcup\eta$, then there is an n, $0<n<\omega$, and $\eta'=\bigcup\eta'$ such that $\eta=\eta'+n$. Moreover, since $\eta<\beta\leqslant\omega^{\alpha+1}$, there is a $k\in\omega$ and $\eta''<\omega^{\alpha}$ such that $\eta=\omega^{\alpha}k+\eta''$. Then again using [3], Theorem 9.1.6, we obtain

$$(\omega^{\alpha}m+\eta)\,\eta=\omega^{\alpha}\eta'+\omega^{\alpha}(k+m)\,n+\eta''<\omega^{\alpha}\beta\,.$$

In either case, we obtain $\gamma\in\omega^{\alpha}\beta$ which implies $\sum_{\xi<\beta}(\omega^{\alpha}m+\xi)\leqslant\omega^{\alpha}\beta$, and completes the proof of the theorem.

THEOREM 2.6. *If* $\alpha>0$, $\beta=\bigcup\beta\leqslant\omega^{\alpha+1}$, *and* $0<m<\omega$, *then*

$$T(\omega^{\alpha}m+\beta)=T(\omega^{\alpha}m)+\omega^{\alpha}\beta\,.$$

Proof. 2.3 and 2.5.

Now it follows from 1.21, 2.4, and 2.6, that to calculate all infinite triangular numbers, it is sufficient to calculate $T(\omega^{\alpha}m)$ for all $\alpha>0$ and $m\in\omega$.

THEOREM 2.7. *If* $\alpha>0$ *and* $0<m<\omega$ *then*

$$T(\omega^{\alpha}m)=T(\omega^{\alpha})+\omega^{\alpha\cdot 2}(m-1)\,.$$

Proof.

$$\begin{aligned}T(\omega^{\alpha}m)&=T\big(\omega^{\alpha}+\omega^{\alpha}(m-1)\big)\\&=T(\omega^{\alpha})+\omega^{\alpha}\big(\omega^{\alpha}(m-1)\big)\qquad[2.6]\\&=T(\omega^{\alpha})+\omega^{\alpha\cdot 2}(m-1)\,.\end{aligned}$$

Now, it remains to evaluate $T(\omega^{\alpha})$ for $\alpha>0$.

THEOREM 2.8. $T(\omega^{\alpha+1})=\omega^{\alpha\cdot 2+1}$.

Proof.

$$\begin{aligned}T(\omega^{\alpha+1})&=T(\omega^{\alpha}+\omega^{\alpha+1})\\&=T(\omega^{\alpha})+\omega^{\alpha\cdot 2+1}\qquad[2.6]\\&=\omega^{\alpha\cdot 2+1}.\qquad[2.2]\end{aligned}$$

The last case to consider is the value of $T(\omega^{\alpha})$ when α is a limit ordinal. In this case, there exist $\beta>0$ and γ such that $\alpha=\omega^{\beta}(\gamma+1)$.

THEOREM 2.9. $T(\omega^{\omega^{\beta}(\gamma+1)})=\omega^{\omega^{\beta}(\gamma\cdot 2+1)}$.

Proof. The proof is by transfinite induction on β. If $\beta=0$ the theorem follows from 2.8. Suppose the theorem is true for all $\delta<\beta$, and $\beta=\delta+1$.

$$\begin{aligned}T(\omega^{\omega^{\delta+1}(\gamma+1)})&=T(\omega^{\omega^{\delta}(\omega\gamma+\omega)})\\&=\bigcup_{n<\omega}T(\omega^{\omega^{\delta}(\omega\gamma+n)})\\&=\bigcup_{n<\omega}T(\omega^{\omega^{\delta}(\omega\gamma+n)+1})\\&=\bigcup_{n<\omega}\omega^{\omega^{\delta}(\omega\cdot\gamma\cdot 2+n)+1}\qquad[2.8]\\&=\omega^{\omega^{\delta}(\omega\cdot\gamma\cdot 2+\omega)}\\&=\omega^{\omega^{\delta+1}(\gamma\cdot 2+1)}\,.\end{aligned}$$

If $\beta=\bigcup\beta\neq 0$, then

$$\begin{aligned}T(\omega^{\omega^{\beta}(\gamma+1)})&=T(\omega^{\omega^{\beta}\gamma+\omega^{\beta}})\\&=\bigcup_{\xi<\beta}T(\omega^{\omega^{\beta}\gamma+\omega^{\xi}})\\&=\bigcup_{\xi<\beta}T(\omega^{\omega^{\xi}(\omega^{\beta-\xi}\cdot\gamma+1)})\\&=\bigcup_{\xi<\beta}\omega^{\omega^{\xi}(\omega^{\beta-\xi}\cdot\gamma\cdot 2+1)}\qquad[\text{Ind. Hyp.}]\\&=\bigcup_{\xi<\beta}\omega^{\omega^{\beta}\cdot\gamma\cdot 2+\omega^{\xi}}\\&=\omega^{\omega^{\beta}(\gamma\cdot 2+1)}\,.\end{aligned}$$

Thus, it follows by transfinite induction that the theorem holds.

If $n<\omega$ then $T(n)=\frac{1}{2}n(n-1)$. Thus, for $n\in\omega$, $T(n)=n$ if and only if $n=0$ or $n=3$. For infinite values of α, we obtain the fixed points of $T(\alpha)$ from 2.4, 2.6, 2.7, 2.8, and 2.9. We get the following result.

THEOREM 2.10.

$$\begin{aligned}\{\alpha\colon\ T(\alpha)=\alpha\}&=\{\omega^{\omega^{\beta}}\colon\ \beta\in\Omega\}\cup\{0,3\}\\&=M(O_1)\cup\{0,3\}\,.\end{aligned}$$

§ 3. The functions Ξ, $\gamma=1,2,3$.

In this section we study the functions Ξ_1, Ξ_2 and Ξ_3. Ξ_1 is the factorial or gamma function.

DEFINITION 3.1. $\Gamma(\alpha)=\prod_{0<\xi<\alpha}\xi=\Xi_1(\alpha)$.

THEOREM 3.2. *If* $\alpha\geqslant 4$ *then* $\alpha\leqslant\Gamma(\alpha)\leqslant\alpha^{\alpha}$.

Proof. 1.19 and 1.3 (ii).

THEOREM 3.3. $\Gamma(\alpha+\beta) = \Gamma(\alpha)\cdot\prod_{\xi<\beta}(\alpha+\xi)$, *if* $\alpha \neq 0$.

Proof. 3.1.

THEOREM 3.4. *If* $\alpha = \bigcup \alpha \neq 0$ *and* $0 < n < \omega$ *then*

$$\Gamma(\alpha+n) = \Gamma(\alpha)\cdot[\alpha^n + \alpha^{n-1}(n-1) + \alpha^{n-2}(n-2) + \ldots + \alpha].$$

Proof. 3.3 and traditional properties of ordinal arithmetic.

THEOREM 3.5. *If* $\alpha > 0$, $\beta = \bigcup \beta \leqslant \omega^{\alpha+1}$ *and* $0 < m < \omega$ *then*

$$\prod_{\xi<\beta}(\omega^\alpha m + \xi) = \omega^{\alpha\beta}.$$

Proof. The proof is similar to the proof of 2.5—replace "$\sum$" by "$\prod$" and use Theorems 9.1.7 and 9.1.8 of [3].

THEOREM 3.6. *If* $\alpha > 0$, $\beta = \bigcup \beta \leqslant \omega^{\alpha+1}$, *and* $0 < m < \omega$ *then*

$$\Gamma(\omega^\alpha m + \beta) = \Gamma(\omega^\alpha m)\cdot\omega^{\alpha\beta}.$$

Proof. 3.3 and 3.5.

THEOREM 3.7. *If* $\alpha > 0$ *and* $0 < m < \omega$ *then*

$$\Gamma(\omega^\alpha m) = \Gamma(\omega^\alpha)\,\omega^{\alpha\cdot\omega^\alpha(m-1)}.$$

Proof. $\Gamma(\omega^\alpha m) = \Gamma\big(\omega^\alpha + \omega^\alpha(m-1)\big) = \Gamma(\omega^\alpha)\cdot\omega^{\alpha\cdot\omega^\alpha(m-1)}$ [3.6].

THEOREM 3.8. $\Gamma(\omega) = \omega$.

Proof. $\Gamma(\omega) = \prod_{0<n<\omega} n = \bigcup_{m\in\omega}\prod_{0<n<m} n = \omega$.

THEOREM 3.9. *If* $\alpha > 0$ *then* $\Gamma(\omega^{\alpha+1}) = \omega^{\alpha\cdot\omega^{\alpha+1}}$.

Proof.

$$\begin{aligned} \Gamma(\omega^{\alpha+1}) &= \Gamma(\omega^\alpha + \omega^{\alpha+1}) \\ &= \Gamma(\omega^\alpha)\cdot\omega^{\alpha\cdot\omega^{\alpha+1}} && [3.6] \\ &= \omega^{\alpha\cdot\omega^{\alpha+1}}. && [3.2] \end{aligned}$$

COROLLARY 3.10. *If* $1 < n < \omega$ *then* $\Gamma(\omega^n) = \omega^{\omega^n}$.

COROLLARY 3.11. $\Gamma(\omega^\omega) = \omega^{\omega^\omega}$.

Proof. 3.10 and the continuity of Γ.

THEOREM 3.12. *If* $\gamma > 0$ *and* $\delta = \omega^{\omega^\beta(\gamma+1)}$ *then* $\Gamma(\delta) = \delta^\delta$.

Proof. The proof is similar to the proof of 2.9.

Now, the only case that remains is the value of $\Gamma(\omega^{\omega^\beta})$ for $\beta > 1$.

THEOREM 3.13. *If* $\beta > 0$ *then* $\Gamma(\omega^{\omega^\beta}) = \omega^{\omega^{\omega^\beta}}$.

Proof. The proof is by transfinite induction on β. If $\beta = 1$ the theorem follows from 3.11. If β is a limit ordinal the theorem follows from the continuity of Γ. Finally, if $\beta = \gamma+1$ then

$$\begin{aligned} \Gamma(\omega^{\omega^{\gamma+1}}) &= \Gamma(\omega^{\omega^\gamma\cdot\omega}) \\ &= \bigcup_{n<w}\Gamma(\omega^{\omega^\gamma\cdot n}) \\ &= \bigcup_{n<\omega}\omega^{\omega^\gamma\cdot\omega^{\omega^\gamma\cdot n}} && [3.12] \\ &= \omega^{\omega^\gamma\cdot\omega^{\omega^\gamma\cdot\omega}} \\ &= \omega^{\omega^{\omega^{\gamma+1}}}. \end{aligned}$$

THEOREM 3.14. $\{\alpha\colon \Gamma(\alpha) = \alpha\} = M(O_2) \cup \{1\}$.

Thus, the fixed points of Γ larger than 1 are the main numbers of exponentiation—ω and the epsilon numbers.

The function Ξ_2 is expressible in terms of the Γ function.

THEOREM 3.15. *If* $\alpha > 2$ *then* $\Xi_2(\alpha) = 2^{\frac{1}{2}\Gamma(\alpha)}$.

Proof. 1.16 and 1.3.

Therefore, Ξ_2 and Γ have the same fixed points.

THEOREM 3.16. $\{\alpha\colon \Xi_2(\alpha) = \alpha\} = M(O_2) \cup \{1\}$.

The function Ξ_3 behaves like the functions $\Xi_{2\gamma+1}$ with $\gamma > 1$, but some of the theorems used in the next section to derive the results for $\Xi_{2\gamma+1}$ with $\gamma > 1$ do not hold when $\gamma = 1$. Thus we treat Ξ_3 as a special case here.

THEOREM 3.17. $\Xi_3\big(\omega(1+\delta)\big) = \mu(\delta, O_2)$.

Proof. The theorem is true if $\delta = 0$ since $\Xi_3(\omega) = \omega = \mu(0, O_2)$. Suppose $\delta > 0$. We shall show first that $\Xi_3\big(\omega(1+\delta)\big) \in M(O_2)$. Suppose $\alpha, \beta < \Xi_3\big(\omega(1+\delta)\big)$. Then by the continuity of Ξ_3 there is an η such that $\eta = \bigcup \eta \neq 0$, $\eta < \omega(1+\delta)$ and an $n < \omega$ such that $\alpha, \beta < \Xi_3(\eta+n)$. Then,

$$\begin{aligned} \alpha O_2 \beta &= \alpha^\beta && [1.3\ (\text{ii})] \\ &< \Xi_3(\eta+n)^{\Xi_3(\eta+n)} \\ &< \Xi_3(\eta+n+1) && [1.3\ (\text{iii}),\ 1.16\ (\text{iii})] \\ &< \Xi_3\big(\omega(1+\delta)\big). && [1.18] \end{aligned}$$

Therefore, by 1.9 (i) it follows that $\Xi_3\big(\omega(1+\delta)\big) \in M(O_2)$.

We shall prove next that

(1) $$\Xi_3\big(\omega(1+\delta)\big) \leqslant \mu(\delta, O_2).$$

Then, since $\Xi_3(\omega) = \mu(0, O_2)$, $\Xi_3\big(\omega(1+\delta)\big) \in M(O_2)$, and Ξ_3 is an increasing function, it follows that $\Xi_3\big(\omega(1+\delta)\big) = \mu(\delta, O_2)$.

The proof of (1) is by transfinite induction. We have shown above that (1) is true if $\delta = 0$. Suppose it is true for all $\delta' < \delta$. If $\delta = \bigcup \delta \neq 0$ then (1) follows from the continuity of Ξ_3 and μ, 1.16 (iv) and 1.11 (ii). Suppose $\delta = \delta'+1$. Let $\eta = \omega(1+\delta')$. Then by the induction hypothesis

$$\Xi_3(\eta) < \mu(\delta, O_2)\,.$$

Suppose, for $n \in \omega$, $\Xi_3(\eta+n) < \mu(\delta, O_2)$. Then

$$\begin{aligned}\Xi_3(\eta+n+1) &= \Xi_3(\eta+n)\,O_3(\eta+n) && [1.16\ (\mathrm{iii})]\\ &= \Xi_3(\eta+n)^{\Xi_3(\eta+n)(\eta+n)}. && [1.3\ (\mathrm{iii})]\end{aligned}$$

The elements of $M(O_2)$ are ω and the epsilon numbers. It follows from the definition of an epsilon number (see for example [3], pp. 242–246) that $\Xi_3(\eta+n+1) < \mu(\delta, O_2)$. Thus, $\Xi_3(\eta+n) < \mu(\delta, O_2)$ for all $n \in \omega$. Since $\Xi_3(\omega(1+\delta)) = \bigcup_{n<\omega} \Xi_3(\eta+n)$ by 1.16 (iv), it follows that $\Xi_3(\omega(1+\delta)) \leqslant \mu(\delta, O_2)$.

Next, we shall show that the fixed points of Ξ_3 are $M(O_4) \cup \{1\}$.

THEOREM 3.18. $\{\alpha\colon \Xi_3(\alpha) = \alpha\} = M(O_4) \cup \{1\}$.

Proof. By 1.16 (ii), (iii) and 1.2 (ii) we obtain $\Xi_3(0) = \Xi_3(1) = \Xi_3(2) = 1$ and $\Xi_3(3) = 2$. Thus it follows from 1.20 that if $\Xi_3(\alpha) = \alpha$ then either $\alpha = 1$ or $\alpha = \bigcup \alpha \neq 0$. If α is a limit ordinal then there is a δ such that $\alpha = \omega(1+\delta)$. Suppose, that $\alpha = \omega(1+\delta) = \Xi_3(\alpha)$. Then

$$\begin{aligned}\alpha &= \mu(\delta, O_2) && [3.17]\\ &= 2\,O_4\,\omega(1+\delta) && [1.12]\\ &= 2\,O_4\,\alpha\,.\end{aligned}$$

Therefore, it follows from 1.10 (ii) that $\alpha \in M(O_4)$. The argument is reversible. Therefore the theorem follows.

§ 4. The function Ξ_γ, $\gamma > 3$.

We consider first the case that γ is a limit ordinal. We shall show that in this case, except for the first few values of α, $\Xi_\gamma(\alpha) \in M_\gamma$. (See 1.9 (iv)).

THEOREM 4.1. *If $\gamma = \bigcup \gamma \neq 0$ and $\alpha > 2$ then*

$$\Xi_\gamma(1+\alpha) = \mu_\gamma\big(T(\alpha)-3\big)\,.$$

Proof. The proof is by transfinite induction on α. If $\alpha = 3$, $\Xi_\gamma(1+\alpha) = 2\,O_\gamma 3 = \mu_\gamma(0) = \mu_\gamma\big(T(3)-3\big) =$ the smallest element of M_γ (1.9 (v), 1.13 and 1.14). Suppose the theorem is true for all $\beta < \alpha$. If α is a limit ordinal then the theorem follows from the continuity of Ξ_γ, T, and μ_γ (1.16 (iv) and 1.11 (iii)).

Suppose $\alpha = \beta+1$ and $\beta > 2$.

$$\begin{aligned}\Xi_\gamma(1+\beta+1) &= \Xi_\gamma(1+\beta)\,O_\gamma(1+\beta) && [1.16\ (\mathrm{iii})]\\ &= \mu_\gamma\big(T(\beta)-3\big)\,O_\gamma(1+\beta) && [\text{Ind. Hyp.}]\\ &= \mu_\gamma\big(\big(T(\beta)-3\big)+\beta\big) && [1.13,\ 1.14]\\ &= \mu_\gamma\big(\big(T(\beta)+\beta\big)-3\big)\\ &= \mu_\gamma\big(T(\beta+1)-3\big)\,.\end{aligned}$$

Thus, we see that if $\gamma = \bigcup \gamma \neq 0$, and $3 < \alpha < \omega$ then $\Xi_\gamma(\alpha)$ is the $\big(T(\alpha-1)-3\big)$th element of M_γ and if $\alpha \geqslant \omega$ then $\Xi_\gamma(\alpha)$ is the $T(\alpha)$th element of M_γ.

THEOREM 4.2. $\Xi_{2\gamma+2}(\omega) = \mu(0, O_{2\gamma})$.

Proof. If $\gamma = 0$, $\Xi_{2\gamma+2}(\omega) = \omega = \mu(0, O_{2\gamma})$ (3.15 and 3.8). Suppose $\gamma > 0$, then by 1.12 and 1.11 (i)

$$\mu(0, O_{2\gamma}) = 2\,O_{2\gamma+2}\,\omega = \bigcup_{n<\omega} 2\,O_{2\gamma+2}\,n\,.$$

But, by 1.16

$$\Xi_{2\gamma+2}(\omega) = \bigcup_{n<\omega} \Xi_{2\gamma+2}(n+1) = \bigcup_{n<\omega} \Xi_{2\gamma+2}(n)\,O_{2\gamma+2}\,n\,.$$

Thus, it follows from the monotonicity law 1.4 (iii) that $\mu(0, O_{2\gamma}) \leqslant \Xi_{2\gamma+2}(\omega)$.

Conversely, $\Xi_{2\gamma+2}(0) = \Xi_{2\gamma+2}(1) = 1 < \mu(0, O_{2\gamma})$.

Suppose $\Xi_{2\gamma+2}(n) < \mu(0, O_{2\gamma})$ then $\Xi_{2\gamma+2}(n+1) = \Xi_{2\gamma+2}(n)\,O_{2\gamma+2}\,n < \mu(0, O_{2\gamma})$ by 1.12. Consequently, $\Xi_{2\gamma+2}(n) < \mu(0, O_{2\gamma})$ for all $n \in \omega$, so $\Xi_{2\gamma+2}(\omega) = \bigcup_{n<\omega} \Xi_{2\gamma+2}(n) \leqslant \mu(0, O_{2\gamma})$.

THEOREM 4.3. *If $\gamma \geqslant 1$ and $\mu(\lambda, O_{2\gamma})$ is the largest element of $M(O_{2\gamma})$ which does not exceed $\Xi_{2\gamma+2}(\alpha)$, $\omega \leqslant \alpha = \omega\cdot\varepsilon+n$, and $\beta = \omega\cdot\delta+m$, $n, m \in \omega$, then*

$$\begin{aligned}\mu\big(\lambda+\sum_{\xi<\delta}(\varepsilon+\xi)\,\omega+(\varepsilon+\delta)\,m,\ O_{2\gamma}\big) &\leqslant \Xi_{2\gamma+2}(\alpha+\beta)\\ &\leqslant \mu\big(\lambda+1+\sum_{\xi<\delta}(1+\varepsilon+\xi)\,\omega+(1+\varepsilon+\delta)\,m,\ O_{2\gamma}\big)\,.\end{aligned}$$

Proof. The proof is by transfinite induction on β. The theorem is true if $\beta = 0$. Suppose the theorem is true for all $\beta' < \beta$. If β is a limit ordinal then the theorem follows from the continuity of μ and $\Xi_{2\gamma+2}$.

Suppose $\beta = \beta'+1$ where $\beta' = \omega\cdot\delta+(m-1)$. Then by 1.16 (iii)

$$(1)\qquad \Xi_{2\gamma+2}(\alpha+\beta) = \Xi_{2\gamma+2}(\alpha+\beta')\,O_{2\gamma+2}(\alpha+\beta')\,.$$

By the induction hypothesis,

$$(2)\qquad \mu_0 = \mu\big(\lambda+\sum_{\xi<\delta}(\varepsilon+\xi)\,\omega+(\varepsilon+\delta)(m-1),\ O_{2\gamma}\big) \leqslant \Xi_{2\gamma+2}(\alpha+\beta')$$

and

(3) $\Xi_{2\gamma+2}(\alpha+\beta') \leqslant \mu\big(\lambda+1+\sum_{\xi<\delta}(1+\varepsilon+\xi)\,\omega+(1+\varepsilon+\delta)(m-1),\ O_{2\gamma}\big) = \mu_1 .$

It follows from 1.12 and 1.4 that for every $\eta > 0$, σ, and $k \in \omega$

(4) $$\mu(\sigma+\eta,\ O_{2\gamma}) \leqslant \mu(\sigma,\ O_{2\gamma})\, O_{2\gamma+2}(\omega\eta+k)$$

and

(5) $$\mu(\sigma,\ O_{2\gamma})\, O_{2\gamma+2}(\omega\eta+k) \leqslant \mu(\sigma+1+\eta,\ O_{2\gamma}) .$$

Now, $\alpha = \omega\cdot\varepsilon+n$ where $\varepsilon > 0$ and $\beta' = \omega\cdot\delta+(m-1)$. Consequently, using (1), (2), and (4) (in (4) take $\eta = \varepsilon+\delta$ and $\sigma = \lambda+\sum_{\xi<\delta}(\varepsilon+\xi)\omega+ +(\varepsilon+\delta)(m-1)$), we obtain

(6) $$\mu\big(\lambda+\sum_{\xi<\delta}(\varepsilon+\xi)\,\omega+(\varepsilon+\delta)\,m,\ O_{2\gamma}\big) \leqslant \mu_0\, O_{2\gamma+2}(\alpha+\beta') \leqslant \Xi_{2\gamma+2}(\alpha+\beta) .$$

Similarly, using (1), (3), and (5) (in (5) take $\eta = 1+\varepsilon+\delta$ and $\sigma = \lambda+ +1+\sum_{\xi<\delta}(1+\varepsilon+\xi)\,\omega+(1+\varepsilon+\delta)(m-1)$), we obtain

(7) $$\begin{aligned}\Xi_{2\gamma+2}(\alpha+\beta) &\leqslant \mu_1\, O_{2\gamma+2}(\alpha+\beta') \\ &\leqslant \mu\big(\lambda+1+\sum_{\xi<\delta}(1+\varepsilon+\xi)\,\omega+(1+\varepsilon+\delta)\,m,\ O_{2\gamma}\big) .\end{aligned}$$

The theorem follows from (6) and (7).

In the case that β is a limit ordinal Theorem 4.3 has a simple form.

COROLLARY 4.4. *If* $\gamma \geqslant 1$ *then*

$$\Xi_{2\gamma+2}\big(\omega(1+\delta)\big) = \mu\big(\sum_{\xi<\delta}(1+\xi)\,\omega,\ O_{2\gamma}\big) .$$

Proof. If $\delta = 0$, the corollary follows from 4.2. If $\delta > 0$, in 4.3 let $\varepsilon = 1$, and $m = n = 0$, then the corollary follows from 4.3 using traditional properties of ordinal arithmetic.

To make 4.4 more meaningful we shall evaluate $\sum_{\xi<\delta}(1+\xi)\,\omega$.

THEOREM 4.5. *If* $\delta < \omega$ *then* $\sum_{\xi<\delta}(1+\xi)\,\omega = \omega\cdot\delta$.

Proof. The proof is by induction on δ.

THEOREM 4.6. $\sum_{\xi<\omega}(1+\xi)\,\omega = \omega^2$.

Proof. 4.5 and continuity.

THEOREM 4.7. *Suppose* $\delta = \omega^{\delta_0}d_0+\ldots+\omega^{\delta_n}d_n$ *is the normal form of* δ. (1.21)

(i) *If* $\delta_0 = \bigcup\delta_0 \neq 0$ *then* $\sum_{\xi<\delta}(1+\xi)\,\omega = \omega^{\delta_0}+\omega^{\delta_0+1}(\delta-\omega^{\delta_0})$.

(ii) *If* $\delta_0 \neq \bigcup\delta_0$ *then* $\sum_{\xi<\delta}(1+\xi)\,\omega = \omega^{\delta_0\cdot 2}+\omega^{\delta_0+1}(\delta-\omega^{\delta_0})$.

Proof. In both cases, since $\delta_0 > 0$, $\delta \geqslant \omega$ so

$$\begin{aligned}\sum_{\xi<\delta}(1+\xi)\,\omega &= \sum_{\xi<\delta}\xi\omega \\ &= \sum_{\xi<\omega^{\delta_0}}\xi\omega+\sum_{\omega^{\delta_0}\leqslant\xi<\delta}\xi\omega .\end{aligned}$$

Let us look at the second sum first. It follows from [3], Theorem 9.1.6 (b) that for all ξ such that $\omega^{\delta_0} \leqslant \xi < \delta$, $\xi\cdot\omega = \omega^{\delta_0+1}$. Thus,

$$\sum_{\omega^{\delta_0}\leqslant\xi<\delta}\xi\omega = \omega^{\delta_0+1}(\delta-\omega^{\delta_0}) .$$

It follows by an easy transfinite induction on δ_0, that the first sum,

$$\sum_{\xi<\omega^{\delta_0}}\xi\cdot\omega = \sum_{\xi<\delta_0}(\omega^{\xi+1}\cdot\omega^{\xi+1}) .$$

If $\delta_0 = \omega$,

$$\sum_{\xi<\delta_0}(\omega^{\xi+1}\cdot\omega^{\xi+1}) = \sum_{\xi<\delta_0}\omega^{\xi\cdot 2+2} = \omega^{\delta_0}.$$

If $\delta_0 = \bigcup\delta_0 > \omega$,

$$\begin{aligned}\sum_{\xi<\delta_0}(\omega^{\xi+1}\cdot\omega^{\xi+1}) &= \sum_{\xi<\omega}(\omega^{\xi+1}\cdot\omega^{\xi+1})+\sum_{\omega\leqslant\xi<\delta_0}(\omega^{\xi+1}\cdot\omega^{\xi+1}) \\ &= \omega^{\omega}+\sum_{\omega\leqslant\xi<\delta_0}\omega^{\xi\cdot 2+1} \\ &= \omega^{\omega}+\omega^{\delta_0} = \omega^{\delta_0}.\end{aligned}$$

Suppose $\delta_0 = \varepsilon+n$ where $\varepsilon = \bigcup\varepsilon$ and $0 < n < \omega$. If $\varepsilon = 0$, then

$$\begin{aligned}\sum_{\xi<\delta_0}(\omega^{\xi+1}\cdot\omega^{\xi+1}) &= \sum_{\xi<n}\omega^{\xi\cdot 2+2} \\ &= \omega^{(n-1)2+2} = \omega^{n\cdot 2} = \omega^{\delta_0\cdot 2}.\end{aligned}$$

Finally, if $\varepsilon = \bigcup\varepsilon \neq 0$ then

$$\begin{aligned}\sum_{\xi<\delta_0}(\omega^{\xi+1}\cdot\omega^{\xi+1}) &= \sum_{\xi<\varepsilon}(\omega^{\xi+1}\cdot\omega^{\xi+1})+\sum_{\varepsilon\leqslant\xi<\delta_0}(\omega^{\xi+1}\cdot\omega^{\xi+1}) \\ &= \omega^{\varepsilon}+\sum_{\varepsilon\leqslant\xi<\delta_0}\omega^{\xi\cdot 2+1} \\ &= \omega^{\varepsilon}+\omega^{(\varepsilon+n-1)2+1} \\ &= \omega^{\varepsilon\cdot 2+n} = \omega^{\delta_0\cdot 2}.\end{aligned}$$

Now to complete our discussion of the accumulation functions we consider the function $\Xi_{2\gamma+1}$.

THEOREM 4.8. $\Xi_{2\gamma+1}(\omega) = \mu(0, O_{2\gamma})$.

Proof. The theorem is true if γ is finite, for in this case $\Xi_{2\gamma+1}(\omega) = \omega = \mu(0, O_{2\gamma})$.

Suppose $\gamma \geqslant \omega$. Then it follows from 1.6 (i) that for $3 < n < \omega$, $\Xi_{2\gamma+1}(n)$ is a limit ordinal. Suppose $\alpha, \beta < \Xi_{2\gamma+1}(\omega)$. Then, by the continuity of $\Xi_{2\gamma+1}$, there is an n, $3 < n < \omega$, such that $\alpha, \beta < \Xi_{2\gamma+1}(n)$. Therefore,

$$\begin{aligned} \alpha O_{2\gamma}\beta &< \Xi_{2\gamma+1}(n) O_{2\gamma}\big(\Xi_{2\gamma+1}(n)\big)(n-1) \\ &= \Xi_{2\gamma+1}(n) O_{2\gamma+1} n && [1.8\ \text{(ii)}] \\ &= \Xi_{2\gamma+1}(n+1) && [1.16\ \text{(iii)}] \\ &< \Xi_{2\gamma+1}(\omega)\,. \end{aligned}$$

Thus, it follows from 1.9 (i), that $\Xi_{2\gamma+1}(\omega) \in M(O_{2\gamma})$.

Then, by a proof similar to the second part of the proof of 4.2, we obtain that for each $n \in \omega$, $\Xi_{2\gamma+1}(n) < \mu(0, O_{2\gamma})$. Consequently, by continuity, $\Xi_{2\gamma+1}(\omega) \leqslant \mu(0, O_{2\gamma})$. But we proved $\Xi_{2\gamma+1}(\omega) \in M(O_{2\gamma})$, so we must have $\Xi_{2\gamma+1}(\omega) = \mu(0, O_{2\gamma})$.

The proof of the next theorem is similar to the proof of 3.17.

THEOREM 4.9. *If* $\gamma > 1$ *then*, $\Xi_{2\gamma+1}\big(\omega(1+\delta)\big) = \mu(\delta, O_{2\gamma})$.

Proof. The theorem is true if $\delta = 0$ by 4.8. Suppose $\delta > 0$. Using a proof similar to the proof of the first part of 4.8 it can be shown that

(1) $$\Xi_{2\gamma+1}\big(\omega(1+\delta)\big) \in M(O_{2\gamma})\,.$$

Next, we shall prove by transfinite induction that for all δ

(2) $$\Xi_{2\gamma+1}\big(\omega(1+\delta)\big) \leqslant \mu(\delta, O_{2\gamma})\,.$$

The theorem follows from (1) and (2) because $\Xi_{2\gamma+1}\big(\omega(1+\delta)\big)$ is a strictly increasing function of δ. (1.18)

If $\delta = 0$, then the conjecture (2) holds by 4.8. Suppose (2) holds for all $\delta' < \delta$. If δ is a limit ordinal then (2) follows from the continuity of $\Xi_{2\gamma+1}$ and μ.

Suppose $\delta = \delta'+1$. Then

(3) $$\Xi_{2\gamma+1}\big(\omega(1+\delta)\big) = \Xi_{2\gamma+1}\big(\omega(1+\delta')+\omega\big) = \bigcup_{n<\omega} \Xi_{2\gamma+1}\big(\omega(1+\delta')+n\big)\,.$$

By the induction hypothesis,

(4) $$\Xi_{2\gamma+1}\big(\omega(1+\delta')\big) \leqslant \mu(\delta', O_{2\gamma}) < \mu(\delta, O_{2\gamma})\,.$$

It follows from 1.16 (ii) that for $n \in \omega$,

(5) $$\Xi_{2\gamma+1}\big(\omega(1+\delta')+n+1\big) = \Xi_{2\gamma+1}\big(\omega(1+\delta')+n\big) O_{2\gamma+1}\big(\omega(1+\delta')+n\big)\,.$$

Since $M(O_{2\gamma}) = M(O_{2\gamma+1})$ (1.14 (i)) and $\omega(1+\delta')+n \leqslant \Xi_{2\gamma+1}\big(\omega(1+\delta')+n\big)$ (1.18), it follows from (5) and 1.9 (i) that if $\Xi_{2\gamma+1}\big(\omega(1+\delta')+n\big) < \mu(\delta, O_{2\gamma})$ then so is $\Xi_{2\gamma+1}\big(\omega(1+\delta')+n+1\big)$. Thus, since (4) holds, it follows by mathematical induction that $\Xi_{2\gamma+1}\big(\omega(1+\delta')+n\big) < \mu(\delta, O_{2\gamma})$ for all $n \in \omega$. Therefore, it follows from (3) that $\Xi_{2\gamma+1}\big(\omega(1+\delta)\big) \leqslant \mu(\delta, O_{2\gamma})$.

Thus, it follows from 4.9, that if $\gamma > 1$, the function $\psi_\gamma(\delta) = \Xi_{2\gamma+1}\big(\omega(1+\delta)\big)$ enumerates the elements of $M(O_{2\gamma})$.

We conclude by discussing the fixed points of Ξ_γ with $\gamma > 3$. First, we prove a preliminary lemma.

LEMMA 4.10.

(i) *If* $\alpha > 1$, $\gamma = \bigcup \gamma \neq 0$, *and* $\alpha = \Xi_\gamma(\alpha)$ *then* $\alpha = T(\alpha)$.

(ii) *If* $\alpha > \omega$, $\gamma > 0$, *and* $\alpha = \Xi_{2\gamma+2}(\alpha)$ *then* $\sum_{\xi<\alpha}(1+\xi)\omega = \alpha$.

Proof. Part (i) follows from 1.20, 4.1 and 2.9 and part (ii) follows from 1.20, 4.4 and 4.7.

THEOREM 4.11. *If* $\gamma = \bigcup \gamma \neq 0$ *then*

$$\{\alpha\colon \Xi_\gamma(\alpha) = \alpha\} = M(O_\gamma) \cup \{1\}\,.$$

Proof. If $\gamma = \bigcup \gamma \neq 0$ and $\Xi_\gamma(\alpha) = \alpha$ then it follows from 1.16 and 1.20 that either $\alpha = 1$ or α is a limit ordinal. Suppose then that α is infinite and $\Xi_\gamma(\alpha) = \alpha$. By 4.1,

$$\begin{aligned} \alpha &= \mu_\gamma\big(T(\alpha)\big) \\ &= \mu_\gamma(\alpha) && [4.10\ \text{(i)}] \\ &= 3\,O_\gamma(2+\alpha) && [1.13] \\ &= 3\,O_\gamma\alpha\,. \end{aligned}$$

Therefore, it follows from 1.10 (ii) that $\alpha \in M(O_\gamma)$. The argument is reversible.

THEOREM 4.12. *If* $\gamma > 0$ *then*

$$\{\alpha\colon \Xi_{2\gamma+2}(\alpha) = \alpha\} = M(O_{2\gamma+2}) \cup \{1\}\,.$$

Proof. The proof is similar to the proof of 4.11. Use 4.4, 4.10 (ii) and 1.12 instead of 4.1, 4.10 (i) and 1.13 respectively.

THEOREM 4.13. *If* $\gamma > 1$ *then*

$$\{\alpha\colon \Xi_{2\gamma+1}(\alpha) = \alpha\} = M(O_{2\gamma+2}) \cup \{1\}\,.$$

Proof. The proof is similar to the proof of 4.11 with 4.1 and 1.13 replaced by 4.9 and 1.12. (See also the proof of 3.18.)

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Reçu par la Rédaction le 27. 1. 1970