On prime binary relational structures

by

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Introduction. The operation of cardinal multiplication, defined on a class $\mathcal{K}$ of relational structures, can be thought of as a generalization of the usual multiplication of positive integers. In this generalization, the role of the *prime numbers* is usually played by the $\mathcal{K}$-*indecomposable structures*; i.e., the definition "p is *prime* if and only if $p \neq 1$ and either $q = 1$ or $r = 1$ whenever $p = qr$" is generalized to the definition "$\mathfrak{X}$ is $\mathcal{K}$-*indecomposable* if and only if $\mathfrak{X} \in \mathcal{K}$, $\mathfrak{X}$ has more than one element, and either $\mathfrak{Y}$ or $\mathfrak{Z}$ has only one element whenever $\mathfrak{Y}, \mathfrak{Z} \in \mathcal{K}$ and $\mathfrak{X} \cong \mathfrak{Y} \times \mathfrak{Z}$". However, it is also true that a positive integer p is prime if and only if $p \neq 1$ and either $p|q$ or $p|r$ whenever $p|qr$; here $|$ is the divisibility relation. Generalizing this, we can make the following definition, where $\mathfrak{X}|\mathfrak{Y}$ means that $\mathfrak{X} \times \mathfrak{W} \cong \mathfrak{Y}$ for some structure $\mathfrak{W}$:

A structure $\mathfrak{X}$ is $\mathcal{K}$-*prime* if and only if

(i) $\mathfrak{X} \in \mathcal{K}$;

(ii) $\mathfrak{X}$ has more than one element; and

(iii) if $\mathfrak{Y}, \mathfrak{Z} \in \mathcal{K}$ and $\mathfrak{X}|(\mathfrak{Y} \times \mathfrak{Z})$, then either $\mathfrak{X}|\mathfrak{Y}$ or $\mathfrak{X}|\mathfrak{Z}$.

The general question of comparing the notions of $\mathcal{K}$-indecomposability and $\mathcal{K}$-primeness was first posed by Tarski. Although it is easy to show that the two notions are not generally identical, one might think that they do not differ by very much. However, Ralph McKenzie showed (see [4]) that if $\langle n_1, \ldots, n_k \rangle$ is any sequence of natural numbers, not all zero, and $\mathcal{K}$ is the class of all *algebras* $\langle X, f_1, \ldots, f_k \rangle$, in which f_i is an n_i-ary operation on X, then there are *no $\mathcal{K}$-prime structures at all*. This result may be somewhat surprising, but it is really what should be expected. For, given $\mathfrak{X} \in \mathcal{K}$, $\mathfrak{X}$ having more than one element, consider how we would show that $\mathfrak{X}$ is *not* $\mathcal{K}$-indecomposable. (For simplicity, assume that isomorphs of members of $\mathcal{K}$ are necessarily members of $\mathcal{K}$.) To show that $\mathfrak{X}$ is not $\mathcal{K}$-indecomposable, we must find non-trivial $\mathfrak{Y}, \mathfrak{Z} \in \mathcal{K}$ such that $\mathfrak{X} \cong \mathfrak{Y} \times \mathfrak{Z}$. If such $\mathfrak{Y}$ and $\mathfrak{Z}$ exist, they can be found within the set of cardinal factors of $\mathfrak{X}$; if $\mathfrak{X}$ is small, then this set will be small and the search is very likely to fail. On the other hand, to show that $\mathfrak{X}$ is not $\mathcal{K}$-prime, we must find $\mathfrak{Y}, \mathfrak{Z} \in \mathcal{K}$ such that $\mathfrak{X}|(\mathfrak{Y}\times\mathfrak{Z})$ but neither $\mathfrak{X}|\mathfrak{Y}$ nor $\mathfrak{X}|\mathfrak{Z}$. The search for such $\mathfrak{Y}$ and $\mathfrak{Z}$ will not fail unless it fails within the intersection of $\mathcal{K}$ with the class of cardinal factors of *cardinal multiples* of $\mathfrak{X}$; if $\mathcal{K}$ is large, this class will be large even if $\mathfrak{X}$ is small, and the search is very likely to succeed. Thus we may reasonably expect that for very general classes $\mathcal{K}$, such as the classes of all algebras just mentioned, there will be considerably fewer $\mathcal{K}$-prime structures than $\mathcal{K}$-indecomposable ones. For highly restricted classes $\mathcal{K}$, however, the two notions may be in closer agreement.

In this paper, we characterize the $\mathcal{K}$-prime structures for various classes $\mathcal{K}$ of *binary relational structures*, i.e. pairs $\langle X, R\rangle$ in which X is a non-empty set and $R \subseteq X\times X$. The *cardinal product* of two such structures $\langle X, R\rangle$ and $\langle Y, S\rangle$ is the structure $\langle X\times Y, R\otimes S\rangle$, in which $R\otimes S$ is the relation such that $\langle x, y\rangle(R\otimes S)\langle w, z\rangle$ if and only if both xRw and ySz. It will be seen that the class $\mathcal{B}$ of *all* binary relational structures has *no* primes, but that the class $\mathcal{R}\subseteq\mathcal{B}$ of structures with *reflexive* relations does have primes, which we shall characterize. This raises the question of where the "borderline" lies; i.e. as we define classes $\mathcal{K}\subseteq\mathcal{B}$ by increasingly stronger natural definitions, at what stage between $\mathcal{B}$ and $\mathcal{R}$ do primes appear? This question is answered in § 1. We then turn to classes containing only *finite* structures. It will be seen that the property of *connectedness* is of some importance in the study of these problems. (A structure is *connected* if it is not the cardinal sum of two other structures.) As a preliminary, we shall obtain, for various $\mathcal{K}\subseteq\mathcal{B}$, a result which has as a corollary an interesting condition on two connected members of $\mathcal{K}$ which is necessary and sufficient for their cardinal product to be connected. As another corollary, we shall get a strong *necessary* condition for primeness in several classes of finite structures, which we shall use to show that there are *no* primes in the class of all finite *unary algebras* (i.e. structures $\langle X, R\rangle$ in which X is finite and R is a unary operation on X). This result is of interest because R. McKenzie showed in [4] that for $n > 1$ there *are* primes in the class of all algebras $\langle X, f\rangle$ in which X is finite and f is an n-ary operation on X.

In the class of finite, connected binary relational structures with *reflexive* relations, the prime structures are the same as the indecomposable structures. A proof of this fact will not be given here, as it follows from results in a forthcoming paper by R. McKenzie. In the present paper, however, we shall make use of the just-stated fact to characterize the primes in the class of *all* finite binary relational structures with reflexive relations.

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We shall use the following notation. If $R\subseteq X\times X$ and $Z\subseteq X$, then define R^*Z to be $\{x\in X\colon (\exists y\in Z)(yRx)\}$; also, define $\breve{R}$ to be the *inverse* of R, i.e. $x\breve{R}y$ if and only if yRx. Thus, if $R\subseteq X\times X$, the *domain* of R is $\breve{R}^*X$ and the *range* of R is R^*X. If also $S\subseteq X\times X$, then $R|S$ will denote the *relative product* of R and S, i.e. $x(R|S)y$ if and only if there is some $z\in X$ such that xRz and zRy. If X is a set, let $\mathrm{Id}_X = \{\langle x, x\rangle\colon x\in X\}$. If $R\subseteq X\times X$, let $R^{(0)} = \mathrm{Id}_X$ and let $R^{(n+1)} = R^{(n)}|R$ for each natural number n. If $\mathcal{K}\subseteq\mathcal{B}$, then $\mathrm{Pr}(\mathcal{K})$ will denote the class of $\mathcal{K}$-prime structures and $\mathrm{In}(\mathcal{K})$ will denote the class of $\mathcal{K}$-indecomposable structures. If $\mathfrak{X} = \langle X, R\rangle \in\mathcal{B}$, we call X the *universe* of $\mathfrak{X}$, and denote it by $|\mathfrak{X}|$.

We consider an ordinal number to be the set of all smaller ordinal numbers; cardinal numbers are conceived as initial ordinals. If $\varkappa, \lambda$ are cardinals, then $\varkappa\times\lambda$ will denote their Cartesian product and $\varkappa\cdot\lambda$ will denote their cardinal product (i.e. the cardinality of $\varkappa\times\lambda$). If $\varkappa$ is a cardinal number, then $\varkappa^+$ will denote the next larger cardinal number. The cardinal number of a set X will be denoted by $\#(X)$. We say $\varkappa|\lambda$ if and only if $\varkappa\cdot\mu = \lambda$ for some cardinal μ; of course, if λ is infinite we have $\varkappa|\lambda$ if and only if $\varkappa\leqslant\lambda$.

§ 1. Classes containing infinite structures. We consider the following classes:

$$\mathcal{B} = \{\langle X, R\rangle\colon R\subseteq X\times X\},$$
$$\mathcal{B}' = \{\langle X, R\rangle\colon \emptyset\neq R\subseteq X\times X\},$$
$$\mathcal{W} = \{\langle X, R\rangle\colon R\subseteq X\times X \text{ and } (\breve{R}^*X)\cup(R^*X) = X\},$$
$$\mathcal{D} = \{\langle X, R\rangle\colon R\subseteq X\times X \text{ and } \breve{R}^*X = X\},$$
$$\mathcal{S} = \{\langle X, R\rangle\colon R\subseteq X\times X \text{ and } \breve{R}^*X = R^*X = X\},$$
$$\mathcal{R} = \{\langle X, R\rangle\colon R\subseteq X\times X \text{ and } xRx \text{ for every } x\in X\},$$
$$\mathcal{E} = \{\langle X, R\rangle\colon R \text{ is an equivalence relation on } X\},$$
$$\mathcal{U} = \{\langle X, R\rangle\colon R = X\times X\}.$$

Obviously $\mathcal{B}\supseteq\mathcal{B}'\supseteq\mathcal{W}\supseteq\mathcal{D}\supseteq\mathcal{S}\supseteq\mathcal{R}\supseteq\mathcal{E}\supseteq\mathcal{U}$. Note that it is not a foregone conclusion that $\mathrm{Pr}(\mathcal{B})$, $\mathrm{Pr}(\mathcal{B}')$, etc. will be ordered in the reverse order; for in general $\mathcal{K}\subseteq\mathcal{L}$ does not imply $\mathrm{Pr}(\mathcal{K})\supseteq\mathrm{Pr}(\mathcal{L})$, but only $\mathrm{Pr}(\mathcal{K})\supseteq\mathcal{K}\cap\mathrm{Pr}(\mathcal{L})$. However, we shall show that $\mathrm{Pr}(\mathcal{B}) = \mathrm{Pr}(\mathcal{B}') = \mathrm{Pr}(\mathcal{W}) = \mathrm{Pr}(\mathcal{D}) = \emptyset$ and $\mathrm{Pr}(\mathcal{S}) = \mathrm{Pr}(\mathcal{R}) = \mathrm{Pr}(\mathcal{E}) = \mathrm{Pr}(\mathcal{U}) = \mathcal{T}$, where

$$\mathcal{T} = \{\langle X, R\rangle\colon R = X\times X \text{ and } \#(X) \text{ is prime or infinite}\}.$$

We could also consider $\mathcal{D}' = \{\langle X, R\rangle\colon R\subseteq X\times X \text{ and } R^*X = X\}$; but this would yield nothing new, since obviously

$$\mathrm{Pr}(\mathcal{D}') = \{\langle X, R\rangle\colon \langle X, \breve{R}\rangle\in\mathrm{Pr}(\mathcal{D})\}.$$

For each cardinal number $\varkappa$, let $\mathfrak{U}_\varkappa$ be the structure $\langle\varkappa, \varkappa\times\varkappa\rangle$ and let $\mathfrak{I}_\varkappa$ be the structure $\langle\varkappa, \mathrm{Id}_\varkappa\rangle$.

The two extremes on our list of results are trivial.

THEOREM 1.1. $\Pr(\mathcal{B}) = \emptyset$ *and* $\Pr(\mathcal{U}) = \mathcal{T}$.

Proof. Given $\mathfrak{X} \in \mathcal{B}$ with $\#(|\mathfrak{X}|) = \varkappa > 1$, we can choose $\mathfrak{Y} \in \mathcal{B}$ such that $\#(|\mathfrak{Y}|) = \varkappa$ but *not* $\mathfrak{X}|\mathfrak{Y}$. (If $\mathfrak{X} \in \mathcal{U}$, take $\mathfrak{Y} = \mathfrak{I}_\varkappa$; if $\mathfrak{X} \notin \mathcal{U}$, take $\mathfrak{Y} = \mathfrak{U}_\varkappa$.) Letting $\mathfrak{Z} = \langle\{0\}, \emptyset\rangle$, we get $\mathfrak{X}\times\mathfrak{Z} \cong \langle\varkappa, \emptyset\rangle \cong \mathfrak{Y}\times\mathfrak{Z}$; but neither $\mathfrak{X}|\mathfrak{Y}$ nor $\mathfrak{X}|\mathfrak{Z}$. So $\mathfrak{X} \notin \Pr(\mathcal{B})$. Hence $\Pr(\mathcal{B}) = \emptyset$. Furthermore, if $\varkappa, \lambda$ are cardinals, then obviously $\mathfrak{U}_\varkappa|\mathfrak{U}_\lambda$ if and only if $\varkappa|\lambda$; since $\mathfrak{X} \in \mathcal{U}$ if and only if $\mathfrak{X} \cong \mathfrak{U}_\varkappa$ for some $\varkappa$, the result $\Pr(\mathcal{U}) = \mathcal{T}$ follows from simple cardinal arithmetic.

We proceed to the proofs of the other results. The principal tool used here is the following equivalence relation, which is also used by Chang in [1].

DEFINITION 1.2. If $\mathfrak{X} = \langle X, R\rangle \in \mathcal{B}$, we define

$$E(\mathfrak{X}) = \big\{\langle x, y\rangle\colon\ x, y \in X,\ R^*\{x\} = R^*\{y\},\ \text{and}\ \breve{R}^*\{x\} = \breve{R}^*\{y\}\big\}.$$

Obviously $E(\mathfrak{X})$ is an equivalence relation; it identifies points which cannot be told apart by just looking at their relatives. If $x \in X$, then $x/E(\mathfrak{X})$ will denote the $E(\mathfrak{X})$-equivalence class containing x; and $X/E(\mathfrak{X})$ is $\{x/E(\mathfrak{X})\colon x \in X\}$. Obviously $E(\mathfrak{X})|R|E(\mathfrak{X}) = R$, so we can define a relation $R/E(\mathfrak{X})$ on $X/E(\mathfrak{X})$ by saying $\langle x/E(\mathfrak{X}), y/E(\mathfrak{X})\rangle \in R/E(\mathfrak{X})$ if and only if xRy. Let $\mathfrak{X}/E(\mathfrak{X}) = \langle X/E(\mathfrak{X}), R/E(\mathfrak{X})\rangle$. The important facts about $E(\mathfrak{X})$ are given in the following Lemma.

LEMMA 1.3. (a) *If* $\mathfrak{X}, \mathfrak{Y} \in \mathcal{S}$, *then* $E(\mathfrak{X}\times\mathfrak{Y}) = E(\mathfrak{X})\otimes E(\mathfrak{Y})$. (b) *If* $\mathfrak{X} \in \mathcal{U}$ *and* $\mathfrak{Y} \in \mathcal{B}$, *then* $E(\mathfrak{X}\times\mathfrak{Y}) = E(\mathfrak{X})\otimes E(\mathfrak{Y})$. (c) *If* $E(\mathfrak{X}\times\mathfrak{Y}) = E(\mathfrak{X})\otimes E(\mathfrak{Y})$, *then* $|\mathfrak{X}\times\mathfrak{Y}|/E(\mathfrak{X}\times\mathfrak{Y}) = \{C\times D\colon C \in |\mathfrak{X}|/E(\mathfrak{X})$ *and* $D \in |\mathfrak{Y}|/E(\mathfrak{Y})\}$. (d) *If* $\mathfrak{X}, \mathfrak{Y} \in \mathcal{B}$ *and* $\mathfrak{X}/E(\mathfrak{X}) \cong \mathfrak{Y}/E(\mathfrak{Y})$ *under some isomorphism* φ *such that* $\#\big(\varphi(C)\big) = \#(C)$ *for every* $C \in |\mathfrak{X}|/E(\mathfrak{X})$, *then* $\mathfrak{X} \cong \mathfrak{Y}$. (e) *If* $\varkappa$ *is a cardinal and* $\mathfrak{Y} \in \mathcal{B}$, *then* $\mathfrak{U}_\varkappa|\mathfrak{Y}$ *if and only if* $\varkappa|\#(C)$ *for every* $C \in |\mathfrak{Y}|/E(\mathfrak{Y})$.

Proof. (a, b) Suppose $\mathfrak{X} = \langle X, R\rangle$, $\mathfrak{Y} = \langle Y, S\rangle$. For $\langle x, y\rangle, \langle w, z\rangle \in X\times Y$, we have $(R\otimes S)^*\{\langle x, y\rangle\} = (R^*\{x\})\times(S^*\{y\})$ and $(R\otimes S)^*\{\langle w, z\rangle\} = (R^*\{w\})\times(S^*\{z\})$, by definition of $\otimes$. If $\langle x, y\rangle E(\mathfrak{X}\times\mathfrak{Y})\langle w, z\rangle$, then the left-hand sides of these two equations are equal. If $\mathfrak{X}, \mathfrak{Y} \in \mathcal{S}$, we thus get $R^*\{x\} = R^*\{w\}$ and $S^*\{y\} = S^*\{z\}$ since all four sets are non-empty; if $\mathfrak{X} \in \mathcal{U}$ and $\mathfrak{Y} \in \mathcal{B}$, we get the same result, this time because $R^*\{x\} = X = R^*\{w\}$. Arguing similarly with $\breve{R}$ and $\breve{S}$ $[(R\otimes S)\breve{\ } = \breve{R}\otimes\breve{S}]$, we see that $xE(\mathfrak{X})w$ and $yE(\mathfrak{Y})z$. Hence $E(\mathfrak{X}\times\mathfrak{Y}) \subseteq E(\mathfrak{X})\otimes E(\mathfrak{Y})$. The converse is simpler.

(c) Straightforward.

(d) For each $C \in |\mathfrak{X}|/E(\mathfrak{X})$, choose a one-to-one correspondence between C and $\varphi(C)$; the union of these correspondences is clearly an isomorphism between $\mathfrak{X}$ and $\mathfrak{Y}$.

(e) If $\mathfrak{U}_\varkappa|\mathfrak{Y}$, i.e. $\mathfrak{U}_\varkappa\times\mathfrak{Z} \cong \mathfrak{Y}$ for some $\mathfrak{Z} \in \mathcal{B}$, and $C \in |\mathfrak{Y}|/E(\mathfrak{Y})$, then by (b, c) C is equinumerous with the Cartesian product of an $E(\mathfrak{U}_\varkappa)$-class and an $E(\mathfrak{Z})$-class. But the only $E(\mathfrak{U}_\varkappa)$-class is $\varkappa$. Therefore $\varkappa|\#(C)$. Conversely, if $|\mathfrak{Y}|/E(\mathfrak{Y}) = \{C_i\colon i \in I\}$ and $\varkappa|\#(C_i)$ for every $i \in I$, then we may assume $C_i = D_i\times\varkappa$ for some sets D_i. Let $\mathfrak{Z} = \langle Z, T\rangle$, where Z is the union of the sets D_i and aTb if and only if $\langle a, 0\rangle$ and $\langle b, 0\rangle$ are related in $\mathfrak{Y}$ (we assume $0 \in \varkappa$). Then $Z/E(\mathfrak{Z}) = \{D_i\colon i \in I\}$, and matching C_i with D_i gives $\mathfrak{Z}/E(\mathfrak{Z}) \cong \mathfrak{Y}/E(\mathfrak{Y})$. Furthermore, matching $D_i\times\varkappa$ with D_i clearly gives $\mathfrak{U}_\varkappa\times\mathfrak{Z}/E(\mathfrak{U}_\varkappa\times\mathfrak{Z}) \cong \mathfrak{Z}/E(\mathfrak{Z})$. Composing the two isomorphisms and applying (d), we see that $\mathfrak{Y} \cong \mathfrak{U}_\varkappa\times\mathfrak{Z}$. So $\mathfrak{U}_\varkappa|\mathfrak{Y}$.

We now show that $\Pr(\mathcal{B}') = \Pr(\mathcal{W}) = \Pr(\mathcal{D}) = \emptyset$ and $\Pr(\mathcal{S}) = \Pr(\mathcal{R}) = \Pr(\mathcal{E}) = \mathcal{T}$. We do this in three steps. First, we show that $\mathcal{T}$-structures are prime in $\mathcal{S}$, $\mathcal{R}$, and $\mathcal{E}$, but (secondly) *not* prime in $\mathcal{B}'$, $\mathcal{W}$, or $\mathcal{D}$. Thirdly, we show that structures *not* in $\mathcal{T}$ are *not* prime in any of $\mathcal{B}'$, $\mathcal{W}$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{R}$, or $\mathcal{E}$.

THEOREM 1.4. *If* $\mathcal{K}$ *is any of the classes* $\mathcal{S}$, $\mathcal{R}$, *or* $\mathcal{E}$, *then* $\mathcal{T} \subseteq \Pr(\mathcal{K})$.

Proof. The proof given by Chang in [1] for finite $\mathcal{R}$-structures works here. If $\mathfrak{X}, \mathfrak{Y} \in \mathcal{K}$, $\mathfrak{Z} \in \mathcal{T}$, and $\mathfrak{Z}|(\mathfrak{X}\times\mathfrak{Y})$, then we may assume $\mathfrak{Z} = \mathfrak{U}_\varkappa$ for some $\varkappa$. Since $\mathcal{K} \subseteq \mathcal{S}$, Lemma 1.3 (a, c, e) gives $\varkappa|\#(C\times D)$ for every $C \in |\mathfrak{X}|/E(\mathfrak{X})$, $D \in |\mathfrak{Y}|/E(\mathfrak{Y})$. Since $\varkappa$ is prime or infinite, if there is some $C_0 \in |\mathfrak{X}|/E(\mathfrak{X})$ such that *not* $\varkappa|\#(C_0)$, then $\varkappa|\#(C_0\times D)$ implies $\varkappa|\#(D)$ for every $D \in |\mathfrak{Y}|/E(\mathfrak{Y})$. Thus either $\mathfrak{U}_\varkappa|\mathfrak{X}$ or $\mathfrak{U}_\varkappa|\mathfrak{Y}$, by Lemma 1.3(e).

THEOREM 1.5. *If* $\mathfrak{X} \in \mathcal{T}$, *then* $\mathfrak{X} \notin \Pr(\mathcal{D})$ (*whence* $\mathfrak{X} \notin \Pr(\mathcal{W})$ *and* $\mathfrak{X} \notin \Pr(\mathcal{B}')$, *since* $\mathcal{D} \subseteq \mathcal{W} \subseteq \mathcal{B}'$).

Proof. We may assume $\mathfrak{X} = \mathfrak{U}_\varkappa$ for some $\varkappa > 1$. If $\varkappa$ is finite, let $\mu = \varkappa - 1$; otherwise, let $\mu = \varkappa$. Let

$$\mathfrak{Y} = \langle\varkappa \cup \{\varkappa\}, (\varkappa\times\varkappa) \cup \{\langle\varkappa, a\rangle\colon a \in \varkappa\}\rangle$$

and

$$\mathfrak{Z} = \langle\mu \cup \{\mu\}, \{\langle\beta, \mu\rangle\colon \beta \in \mu \cup \{\mu\}\}\rangle.$$

It is clear that $\mathfrak{Y}, \mathfrak{Z} \in \mathcal{D}$ and also that $|\mathfrak{Y}|/E(\mathfrak{Y}) = \{\varkappa, \{\varkappa\}\}$ and $|\mathfrak{Z}|/E(\mathfrak{Z}) = \{\mu, \{\mu\}\}$. Since $\varkappa > 1$, Lemma 1.3(e) gives us that neither $\mathfrak{U}_\varkappa|\mathfrak{Y}$ nor $\mathfrak{U}_\varkappa|\mathfrak{Z}$. On the other hand,

$$|\mathfrak{Y}\times\mathfrak{Z}|/E(\mathfrak{Y}\times\mathfrak{Z}) = \{\varkappa\times\{\mu\},\ (\varkappa\times\mu) \cup (\{\varkappa\}\times\mu) \cup (\{\varkappa\}\times\{\mu\})\};$$

and $\#(\varkappa\times\{\mu\}) = \varkappa$ and $\#\big((\varkappa\times\mu) \cup (\{\varkappa\}\times\mu) \cup (\{\varkappa\}\times\{\mu\})\big) = (\varkappa\cdot\mu) + \mu + 1$, the latter being equal to $\varkappa^2$ if $\varkappa$ is finite and $\varkappa$ otherwise. So $\mathfrak{U}_\varkappa|(\mathfrak{Y}\times\mathfrak{Z})$ by Lemma 1.3(e), and $\mathfrak{U}_\varkappa \notin \Pr(\mathcal{D})$.

THEOREM 1.6. *If* $\mathcal{K}$ *is any of the classes* $\mathcal{B}'$, $\mathcal{W}$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{R}$, *or* $\mathcal{E}$, *then* $\Pr(\mathcal{K}) \subseteq \mathcal{T}$.

Proof. If $\mathfrak{X} = \langle X, R\rangle \in \mathcal{B}$, define $\boldsymbol{D}(\mathfrak{X}) = \breve{R}^*X \sim R^*X$ and $\boldsymbol{S}(\mathfrak{X}) = \breve{R}^*X \cap R^*X$. If $\mathfrak{X}, \mathfrak{Y} \in \mathcal{B}$, we have the following obvious facts:

(i) $\boldsymbol{D}(\mathfrak{X}\times\mathfrak{Y}) = \big(\boldsymbol{D}(\mathfrak{X})\times\boldsymbol{D}(\mathfrak{Y})\big) \cup \big(\boldsymbol{D}(\mathfrak{X})\times\boldsymbol{S}(\mathfrak{Y})\big) \cup \big(\boldsymbol{S}(\mathfrak{X})\times\boldsymbol{D}(\mathfrak{Y})\big)$.

(ii) If $x \in \boldsymbol{D}(\mathfrak{X})$, then $x/E(\mathfrak{X}) \subseteq \boldsymbol{D}(\mathfrak{X})$.

(iii) $\boldsymbol{S}(\mathfrak{X}\times\mathfrak{Y}) = \boldsymbol{S}(\mathfrak{X})\times\boldsymbol{S}(\mathfrak{Y})$.

(iv) If $x \in \boldsymbol{S}(\mathfrak{X})$, then $x/E(\mathfrak{X}) \subseteq \boldsymbol{S}(\mathfrak{X})$.

Now suppose $\mathfrak{X} = \langle X, R\rangle \in \mathrm{Pr}(\mathcal{K})$. Since $\mathcal{K} \subseteq \mathcal{B}'$, we must have $\breve{R}^*X \neq \emptyset$ and thus either $\boldsymbol{S}(\mathfrak{X}) \neq \emptyset$ or $\boldsymbol{D}(\mathfrak{X}) \neq \emptyset$. If $\boldsymbol{S}(\mathfrak{X}) \neq \emptyset$, choose $x_0 \in \boldsymbol{S}(\mathfrak{X})$; otherwise, choose $x_0 \in \boldsymbol{D}(\mathfrak{X})$. Let $\varkappa = \#(X)$ and let $\mu = 1$ if $\varkappa$ is finite; otherwise, let $\mu = \varkappa^+$. Pick some set Z such that $\#(Z) = \mu$ and $Z \cap X = \emptyset$, and add Z to $x_0/E(\mathfrak{X})$; i.e. define $\mathfrak{X}' = \langle X \cup Z, R \cup S \cup T\rangle$, where $S = (Z \times R^*\{x_0\}) \cup (\breve{R}^*\{x_0\} \times Z)$ and $T = Z\times Z$ if $x_0 R x_0$, $T = \emptyset$ otherwise. The result of this construction is that $x_0/E(\mathfrak{X}') = Z \cup \big(x_0/E(\mathfrak{X})\big)$ and $w/E(\mathfrak{X}') = w/E(\mathfrak{X})$ for $w \in X \sim \big(x_0/E(\mathfrak{X})\big)$. Furthermore, by Lemma 1.3(b, c) and the fact that $\nu/E(\mathfrak{U}_\nu) = \{\nu\}$ for any cardinal ν, we have

$$|\mathfrak{X}\times\mathfrak{U}_\nu|/E(\mathfrak{X}\times\mathfrak{U}_\nu) = \{C\times\nu\colon\ C \in |\mathfrak{X}|/E(\mathfrak{X})\}$$

and

$$|\mathfrak{X}'\times\mathfrak{U}_\nu|/E(\mathfrak{X}'\times\mathfrak{U}_\nu) = \{D\times\nu\colon\ D \in |\mathfrak{X}'|/E(\mathfrak{X}')\}$$

for any cardinal ν. Matching $w/E(\mathfrak{X})$ with $w/E(\mathfrak{X}')$ for every $w \in X$ obviously gives an isomorphism between $(\mathfrak{X}\times\mathfrak{U}_\nu)/E(\mathfrak{X}\times\mathfrak{U}_\nu)$ and $(\mathfrak{X}'\times\mathfrak{U}_\nu)/E(\mathfrak{X}'\times\mathfrak{U}_\nu)$; now fix ν to be the first infinite cardinal larger than $\varkappa$. Then $\#\Big(\big(x_0/E(\mathfrak{X})\big)\times\nu\Big) = \nu = \#\Big(\big(Z\cup\big(x_0/E(\mathfrak{X})\big)\big)\times\nu\Big)$, since $\#\Big(Z\cup\big(x_0/E(\mathfrak{X})\big)\Big) \leqslant \mu+\varkappa \leqslant \nu$ and ν is infinite; and of course $\#\big(w/E(\mathfrak{X})\big) = \#\big(w/E(\mathfrak{X}')\big)$ for all $w \in X \sim \big(x_0/E(\mathfrak{X})\big)$. So Lemma 1.3(d) gives $\mathfrak{X}\times\mathfrak{U}_\nu \cong \mathfrak{X}'\times\mathfrak{U}_\nu$; so we have $\mathfrak{X}|(\mathfrak{X}'\times\mathfrak{U}_\nu)$, and clearly $\mathfrak{X}' \in \mathcal{K}$ and $\mathfrak{U}_\nu \in \mathcal{U} \subseteq \mathcal{K}$. Since $\mathfrak{X} \in \mathrm{Pr}(\mathcal{K})$, we have either $\mathfrak{X}|\mathfrak{X}'$ or $\mathfrak{X}|\mathfrak{U}_\nu$. In the latter case, obviously $\mathfrak{X} \in \mathcal{U}$, so $\mathfrak{X} \in \mathrm{Pr}(\mathcal{K}) \cap \mathcal{U} \subseteq \mathrm{Pr}(\mathcal{U}) = \mathcal{I}$ and we are done. So suppose $\mathfrak{X}|\mathfrak{X}'$, or $\mathfrak{X}\times\mathfrak{Z} \cong \mathfrak{X}'$ for some $\mathfrak{Z} \in \mathcal{B}$. Then $\mu \neq 1$; so $\varkappa$ is infinite, and $\#(|\mathfrak{X}'|) = \#(|\mathfrak{Z}|) = \varkappa^+$. Let W be the image of $x_0/E(\mathfrak{X}')$ in $X\times|\mathfrak{Z}|$ under the isomorphism. Now we claim that $\boldsymbol{S}(\mathfrak{X}) \neq \emptyset$. For, if $\boldsymbol{S}(\mathfrak{X}) = \emptyset$, then $x_0 \in \boldsymbol{D}(\mathfrak{X})$ by construction, and also $R^*X \sim \breve{R}^*X \neq \emptyset$ since $\mathfrak{X} \in \mathcal{B}'$. By (ii) above, $x_0/E(\mathfrak{X}') \subseteq \boldsymbol{D}(\mathfrak{X}')$ (obviously $x_0 \in \boldsymbol{D}(\mathfrak{X}')$); so (i) gives

$$W \subseteq \boldsymbol{D}(\mathfrak{X}\times\mathfrak{Z}) = \boldsymbol{D}(\mathfrak{X})\times\big(\boldsymbol{D}(\mathfrak{Z})\cup\boldsymbol{S}(\mathfrak{Z})\big)$$

since $\boldsymbol{S}(\mathfrak{X}) = \emptyset$. But then $(R^*X \sim \breve{R}^*X)\times|\mathfrak{Z}|$ is a non-empty subset of $X\times|\mathfrak{Z}|$, disjoint from W, with cardinality greater then $\varkappa$. But this is impossible, since $(X\times|\mathfrak{Z}|)\sim W$ is equinumerous to $X \sim \big(x_0/E(\mathfrak{X})\big)$, which has cardinality at most $\varkappa$. This contradiction shows that $\boldsymbol{S}(\mathfrak{X}) \neq \emptyset$ and, consequently, that $x_0 \in \boldsymbol{S}(\mathfrak{X})$. Obviously, then, $x_0 \in \boldsymbol{S}(\mathfrak{X}')$, and (iv) above gives $x_0/E(\mathfrak{X}') \subseteq \boldsymbol{S}(\mathfrak{X}')$. Thus $W \subseteq \boldsymbol{S}(\mathfrak{X}\times\mathfrak{Z}) = \boldsymbol{S}(\mathfrak{X})\times\boldsymbol{S}(\mathfrak{Z})$. Furthermore, we even have $\boldsymbol{S}(\mathfrak{X}) = X$; for otherwise, $\big(X\sim\boldsymbol{S}(\mathfrak{X})\big)\times|\mathfrak{Z}|$ would be a non-empty subset of $X\times|\mathfrak{Z}|$, disjoint from W, having cardinality greater than $\varkappa$, leading to a contradiction similar to the one above. Since $\boldsymbol{S}(\mathfrak{X}) = X$, we have $\mathfrak{X} \in \mathcal{S}$, whence $\mathfrak{X}' \in \mathcal{S}$ and necessarily $\mathfrak{Z} \in \mathcal{S}$. But then $W = C\times D$ for some $C \in X/E(\mathfrak{X})$, $D \in |\mathfrak{Z}|/E(\mathfrak{Z})$ by Lemma 1.3(a, c). If $X \neq C$, then $(X\sim C)\times|\mathfrak{Z}|$ is a non-void subset of $X\times|\mathfrak{Z}|$, disjoint from W, with cardinality greater than $\varkappa$; this is impossible as before. So $X = C$, or $X \in X/E(\mathfrak{X})$; consequently $E(\mathfrak{X}) = X\times X$ and $\mathfrak{X} \in \mathcal{U}$, whence $\mathfrak{X} \in \mathcal{I}$ as before.

We say $\mathfrak{X} \in \mathcal{B}$ is *connected* if and only if it is not the cardinal sum of two structures; i.e. if there do not exist structures $\langle Y, S\rangle, \langle Z, T\rangle \in \mathcal{B}$ such that $Y \cap Z = \emptyset$ and $\mathfrak{X} = \langle Y\cup Z, S\cup T\rangle$. If $\mathcal{K} \subseteq \mathcal{B}$, let $\mathcal{K}_C$ denote the class of all connected structures in $\mathcal{K}$. The above proofs also show that $\mathrm{Pr}(\mathcal{W}_C) = \mathrm{Pr}(\mathcal{D}_C) = \emptyset$ and $\mathrm{Pr}(\mathcal{S}_C) = \mathrm{Pr}(\mathcal{R}_C) = \mathcal{I}$. (Obviously $\mathcal{B}_C = \mathcal{B}'_C = \mathcal{W}_C$ and $\mathcal{E}_C = \mathcal{U}_C = \mathcal{U}$.)

2. Connectivity in cardinal products. We now turn our attention to classes of finite structures; if $\mathcal{K} \subseteq \mathcal{B}$, let $\mathcal{K}^f$ be the class of all finite structures in $\mathcal{K}$. (Then $\mathcal{K}_C^f$ is the class of connected finite structures in $\mathcal{K}$.) Suppose $\mathfrak{X} = \langle X, R\rangle \in \mathcal{B}$ and $\emptyset \neq Y \subseteq X$; then both Y and the substructure $\mathfrak{Y} = \langle Y, R\cap(Y\times Y)\rangle$ will be called *ideals* of $\mathfrak{X}$ if and only if $\mathfrak{Y}$ is connected and $R^*Y \cup \breve{R}^*Y \subseteq Y$. Thus the ideals of $\mathfrak{X}$ are its maximal connected substructures. Let $L(\mathfrak{X})$ denote the cardinal number of the set of all ideals of $\mathfrak{X}$. In this section we consider the question: given $\mathfrak{X}, \mathfrak{Y} \in \mathcal{D}^f$, how many ideals will $\mathfrak{X}\times\mathfrak{Y}$ have? Since any structure is a cardinal sum of connected structures (its ideals) and cardinal multiplication is distributive over cardinal addition (to within isomorphism), it suffices to answer the question for $\mathfrak{X}, \mathfrak{Y} \in \mathcal{D}_C^f$. The answer derived here will be used in § 3 to deduce more about prime structures.

Definition 2.1. Let α, $\breve{}$, and $|$ be fixed symbols. The set $\boldsymbol{T}$ of strings of these symbols is defined inductively as follows: (i) $\{\alpha, \alpha\breve{}\} \subseteq \boldsymbol{T}$; (ii) if $\sigma, \tau \in \boldsymbol{T}$, then $\sigma|\tau \in \boldsymbol{T}$; (iii) $\boldsymbol{T}$ has no members not required by (i) and (ii). If $\tau \in \boldsymbol{T}$ and R is a binary relation, then $\tau(R)$ will denote the relation named by τ when α is interpreted as R, $\breve{}$ as inversion, and $|$ as relative product. The *rank* of any $\tau \in \boldsymbol{T}$, denoted by $\varrho(\tau)$, is the integer named by τ when α is interpreted as 1, $\alpha\breve{}$ as -1, and $|$ as addition. The *inverse* of $\tau \in \boldsymbol{T}$, denoted by τ^{-1}, is defined by induction on $\boldsymbol{T}$ as follows: (i) $\alpha^{-1} = \alpha\breve{}$; (ii) $(\alpha\breve{})^{-1} = \alpha$; (iii) $(\sigma|\tau)^{-1} = \tau^{-1}|\sigma^{-1}$. Finally, if $\tau \in \boldsymbol{T}$ define $\tau^{(1)} = \tau$, $\tau^{(n+1)} = \tau^{(n)}|\tau$ for all $n > 0$.

As an example of this notion, we see that $\langle X, R\rangle$ is connected if and only if for every $x, y \in X$ there is some $\tau \in \boldsymbol{T}$ with $\langle x, y\rangle \in \tau(R)$. The following facts are easily verified.

LEMMA 2.2. (a) *If R, S are relations and $\tau \in \boldsymbol{T}$, then $\tau(R\otimes S) = \tau(R)\otimes\otimes\tau(S)$.* (b) *If $\langle X, R\rangle \in \mathcal{S}$, $x \in X$, and $\tau \in \boldsymbol{T}$, then $\langle x, y\rangle \in \tau(R)$ for some $y \in X$.*

Now we note some facts about $\mathfrak{D}_C^f$-structures. If $n = \{0, 1, \ldots, n-1\}$ is a positive natural number, we define $\mathfrak{C}_n$ to be the structure $\langle n, R\rangle$ in which iRj if and only if $i+1 \equiv j \pmod{n}$. Let ψ_1 and ψ_2 be the two projection functions defined on any Cartesian product; e.g., $\psi_1\langle x, y\rangle = x$, $\psi_2\langle x, y\rangle = y$.

LEMMA 2.3. (a) *If $\mathfrak{X} = \langle X, R\rangle \in \mathfrak{D}_C^f$, then for some $n > 0$, $\mathfrak{X}$ has a substructure $\langle Y, R \cap (Y\times Y)\rangle$ isomorphic to $\mathfrak{C}_n$.* (b) *Suppose $\langle X, R\rangle$, $\langle Y, S\rangle \in \mathfrak{D}_C^f$, $Z \subseteq X\times Y$ is an ideal of $\langle X, R\rangle\times\langle Y, S\rangle$, and xRw. Then there are $p, q \in Z$ such that $p(R\otimes S)q$, $\psi_1(p) = x$, and $\psi_1(q) = w$; and, consequently, the projection of Z on X is all of X.*

Proof. (a) Choose any $x_0 \in X$; given $x_i \in X$, since $x_i \in \breve{R}^*X$ we can choose $x_{i+1} \in X$ so that x_iRx_{i+1}. This gives an infinite sequence of points in X, each related to the next. Since X is finite, this sequence must contain closed cycles; the shortest such cycle will be isomorphic to some $\mathfrak{C}_m$. (b) Using (a), let $\mathfrak{Z}_0$ be a substructure of $\langle X, R\rangle\times\langle Y, S\rangle$ such that $|\mathfrak{Z}_0| \subseteq Z$ and $\mathfrak{Z}_0 \cong \mathfrak{C}_k$ for some $k > 0$. Let Y_0 be the projection of $|\mathfrak{Z}_0|$ on Y; if $\mathfrak{Y}_0 = \langle Y_0, S\cap(Y_0\times Y_0)\rangle$, clearly $\mathfrak{Y}_0 \in \mathcal{S}$. Choose some $u \in |\mathfrak{Z}_0|$, and suppose xRw. Since $\langle X, R\rangle$ is connected, there is some $\tau \in \boldsymbol{T}$ such that $\langle w, \psi_1(u)\rangle \in \tau(R)$. Since $\mathfrak{Y}_0 \in \mathcal{S}$, by Lemma 2.2(b) there exist $c, d \in Y_0$ such that $\langle c, \psi_2(u)\rangle \in \tau(S)$ and dSc. By Lemma 2.2(a), $\langle\langle w, c\rangle, u\rangle \in \tau(R\otimes S)$ and $\langle x, d\rangle(R\otimes S)\langle w, c\rangle$. Since $u \in Z$ and Z is an ideal, both $\langle x, d\rangle$ and $\langle w, c\rangle$ must belong to Z. So let $p = \langle x, d\rangle$ and $q = \langle w, c\rangle$.

DEFINITION 2.4. If $\mathfrak{X} = \langle X, R\rangle \in \mathcal{B}$ and $x \in X$, define

$$K^R(x) = \{\varrho(\tau)\colon \tau \in \boldsymbol{T} \text{ and } \langle x, x\rangle \in \tau(R)\}$$

and

$$K(\mathfrak{X}) = \bigcup\{K^R(x)\colon x \in X\}.$$

LEMMA 2.5. *If $\mathfrak{X} = \langle X, R\rangle \in \mathfrak{D}_C^f$, then $K^R(x) = K(\mathfrak{X})$ for every $x \in X$; also, $K(\mathfrak{X})$ is a subgroup of the group of integers and $K(\mathfrak{X}) \neq \{0\}$.*

Proof. To show the first assertion, it suffices to show that $K^R(x) \subseteq K^R(y)$ for every $x, y \in X$. But since $\mathfrak{X}$ is connected, for any given $x, y \in X$ we have $\langle x, y\rangle \in \sigma(R)$ for some $\sigma \in \boldsymbol{T}$; so if $n \in K^R(x)$ by virtue of $\tau \in \boldsymbol{T}$, we have $\langle y, y\rangle \in (\sigma(R))^{\smile}|\tau(R)|\sigma(R)$, or $\langle y, y\rangle \in (\sigma^{-1}|\tau|\sigma)(R)$, and $\varrho(\sigma^{-1}|\tau|\sigma)$ is $-\varrho(\sigma)+\varrho(\tau)+\varrho(\sigma) = n$. So $n \in K^R(y)$. Now to finish we need only show that each $K^R(x)$ is a non-trivial subgroup of the integers. But this is easy; if $m, n \in K^R(x)$ by virtue of σ, τ, respectively, then $-m \in K^R(x)$ by virtue of σ^{-1}, and $m+n \in K^R(x)$ by virtue of $\sigma|\tau$. By Lemma 2.3(a), some $K^R(x)$ contains a positive integer and so is not $\{0\}$.

DEFINITION 2.6. If $\mathfrak{X} \in \mathfrak{D}_C^f$, define $k(\mathfrak{X})$ to be the (unique) non-negative generator of the group $K(\mathfrak{X})$; then $K(\mathfrak{X})$ consists of all multiples of $k(\mathfrak{X})$.

THEOREM 2.7. *If $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{D}_C^f$, then $L(\mathfrak{X}\times\mathfrak{Y}) = $ g.c.d. $\{k(\mathfrak{X}), k(\mathfrak{Y})\}$.*

Proof. Let $\mathfrak{X} = \langle X, R\rangle$, $\mathfrak{Y} = \langle Y, S\rangle$, $m = k(\mathfrak{X})$, $n = k(\mathfrak{Y})$, and $s = $ g.c.d. $\{m, n\}$. By Lemma 2.3(a), choose $C \subseteq X$, $D \subseteq Y$ such that $\langle C, R\cap(C\times C)\rangle \cong \mathfrak{C}_e$ and $\langle D, S\cap(D\times D)\rangle \cong \mathfrak{C}_f$ for some $e, f > 0$. Obviously $e \in K(\mathfrak{X})$, so by Definition 2.6 we have $e = pm$ for some p; similarly, $f = qn$ for some q. Let $C = \{c_0, c_1, \ldots, c_{pm-1}\}$ and $D = \{d_0, d_1, \ldots \ldots, d_{qn-1}\}$ be enumerations such that c_iRc_j if and only if $j \equiv i+1 \pmod{pm}$, and d_iSd_j if and only if $j \equiv i+1 \pmod{qn}$. Consider the set $\{\langle c_0, d_i\rangle\colon i < qn\}$.

CLAIM 1. *The s ideals of $\mathfrak{X}\times\mathfrak{Y}$ containing the points $\langle c_0, d_0\rangle, \langle c_0, d_1\rangle, \ldots \ldots, \langle c_0, d_{s-1}\rangle$ are mutually disjoint.*

Proof. Suppose on the contrary that $\langle c_0, d_i\rangle$ is connected to $\langle c_0, d_j\rangle$ for some $0 \leqslant i < j < s$; that is, $\langle\langle c_0, d_i\rangle, \langle c_0, d_j\rangle\rangle \in \tau(R\otimes S)$ for some $\tau \in \boldsymbol{T}$. By Lemma 2.2(a), this means $\langle c_0, c_0\rangle \in \tau(R)$ and $\langle d_i, d_j\rangle \in \tau(S)$. In turn, $\langle c_0, c_0\rangle \in \tau(R)$ implies $\varrho(\tau) \in K(\mathfrak{X})$ by Definition 2.4, whence $\varrho(\tau) \equiv 0 \pmod{m}$ since $K(\mathfrak{X})$ consists of multiples of m. At the same time, $\langle d_j, d_i\rangle \in \breve{S}^{(j-i)}$ since $\langle D, S\cap(D\times D)\rangle \cong \mathfrak{C}_f$. So $\langle d_i, d_i\rangle \in \tau(S)|\breve{S}^{(j-i)}$ and thus $\varrho(\tau)-(j-i) \in K(\mathfrak{Y})$, or $\varrho(\tau) \equiv j-i \pmod{n}$. The two congruences $\varrho(\tau) \equiv 0 \pmod{m}$ and $\varrho(\tau) \equiv j-i \pmod{n}$ together imply that $j-i$ is a multiple of s; but this is a contradiction, since $0 < j-i < s$.

CLAIM 2. *If $i < qn$, then the points $\langle c_0, d_i\rangle$ and $\langle c_0, d_{i+s(\mathrm{mod}\, qn)}\rangle$ belong to the same ideal of $\mathfrak{X}\times\mathfrak{Y}$.*

Proof. Since $s = $ g.c.d. $\{m, n\}$, we have $s = um-vn$ for some integers u, v. Since $m = k(\mathfrak{X})$, $um \in K(\mathfrak{X})$; so by Lemma 2.5, there is some $\tau \in \boldsymbol{T}$ such that $\varrho(\tau) = um$ and $\langle c_0, c_0\rangle \in \tau(R)$. Since $\langle D, S\cap(D\times D)\rangle \in \mathcal{S}$, there is by Lemma 2.2(b) some $d_j \in D$ with $\langle d_i, d_j\rangle \in (S\cap(D\times D))$; hence $\varrho(\tau) \equiv j-i \pmod{qn}$ by the same sort of argument as was used in Claim 1; or, $j \equiv i+um \pmod{qn}$. Now by Lemma 2.2(a) we have $\langle\langle c_0, d_i\rangle, \langle c_0, d_{i+um(\mathrm{mod}\, qn)}\rangle\rangle \in \tau(R\otimes S)$. In a similar way, $vn \in K(\mathfrak{Y})$, there is some $\sigma \in \boldsymbol{T}$ such that $\varrho(\sigma) = vn$ and $\langle d_{i+um(\mathrm{mod}\, qn)}, d_{i+um(\mathrm{mod}\, qn)}\rangle \in \sigma(S)$, and, since $\langle C, R\cap(C\times C)\rangle \in \mathcal{S}$, we get as before $\langle c_0, c_{0+vn(\mathrm{mod}\, qn)}\rangle \in \sigma(R)$, whence $\langle\langle c_0, d_{i+um(\mathrm{mod}\, qn)}\rangle, \langle c_{0+vn(\mathrm{mod}\, pm)}, d_{i+um(\mathrm{mod}\, qn)}\rangle\rangle \in \sigma(R\otimes S)$. On the other hand, it is obvious that

$$\langle\langle c_0, d_{i+um-vn(\mathrm{mod}\, qn)}\rangle, \langle c_{0+vn(\mathrm{mod}\, pm)}, d_{i+um(\mathrm{mod}\, qn)}\rangle\rangle \in (R\otimes S)^{(vn)}.$$

Hence $\langle\langle c_0, d_i\rangle, \langle c_0, d_{i+s(\mathrm{mod}\, qn)}\rangle\rangle \in \tau(R\otimes S)|\sigma(R\otimes S)|\big((R\otimes S)^{(vn)}\big)^{\smile}$. This proves Claim 2.

Let L_i be the ideal of $\mathfrak{X}\times\mathfrak{Y}$ containing $\langle c_0, d_i\rangle$, for $i < s$.

CLAIM 3. $\bigcup\{L_i\colon\ i<s\}=X\times Y$.

Proof. For convenience, let $A=\bigcup\{L_i\colon\ i<s\}$. By Claim 2, we have $\{c_0\}\times D\subseteq A$. For any $j<pm$, $k<qn$ we have $\langle\langle c_0, d_{k-j(\mathrm{mod}\,qn)}\rangle, \langle c_j, d_k\rangle\rangle\in(R\otimes S)^{(j)}$; hence $C\times D\subseteq A$. Given any $x\in X$, since $\mathfrak{X}$ is connected there is some $\tau\in\boldsymbol{T}$ such that $\langle x, c_0\rangle\in\tau(R)$; given any $d\in D$, by Lemma 2.2(b) there is some $d'\in D$ such that $\langle d, d'\rangle\in\tau(S)$. Hence for any $x\in X$ and $d\in D$ there is some $\tau\in\boldsymbol{T}$ and $d'\in D$ such that $\langle\langle x, d\rangle, \langle c_0, d'\rangle\rangle\in\tau(R\otimes S)$, by Lemma 2.2(a); and $\langle c_0, d'\rangle\in C\times D\subseteq A$, so $\langle x, d\rangle\in A$ since A is a union of ideals. Hence $X\times D\subseteq A$. Now given any $u\in X\times Y$, let M be the ideal of $\mathfrak{X}\times\mathfrak{Y}$ containing u; since (applying Lemma 2.3(b) to $\mathfrak{Y}\times\mathfrak{X}$) the projection of M on Y must be all of Y, the set $M\cap(X\times D)$ must have a member, say v. Since M is an ideal, there is some $\sigma\in\boldsymbol{T}$ such that $\langle u, v\rangle\in\sigma(R\otimes S)$. Since $v\in X\times D\subseteq A$, this means $u\in A$. This proves $X\times Y\subseteq A$.

By Claim 1, $L(\mathfrak{X}\times\mathfrak{Y})\geqslant s$; by Claim 3, $L(\mathfrak{X}\times\mathfrak{Y})\leqslant s$. So we are done.

COROLLARY 2.8. *If* $\mathfrak{X}=\langle X, R\rangle\in\mathfrak{D}_C^f$, $r>0$, *and* $\mathfrak{Z}_1, \mathfrak{Z}_2$ *are ideals fo* $\mathfrak{X}\times\mathfrak{C}_r$, *then* $\mathfrak{Z}_1\cong\mathfrak{Z}_2$.

Proof. In the proof of Theorem 2.7, let $\mathfrak{Y}=\mathfrak{C}_r=\langle r, S\rangle$. Define $\varphi\colon X\times r\to X\times r$ by $\varphi\langle x, j\rangle=\langle x, j+1\ (\mathrm{mod}\,r)\rangle$. It is clear that φ is an automorphism of $\mathfrak{X}\times\mathfrak{C}_r$, since in $\mathfrak{C}_r$ we have jSk if and only if $\langle j+1\ (\mathrm{mod}\,r), k+1\ (\mathrm{mod}\,r)\rangle\in S$. Thus φ must carry ideals onto ideals. But obviously $\varphi\langle c_0, d_i\rangle=\langle c_0, d_{i+1(\mathrm{mod}\,r)}\rangle$ for all $i<r$, so φ carries each ideal L_i onto the ideal $L_{i+1(\mathrm{mod}\,s)}$ and all the ideals are isomorphic.

COROLLARY 2.9. *If* $\mathfrak{K}$ *is either of the classes* $\mathfrak{D}$, $\mathfrak{S}$ *and* $\mathfrak{X}, \mathfrak{Y}\in\mathfrak{K}_C^f$, *then* $\mathfrak{X}\times\mathfrak{Y}\in\mathfrak{K}_C^f$ *if and only if* g.c.d. $\{k(\mathfrak{X}), k(\mathfrak{Y})\}=1$.

§ 3. Primes in classes of finite structures. It is rather easy to show that $\mathrm{Pr}(\mathfrak{W}^f)=\mathrm{Pr}(\mathfrak{W}_C^f)=\emptyset$, but the proof is not instructive and we omit it here (it is given in [5]). Our first main goal in this section is to show that $\mathrm{Pr}(\mathcal{A}^f)=\mathrm{Pr}(\mathcal{A}_C^f)=\emptyset$, where

$$\mathcal{A}=\{\langle X, R\rangle\colon\ \breve{R}^*X=X \text{ and } R \text{ is a function}\},$$

the class of *unary algebras*. (Mc Kenzie showed in [4] that $\mathrm{Pr}(\mathcal{A})=\mathrm{Pr}(\mathcal{A}_C)=\emptyset$.) We shall obtain this result with the help of two strong necessary conditions for primeness which apply equally to $\mathfrak{D}^f$, $\mathfrak{D}_C^f$, $\mathfrak{S}^f$, and $\mathfrak{S}_C^f$. First, a prime in any of these classes *must be connected*, i.e.

THEOREM 3.1. *If* $\mathfrak{K}$ *is any one of the classes* $\mathfrak{D}$, $\mathfrak{S}$, *or* $\mathcal{A}$, *then* $\mathrm{Pr}(\mathfrak{K}^f)\subseteq\mathrm{Pr}(\mathfrak{K}_C^f)$.

Proof. It suffices to show $\mathrm{Pr}(\mathfrak{K}^f)\subseteq\mathfrak{K}_C^f$; we show the contrapositive of this. Suppose $\mathfrak{X}=\langle X, R\rangle\in\mathfrak{K}^f$ is *not* connected. Then $\mathfrak{X}=\mathfrak{Y}_1+\mathfrak{Y}_2$, a cardinal sum of two structures, where we may assume that $|\mathfrak{Y}_1|$ is an ideal of $\mathfrak{X}$ having the largest possible cardinality. Let n be a prime number greater than $\#(X)$. Let $\mathfrak{Z}=(\mathfrak{Y}_1\times\mathfrak{C}_n)+(\mathfrak{Y}_2\times\mathfrak{I}_n)$. Then since $\mathfrak{C}_n\times\mathfrak{C}_n\cong\mathfrak{C}_n\times\mathfrak{I}_n$, we have

$$\mathfrak{X}\times(\mathfrak{C}_n\times\mathfrak{I}_n)\cong(\mathfrak{Y}_1\times(\mathfrak{C}_n\times\mathfrak{C}_n))+(\mathfrak{Y}_2\times(\mathfrak{C}_n\times\mathfrak{I}_n))\cong\mathfrak{Z}\times\mathfrak{C}_n.$$

So $\mathfrak{X}|(\mathfrak{Z}\times\mathfrak{C}_n)$, and clearly $\mathfrak{Z}, \mathfrak{C}_n\in\mathfrak{K}^f$. Furthermore, *not* $\mathfrak{X}|\mathfrak{C}_n$, since $\mathfrak{X}$ is not connected and clearly any cardinal factor of $\mathfrak{C}_n$ must be connected. Suppose $\mathfrak{X}|\mathfrak{Z}$, or $\mathfrak{X}\times\mathfrak{W}\cong\mathfrak{Z}$ for some $\mathfrak{W}=\langle W, T\rangle$. Then $\mathfrak{Y}_1\times\mathfrak{C}_n$, which is connected by Theorem 2.7 (n is prime and greater than $\#(|\mathfrak{Y}_1|)$) and is thus an ideal of $\mathfrak{Z}$, is isomorphic to some ideal $\mathfrak{A}$ of $\mathfrak{X}\times\mathfrak{W}$. Let X', W' be the projections of $|\mathfrak{A}|$ on X and W, respectively; it follows from Lemma 2.3 that X' and W' are ideals of $\mathfrak{X}$ and $\mathfrak{W}$, respectively. Furthermore, $|\mathfrak{A}|\subseteq X'\times W'$, so $\#(|\mathfrak{A}|)\leqslant\#(X')\cdot\#(W')$; but $\#(|\mathfrak{A}|)=n\cdot\#(|\mathfrak{Y}_1|)$, and $\#(X')\leqslant\#(|\mathfrak{Y}_1|)$ since $\mathfrak{Y}_1$ was among the largest ideals of $\mathfrak{X}$. So $\#(W')\geqslant n$. Let X'' be an ideal of $\mathfrak{X}$ other than X', and let $B\subseteq X''\times W'$ be any ideal of $\mathfrak{X}\times\mathfrak{W}$; then $B\neq|\mathfrak{A}|$. By Lemma 2.3, the projection of B on W is W'; so $\#(B)\geqslant n$ and the isomorphic image of B in $\mathfrak{Z}$ is an ideal, other than $\mathfrak{Y}_1\times\mathfrak{C}_n$, with n or more elements. This is a contradiction, since all the ideals of $\mathfrak{Z}$ other than $\mathfrak{Y}_1\times\mathfrak{C}_n$ are obviously isomorphic to ideals of $\mathfrak{X}$, and $\#(X)<n$. This contradiction shows that actually *not* $\mathfrak{X}|\mathfrak{Z}$. Consequently $\mathfrak{X}\notin\mathrm{Pr}(\mathfrak{K}^f)$.

Second, we have the following strong necessary condition on connected prime structures:

THEOREM 3.2. *If* $\mathfrak{K}$ *is either* $\mathfrak{D}$, $\mathfrak{S}$, *or* $\mathcal{A}$ *and* $\mathfrak{X}\in\mathrm{Pr}(\mathfrak{K}_C^f)$, *then* $k(\mathfrak{X})=1$.

Proof. Suppose $\mathfrak{X}=\langle X, R\rangle\in\mathfrak{K}_C^f$ but $m=k(\mathfrak{X})>1$; we show $\mathfrak{X}\notin\mathrm{Pr}(\mathfrak{K}_C^f)$. Consider the structure $\mathfrak{X}\times\mathfrak{C}_{m^2}$; for convenience, let $\mathfrak{C}_{m^2}=\langle m^2, S\rangle$. Of course $k(\mathfrak{C}_{m^2})=m^2$; so by Theorem 2.7 and Corollary 2.8, $\mathfrak{X}\times\mathfrak{C}_{m^2}$ is the cardinal sum of m mutually isomorphic ideals, or $\mathfrak{X}\times\mathfrak{C}_{m^2}\cong\mathfrak{Y}\times\mathfrak{I}_m$ for some $\mathfrak{Y}\in\mathfrak{K}_C^f$, $\mathfrak{Y}$ an ideal of $\mathfrak{X}\times\mathfrak{C}_{m^2}$.

Now suppose $\langle\langle x, i\rangle, \langle x, i\rangle\rangle\in\sigma(R\otimes S)$ for some $\langle x, i\rangle\in|\mathfrak{Y}|$, $\sigma\in\boldsymbol{T}$. Then $\langle i, i\rangle\in\sigma(S)$ by Lemma 2.2(a), and $\varrho(\sigma)$ must be a multiple of m^2. Consequently, every number in $K(\mathfrak{Y})$ must be a multiple of m^2. On the other hand, since $m=k(\mathfrak{X})$ there is some $y\in X$ and some $\tau\in\boldsymbol{T}$ such that $\varrho(\tau)=m$ and $\langle y, y\rangle\in\tau(R)$; then $\varrho(\tau^{(m)})=m+m+\ldots+m$ (m times) $=m^2$ and $\langle 0, 0\rangle\in\tau^{(m)}(S)$, so $\langle\langle y, 0\rangle, \langle y, 0\rangle\rangle\in\tau^{(m)}(R\otimes S)$. Hence $m^2\in K(\mathfrak{Y})$, which together with the fact that m^2 divides every member of $K(\mathfrak{Y})$ gives us that $k(\mathfrak{Y})=m^2$. Thus, by Theorem 2.7 and Corollary 2.8, the structure $\mathfrak{Y}\times\mathfrak{C}_m$ is a cardinal sum of m mutually isomorphic ideals; if $\mathfrak{Z}$ is one of these ideals, then the projection map $\mathfrak{Z}\to\mathfrak{Y}$ is onto by Lemma 2.3(b). Hence, if $\mathfrak{Z}$ is an ideal of $\mathfrak{Y}\times\mathfrak{C}_m$, then the projection map $\mathfrak{Z}\to\mathfrak{Y}$ is one-one; for if both $\langle y, i\rangle\in|\mathfrak{Z}|$ and $\langle y, j\rangle\in|\mathfrak{Z}|$ for some $y\in|\mathfrak{Y}|$, $i, j<m$, $i\neq j$, then since $\#((\{y\}\times m)\sim\{\langle y, i\rangle, \langle y, j\rangle\})=m-2$ and there are m ideals of $\mathfrak{Y}\times\mathfrak{C}_m$, there must be some ideal $\mathfrak{Z}'$ for which

$|\mathfrak{Z}'| \cap (\{y\} \times m) = \emptyset$. But this contradicts the fact that the projection $\mathfrak{Z}' \to \mathfrak{Y}$ is onto. Now that we know that the projection $\mathfrak{Z} \to \mathfrak{Y}$ is one-one and onto, we see that it is an isomorphism by Lemma 2.3(b). So $\mathfrak{Z} \cong \mathfrak{Y}$ and hence $\mathfrak{Y} \times \mathfrak{C}_m \cong \mathfrak{Y} \times \mathfrak{I}_m$.

Combining the final results of the preceding two paragraphs, we get $\mathfrak{X} \times \mathfrak{C}_{m^2} \cong \mathfrak{Y} \times \mathfrak{C}_m$.

We now show that *not* $\mathfrak{X}|\mathfrak{Y}$. Suppose that on the contrary, $\mathfrak{X} \times \mathfrak{W} \cong \mathfrak{Y}$ for some $\mathfrak{W} = \langle W, T\rangle$; then $\mathfrak{W} \in \mathfrak{D}^f_C$ and $\#(W) = m$, since $\#(|\mathfrak{Y}|) = \#(X) \cdot m^2/m$. By Lemma 2.3(a), let $\mathfrak{B}$ be a substructure of $\mathfrak{W}$ such that $\mathfrak{B} \cong \mathfrak{C}_p$ for some $p > 0$; we may assume $\mathfrak{B} = \mathfrak{C}_p$. As above, choose $y \in X$, $\tau \in \boldsymbol{T}$ such that $\varrho(\tau) = m$ and $\langle y, y\rangle \in \tau(R)$. Then for all $i \in p$ we have $\langle i, i+m(\mathrm{mod}\, p)\rangle \in \tau(T)$, since $\mathfrak{B} = \mathfrak{C}_p$. By Lemma 2.2(a), $\langle\langle y, i\rangle, \langle y, i+m(\mathrm{mod}\, p)\rangle\rangle \in \tau(R \otimes T)$ for all $i \in p$; hence each of the points in the list

$$\langle y, 0\rangle, \langle y, m(\mathrm{mod}\, p)\rangle, \langle y, 2\cdot m(\mathrm{mod}\, p)\rangle, \ldots, \langle y, (m-1)\cdot m(\mathrm{mod}\, p)\rangle$$

is $\tau(R \otimes T)$-related to the next one, and thus the numbers $0, m, 2m, \ldots$ $\ldots, (m-1)m$ are *all different modulo* p (since otherwise there would be some $b \in X \times p$, $q < m$ such that $0 < q$, $\langle b, b\rangle \in \tau^{(q)}(R \otimes T)$, and $\varrho(\tau^{(q)}) = q \cdot m < m^2$; but $\mathfrak{X} \times \mathfrak{B}$ is isomorphically embeddable in $\mathfrak{Y}$, and $k(\mathfrak{Y}) = m^2$). Hence $p \geqslant m$; but $\#(W) = m$ and $\mathfrak{B}$ is a substructure of $\mathfrak{W}$, so $p \leqslant m$. Thus $p = m$ and $\mathfrak{W} = \mathfrak{B}$; i.e., $\mathfrak{W} \cong \mathfrak{C}_m$. But then $L(\mathfrak{X} \times \mathfrak{W}) = m$ by Theorem 2.7, whereas $L(\mathfrak{Y}) = 1$ since $\mathfrak{Y}$ is an ideal. So $\mathfrak{X} \times \mathfrak{W} \ncong \mathfrak{Y}$. This contradiction shows *not* $\mathfrak{X}|\mathfrak{Y}$.

Now we have $\mathfrak{X}|(\mathfrak{Y} \times \mathfrak{C}_m)$, $\mathfrak{Y}$, $\mathfrak{C}_m \in \mathfrak{K}^f_C$, and *not* $\mathfrak{X}|\mathfrak{Y}$. If also *not* $\mathfrak{X}|\mathfrak{C}_m$, then $\mathfrak{X} \notin \mathrm{Pr}(\mathfrak{K}^f_C)$ and we are done. On the other hand, if $\mathfrak{X}|\mathfrak{C}_m$, then it is easy to see (using Lemma 2.3(a)) that $\mathfrak{X} \cong \mathfrak{C}_s$ for some s. But $\mathfrak{C}_s \notin \mathrm{Pr}(\mathfrak{K}^f_C)$, since it is easy to see that $\mathfrak{C}_s \times (\mathfrak{C}_{s^2} \times \mathfrak{I}_s) \cong \mathfrak{C}_{s^2} \times \mathfrak{C}_{s^2}$ but *not* $\mathfrak{C}_s|\mathfrak{C}_{s^2}$.

The next result complements Theorem 3.1.

THEOREM 3.3. *If* $\mathfrak{K}$ *is either* $\mathfrak{D}$, $\mathfrak{S}$, $\mathcal{A}$, *or* $\mathfrak{R}$, *then* $\mathrm{Pr}(\mathfrak{K}^f_C) \subseteq \mathrm{Pr}(\mathfrak{K}^f)$.

Proof. Suppose $\mathfrak{X} \in \mathrm{Pr}(\mathfrak{K}^f_C)$, $\mathfrak{Y}, \mathfrak{Z} \in \mathfrak{K}^f$ and $\mathfrak{X}|(\mathfrak{Y} \times \mathfrak{Z})$. Then $\mathfrak{X} \times \mathfrak{W} \cong \mathfrak{Y} \times \mathfrak{Z}$ for some $\mathfrak{W} \in \mathfrak{K}^f$. Let $\mathfrak{W}_0, \mathfrak{W}_1, \ldots, \mathfrak{W}_m$ be the ideals of $\mathfrak{W}$. By Theorem 3.2 (or trivially if $\mathfrak{K} = \mathfrak{R}$), $k(\mathfrak{X}) = 1$; so by Corollary 2.9 (or trivially if $\mathfrak{K} = \mathfrak{R}$) we have $\mathfrak{X} \times \mathfrak{W}_i \in \mathfrak{K}^f_C$ for all $i \leqslant m$. That is, the ideals of $\mathfrak{X} \times \mathfrak{W}$ are just the structures $\mathfrak{X} \times \mathfrak{W}_i$, $i \leqslant m$. Hence $\mathfrak{X}$ is a cardinal factor of every ideal of $\mathfrak{Y} \times \mathfrak{Z}$. Let $\mathfrak{Y}_0, \mathfrak{Y}_1, \ldots, \mathfrak{Y}_p$ and $\mathfrak{Z}_0, \mathfrak{Z}_1, \ldots, \mathfrak{Z}_q$ be the ideals of $\mathfrak{Y}$ and of $\mathfrak{Z}$, respectively. If $i \leqslant p$, $j \leqslant q$ then $\mathfrak{Y}_i \times \mathfrak{Z}_j$ is a cardinal sum of ideals of $\mathfrak{Y} \times \mathfrak{Z}$, and hence $\mathfrak{Y}_i \times \mathfrak{Z}_j$ is a cardinal multiple of $\mathfrak{X}$; since $\mathfrak{Y}_i, \mathfrak{Z}_j \in \mathfrak{K}^f_C$ and $\mathfrak{X} \in \mathrm{Pr}(\mathfrak{K}^f_C)$, this means that either $\mathfrak{X}|\mathfrak{Y}_i$ or $\mathfrak{X}|\mathfrak{Z}_j$. Now if *not* $\mathfrak{X}|\mathfrak{Y}$ we must have *not* $\mathfrak{X}|\mathfrak{Y}_i$ for some $i \leqslant p$, whence $\mathfrak{X}|\mathfrak{Z}_j$ for all $j \leqslant q$, and $\mathfrak{X}|\mathfrak{Z}$. So either $\mathfrak{X}|\mathfrak{Y}$ or $\mathfrak{X}|\mathfrak{Z}$, and we have shown $\mathfrak{X} \in \mathrm{Pr}(\mathfrak{K}^f)$.

We now know that $\mathrm{Pr}(\mathfrak{D}^f) = \mathrm{Pr}(\mathfrak{D}^f_C)$ and $\mathrm{Pr}(\mathfrak{S}^f) = \mathrm{Pr}(\mathfrak{S}^f_C)$, and we have strong necessary conditions for membership in any one of these four classes. At present we are unable to completely characterize the primes of $\mathfrak{D}^f$, $\mathfrak{D}^f_C$, $\mathfrak{S}^f$, or $\mathfrak{S}^f_C$, although we know, in addition to the above facts, that $\mathrm{Pr}(\mathfrak{S}^f) \nsubseteq \mathrm{Pr}(\mathfrak{D}^f)$ (by the proofs of Theorems 1.4 and 1.5, $\mathfrak{U}_2 \in \mathrm{Pr}(\mathfrak{S}^f) \sim \mathrm{Pr}(\mathfrak{D}^f)$). We can also say that $\mathrm{In}(\mathfrak{R}^f_C) \nsubseteq \mathrm{Pr}(\mathfrak{S}^f_C)$, whence $\mathrm{Pr}(\mathfrak{R}^f_C) \nsubseteq \mathrm{Pr}(\mathfrak{S}^f_C)$ (since $\mathrm{Pr}(\mathfrak{R}^f_C) = \mathrm{In}(\mathfrak{R}^f_C)$, as mentioned in the Introduction); for if

$$\mathfrak{X} = \langle\{0, 1, 2\}, \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \langle 0, 1\rangle, \langle 2, 1\rangle\}\rangle$$

and

$$\mathfrak{Y} = \langle\{0, 1, 2\}, \{\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 1, 1\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle\}\rangle,$$

it is easy to compute that $\mathfrak{X} \times \mathfrak{C}_2 \cong \mathfrak{Y} \times \mathfrak{C}_2$, but neither $\mathfrak{X}|\mathfrak{Y}$ nor $\mathfrak{X}|\mathfrak{C}_2$. So $\mathfrak{X} \notin \mathrm{Pr}(\mathfrak{S}^f_C)$, since $\mathfrak{Y}, \mathfrak{C}_2 \in \mathfrak{S}^f_C$; but $\mathfrak{X} \in \mathrm{In}(\mathfrak{R}^f_C)$.

Now for the promised proof that $\mathrm{Pr}(\mathcal{A}^f) = \mathrm{Pr}(\mathcal{A}^f_C) = \emptyset$. If $\langle A, f\rangle \in \mathcal{A}$ and $x \in A$, we define inductively $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$ for $n \geqslant 0$; if $f(x) = x$, we call x a *fixed point* of $\langle A, f\rangle$. It is easy to see that $\langle A, f\rangle$ is connected if and only if for every $x, y \in A$ there are numbers m, n such that $f^m(x) = f^n(y)$. A *root* is a finite connected unary algebra with a fixed point; by the preceding sentence, a root has exactly one fixed point. It is easy to see that a finite unary algebra $\mathfrak{A}$ is a root if and only if $\mathfrak{A}$ is connected and $k(\mathfrak{A}) = 1$. Consequently, Theorems 3.1, 3.2, and 3.3 show that $\mathrm{Pr}(\mathcal{A}^f) = \mathrm{Pr}(\mathcal{A}^f_C)$ and that connected non-roots are not $\mathcal{A}^f_C$-prime; thus, to show $\mathrm{Pr}(\mathcal{A}^f) = \mathrm{Pr}(\mathcal{A}^f_C) = \emptyset$, it suffices to show that roots are not $\mathcal{A}^f_C$-prime.

THEOREM 3.4. *If* $\mathfrak{A} = \langle A, f\rangle$ *is a root, then* $\mathfrak{A} \notin \mathrm{Pr}(\mathcal{A}^f_C)$.

Proof. Suppose a_0 is the fixed point of $\mathfrak{A}$. For each $x \in A$, define the *depth* of x, or $d(x)$, to be the least integer n such that $f^n(x) = a_0$. Let $d(\mathfrak{A}) = \max\{d(x): x \in A\}$. If $d(\mathfrak{A}) = 0$, then $\#(A) = 1$ and $\mathfrak{A}$ is not prime, by definition. So we may assume $d(\mathfrak{A}) = k+1$ for some $k \geqslant 0$. Choose some $\hat{x} \in A$ so that $d(\hat{x}) = k+1$. Choose some new object $t \notin A$ and extend f to $A \cup \{t\}$ by defining $f(t) = \hat{x}$. Finally, define $p(0) = 0$ and $p(n+1) = n$ for $n \geqslant 0$. Now consider the algebras $\mathfrak{B} = \langle B, g\rangle$, $\mathfrak{C} = \langle C, h\rangle$, and $\mathfrak{D} = \langle D, j\rangle$ given by the following formulas:

$$B = \{\langle r, y_0, \ldots, y_{k+1}, z\rangle: 0 \leqslant r \leqslant k, y_i \in A, d(y_i) \leqslant i, \text{ and } z \in A \cup \{t\}\} \cup$$
$$\cup \{\langle y_0, \ldots, y_k, z\rangle: y_i \in A, d(y_i) \leqslant i, \text{ and } z \in A \cup \{t\}\}.$$

$$C = \{\langle r, y\rangle: 0 \leqslant r \leqslant k \text{ and } y \in A\} \cup \{u\},$$

where u is some fixed *new* object.

$$D = \{\langle y_0, \ldots, y_{k+1}, z\rangle: y_i \in A, d(y_i) \leqslant i, \text{ and } z \in A \cup \{t\}\}.$$

$$g\langle r, y_0, \ldots, y_{k+1}, z\rangle = \langle p(r), f(y_1), \ldots, f(y_{k+1}), f(z), t\rangle .$$

$$g\langle y_0, \ldots, y_k, z\rangle = \begin{cases} \langle k, f(y_1), \ldots, f(y_k), f(y_0), f(z), t\rangle & \text{if } k > 0; \\ \langle 0, f(y_0), f(z), t\rangle & \text{if } k = 0 . \end{cases}$$

$$h\langle r, y\rangle = \langle p(r), f(y)\rangle .$$

$$h(u) = \langle k, a_0\rangle .$$

$$j\langle y_0, \ldots, y_{k+1}, z\rangle = \langle f(y_1), \ldots, f(y_{k+1}), f(z), t\rangle .$$

It is easy to see that $\mathfrak{B}, \mathfrak{C}, \mathfrak{D} \in \mathcal{A}_C^f$; in fact, they are roots. Define Φ: $A \times B \to C \times D$ by:

$$\Phi\langle x, \langle r, y_0, \ldots, y_{k+1}, z\rangle\rangle = \begin{cases} \langle\langle r, y_r\rangle, \langle y_0, \ldots, y_{r-1}, x, y_{r+1}, \ldots, y_{k+1}, z\rangle\rangle & \\ & \text{if } d(x) \leqslant r, \\ \langle\langle r, x\rangle, \langle y_0, \ldots, y_{k+1}, z\rangle\rangle & \text{if } d(x) > r . \end{cases}$$

$$\Phi\langle x, \langle y_0, \ldots, y_k, z\rangle\rangle = \langle u, \langle y_0, \ldots, y_k, x, z\rangle\rangle .$$

A straightforward computation shows that Φ is an isomorphism of $\mathfrak{A} \times \mathfrak{B}$ onto $\mathfrak{C} \times \mathfrak{D}$. (Note that $f(y_0) = f(y_1) = y_0 = a_0$ when $d(y_0) = 0$ and $d(y_1) \leqslant 1$.)

Now $\#(C) = (k+1)\cdot\#(A)+1$, so since $\#(A) > 1$ we get *not* $\#(A)|\#(C)$, and hence *not* $\mathfrak{A}|\mathfrak{C}$. Suppose $\mathfrak{A}|\mathfrak{D}$, or $\mathfrak{A} \times \mathfrak{E} \cong \mathfrak{D}$ for some $\mathfrak{E} = \langle E, e\rangle$; then necessarily $\mathfrak{E}$ is a root. Let $E' = \{w \in E\colon e^{k+1}(w) = e_0\}$ and $D' = \{w \in D\colon j^{k+1}(w) = d_0\}$, where e_0 and $d_0 = \langle a_0, f^k(\hat{x}), \ldots, f(\hat{x}), \hat{x}, t\rangle$ are the fixed points of $\mathfrak{E}$ and $\mathfrak{D}$, respectively; let $\mathfrak{E}', \mathfrak{D}'$ be the subalgebras of $\mathfrak{E}, \mathfrak{D}$ whose universes are E', D'. Since $d(\mathfrak{A}) = k+1$ and $d(\mathfrak{E}') \leqslant k+1$, it is clear that $\mathfrak{A} \times \mathfrak{E}' \cong \mathfrak{D}'$ under the given isomorphism. Now notice that $D' = \{\langle y_0, \ldots, y_{k+1}, t\rangle\colon y_i \in A \text{ and } d(y_i) \leqslant i\}$, since $f^k(\hat{x}) \neq a_0$. Define $R(\mathfrak{A}) = \{\langle u, v\rangle \in A \times A\colon f^k(u) = f^k(v)\}$, and similarly define $R(\mathfrak{E}')$, $R(\mathfrak{D}')$. Obviously $R(\mathfrak{A})$, $R(\mathfrak{E}')$, and $R(\mathfrak{D}')$ are equivalence relations. Moreover, $\langle\langle y_0, \ldots, y_{k+1}, t\rangle, \langle y_0', \ldots, y_{k+1}', t\rangle\rangle \in R(\mathfrak{D}')$ if and only if $\langle y_{k+1}, y_{k+1}'\rangle \in R(\mathfrak{A})$. Thus $\#\big(A/R(\mathfrak{A})\big) = \#\big(D'/R(\mathfrak{D}')\big)$; on the other hand, clearly

$$\#\big(A/R(\mathfrak{A})\big)\cdot\#\big(E'/R(\mathfrak{E}')\big) = \#\big(D'/R(\mathfrak{D}')\big) .$$

So $\#\big(E'/R(\mathfrak{E}')\big) = 1$, i.e. $e^k(w) = e_0$ for *all* $w \in E'$. Hence $d(\mathfrak{E}) \leqslant k$, and $d(\mathfrak{D}) = d(\mathfrak{A} \times \mathfrak{E}) = \max\{d(\mathfrak{A}), d(\mathfrak{E})\} = k+1$. But this is a contradiction, for $j^{k+1}\langle a_0, a_0, a_0, \ldots, a_0\rangle \neq d_0$ and so $d(\mathfrak{D}) > k+1$. Hence *not* $\mathfrak{A}|\mathfrak{D}$, and $\mathfrak{A} \notin \Pr(\mathcal{A}_C^f)$.

As mentioned above, results in a forthcoming paper by R. McKenzie imply that $\Pr(\mathcal{R}_C^f) = \mathrm{In}(\mathcal{R}_C^f)$; and, furthermore, that every structure in $\mathcal{R}_C^f$ has a unique (except for order) factorization into indecomposables. Our second main goal in this section is to characterize $\Pr(\mathcal{R}^f)$; using the just-mentioned results, we shall show that $\Pr(\mathcal{R}^f) = \mathrm{In}(\mathcal{R}_C^f) \cup \mathfrak{J}$, where $\mathfrak{J} = \{\mathfrak{X}\colon \mathfrak{X} \cong \mathfrak{I}_k \text{ for some prime } k\}$. Notice that the members of $\mathfrak{J}$ are the only *disconnected* primes we have found in this study. Since $\mathrm{In}(\mathcal{R}_C^f) = \Pr(\mathcal{R}_C^f)$ and $\Pr(\mathcal{R}_C^f) \subseteq \Pr(\mathcal{R}^f)$ (Theorem 3.3), we have $\mathrm{In}(\mathcal{R}_C^f) \subseteq \Pr(\mathcal{R}^f)$; we proceed to show $\mathfrak{J} \subseteq \Pr(\mathcal{R}^f)$.

LEMMA 3.5. *If $\mathfrak{X} \in \mathfrak{W}^f$ and $k > 0$, we have $\mathfrak{I}_k|\mathfrak{X}$ if and only if for every ideal $\mathfrak{Y}$ of $\mathfrak{X}$, the number of ideals of $\mathfrak{X}$ which are isomorphic to $\mathfrak{Y}$ is a multiple of k.*

Proof. This is obvious.

LEMMA 3.6. *Suppose that each of $\{\mathfrak{X}_i\colon i \in I\}$, $\{\mathfrak{Y}_j\colon j \in J\}$ is a family of mutually non-isomorphic $\mathcal{R}^f$-structures. Then there exist $i_0 \in I$, $j_0 \in J$ such that $\mathfrak{X}_{i_0} \times \mathfrak{Y}_{j_0} \ncong \mathfrak{X}_i \times \mathfrak{Y}_j$ whenever either $i \neq i_0$ or $j \neq j_0$.*

Proof. In [3], László Lovász shows that to each $\mathfrak{Z} \in \mathcal{R}^f$ we may associate a denumerable sequence $\Lambda(\mathfrak{Z})$ of positive integers in such a way that two structures are assigned different sequences if and only if they are non-isomorphic; and for all n, the nth component of the sequence $\Lambda(\mathfrak{Z} \times \mathfrak{W})$ is the product of the nth components of $\Lambda(\mathfrak{Z})$ and $\Lambda(\mathfrak{W})$. If $\mathfrak{Z}, \mathfrak{W} \in \mathcal{R}^f$, let us say $\mathfrak{Z} < \mathfrak{W}$ if and only if either $\#(|\mathfrak{Z}|) < \#(|\mathfrak{W}|)$, or else $\#(|\mathfrak{Z}|) = \#(|\mathfrak{W}|)$ but $\Lambda(\mathfrak{Z})$ is *lexicographically* less than $\Lambda(\mathfrak{W})$; further, say $\mathfrak{Z} \leqslant \mathfrak{W}$ if either $\mathfrak{Z} < \mathfrak{W}$ or $\mathfrak{Z} \cong \mathfrak{W}$. It is easy to see that if $\mathfrak{Z} < \mathfrak{W}$ and $\mathfrak{X} \in \mathcal{R}^f$, we have $(\mathfrak{Z} \times \mathfrak{X}) < (\mathfrak{W} \times \mathfrak{X})$. Furthermore, since $\{\mathfrak{X}_i\colon i \in I\}$ is a family of mutually non-isomorphic structures, it must have a $<$-least member; call it $\mathfrak{X}_{i_0}$. Similarly, $\{\mathfrak{Y}_j\colon j \in J\}$ has a $<$-least member $\mathfrak{Y}_{j_0}$. Thus, if either $i \neq i_0$ or $j \neq j_0$, we have $\mathfrak{X}_{i_0} \leqslant \mathfrak{X}_i$ and $\mathfrak{Y}_{j_0} \leqslant \mathfrak{Y}_j$, and one of these inequalities must be strict; consequently $(\mathfrak{X}_{i_0} \times \mathfrak{Y}_{j_0}) \leqslant (\mathfrak{X}_{i_0} \times \mathfrak{Y}_j) \leqslant (\mathfrak{X}_i \times \mathfrak{Y}_j)$ and at least one of these inequalities must be strict. So the result follows.

THEOREM 3.7. *If k is prime, then $\mathfrak{I}_k \in \Pr(\mathcal{R}^f)$.*

Proof. Of course $\mathfrak{I}_k \in \mathcal{R}^f$ and $k > 1$. Suppose $\mathfrak{Y}, \mathfrak{Z} \in \mathcal{R}^f$ and neither $\mathfrak{I}_k|\mathfrak{Y}$ nor $\mathfrak{I}_k|\mathfrak{Z}$. Let $D(\mathfrak{W})$ be the set of all ideals of $\mathfrak{W}$ for any $\mathfrak{W}$, and if $\mathfrak{V}$ is an ideal of $\mathfrak{W}$, let $s(\mathfrak{V}, \mathfrak{W})$ be the cardinal number of the set of all deals of $\mathfrak{W}$ isomorphic to $\mathfrak{V}$. Now it is obvious that

$$D(\mathfrak{Y} \times \mathfrak{Z}) = \{\mathfrak{Y}' \times \mathfrak{Z}'\colon \mathfrak{Y}' \in D(\mathfrak{Y}) \text{ and } \mathfrak{Z}' \in D(\mathfrak{Z})\} .$$

Let $\mathfrak{Y}_0, \ldots, \mathfrak{Y}_m$ be a list of mutually non-isomorphic ideals of $\mathfrak{Y}$ such that, if $\mathfrak{W}$ is an ideal of $\mathfrak{Y}$ and $s(\mathfrak{W}, \mathfrak{Y})$ is not divisible by k, then $\mathfrak{W}$ is isomorphic to some $\mathfrak{Y}_i$ on the list. Let $\mathfrak{Z}_0, \ldots, \mathfrak{Z}_n$ be a similar list for $\mathfrak{Z}$. Both these lists are non-empty by Lemma 3.5. By Lemma 3.6, we may assume that $\mathfrak{Y}_0 \times \mathfrak{Z}_0 \ncong \mathfrak{Y}_i \times \mathfrak{Z}_j$ whenever either $i \neq 0$ or $j \neq 0$. Now let $\langle\mathfrak{Y}_0', \mathfrak{Z}_0'\rangle, \ldots, \langle\mathfrak{Y}_q', \mathfrak{Z}_q'\rangle$ be a list of all those pairs $\langle\mathfrak{Y}', \mathfrak{Z}'\rangle$ in $D(\mathfrak{Y}) \times D(\mathfrak{Z})$ for which we have that $\mathfrak{Y}' \times \mathfrak{Z}' \cong \mathfrak{Y}_0 \times \mathfrak{Z}_0$ but either $\mathfrak{Y}' \ncong \mathfrak{Y}_0$ or $\mathfrak{Z}' \ncong \mathfrak{Z}_0$. Then for each $r \leqslant q$, by the above either $\mathfrak{Y}_r'$ is not isomorphic to any of $\mathfrak{Y}_0, \ldots, \mathfrak{Y}_m$ or else $\mathfrak{Z}_r'$ is not isomorphic to any of $\mathfrak{Z}_0, \ldots, \mathfrak{Z}_n$; but this means that one of $s(\mathfrak{Y}_r', \mathfrak{Y})$, $s(\mathfrak{Z}_r', \mathfrak{Z})$ is divisible by k. Hence the number of ideals of $\mathfrak{Y}\times\mathfrak{Z}$ which are isomorphic to $\mathfrak{Y}_0\times\mathfrak{Z}_0$, but are not the product of an ideal of $\mathfrak{Y}$ isomorphic to $\mathfrak{Y}_0$ and an ideal of $\mathfrak{Z}$ isomorphic to $\mathfrak{Z}_0$, is divisible by k; i.e.

$$s(\mathfrak{Y}_0\times\mathfrak{Z}_0, \mathfrak{Y}\times\mathfrak{Z}) \equiv s(\mathfrak{Y}_0, \mathfrak{Y})\cdot s(\mathfrak{Z}_0, \mathfrak{Z}) \pmod{k}.$$

But $s(\mathfrak{Y}_0, \mathfrak{Y})\cdot s(\mathfrak{Z}_0, \mathfrak{Z})$ is not divisible by k, so neither is $s(\mathfrak{Y}_0\times\mathfrak{Z}_0, \mathfrak{Y}\times\mathfrak{Z})$. By Lemma 3.5, *not* $\mathfrak{I}_k|(\mathfrak{Y}\times\mathfrak{Z})$. So $\mathfrak{I}_k \in \mathrm{Pr}(\mathfrak{R}^f)$.

Our characterization is completed by the following result, the proof of which was inspired by an example given by Hashimoto and Nakayama in [2].

THEOREM 3.8. $\mathrm{Pr}(\mathfrak{R}^f) \subseteq \mathrm{In}(\mathfrak{R}_C^f) \cup \mathfrak{I}$.

Proof. Suppose $\mathfrak{X} \in \mathrm{Pr}(\mathfrak{R}^f)$ but $\mathfrak{X} \notin \mathrm{In}(\mathfrak{R}_C^f)$; we show $\mathfrak{X} \in \mathfrak{I}$. Obviously $\mathrm{Pr}(\mathfrak{R}^f) \subseteq \mathrm{In}(\mathfrak{R}^f)$ and $\mathfrak{R}_C^f \cap \mathrm{In}(\mathfrak{R}^f) \subseteq \mathrm{In}(\mathfrak{R}_C^f)$, so we must have $\mathfrak{X} \notin \mathfrak{R}_C^f$. Thus $\mathfrak{X}$ has at least two ideals. Now *assume* that $\mathfrak{X} \notin \mathfrak{I}$; then at least one ideal of $\mathfrak{X}$ has an indecomposable factor, say $\mathfrak{W}$, with $\#(|\mathfrak{W}|) > 1$. Since $\mathfrak{X} \in \mathrm{In}(\mathfrak{R}^f)$, $\mathfrak{W}$ cannot be a factor of *all* the ideals of $\mathfrak{X}$. So we have $\mathfrak{X} \cong (\mathfrak{V}\times\mathfrak{W})+\mathfrak{Z}$ for some $\mathfrak{V}, \mathfrak{Z} \in \mathfrak{R}^f$ such that no ideal of $\mathfrak{Z}$ is a multiple of $\mathfrak{W}$ (as before "+" is the cardinal sum). Let $\mathfrak{Y} = \mathfrak{V}\times\mathfrak{W}$. Now we have $\mathfrak{X}|((\mathfrak{Y}^3+\mathfrak{Z}^3)\times(\mathfrak{Y}^2+(\mathfrak{Y}\times\mathfrak{Z})+\mathfrak{Z}^2))$, since in fact

$$(\mathfrak{Y}+\mathfrak{Z})\times(\mathfrak{Y}^4+(\mathfrak{Y}^2\times\mathfrak{Z}^2)+\mathfrak{Z}^4) \cong (\mathfrak{Y}^3+\mathfrak{Z}^3)\times(\mathfrak{Y}^2+(\mathfrak{Y}\times\mathfrak{Z})+\mathfrak{Z}^2)$$

(where $\mathfrak{Y}^n$ is the cardinal nth power of $\mathfrak{Y}$, etc.). Since $\mathfrak{X} \in \mathrm{Pr}(\mathfrak{R}^f)$, we have either $\mathfrak{X}|(\mathfrak{Y}^3+\mathfrak{Z}^3)$ or $\mathfrak{X}|(\mathfrak{Y}^2+(\mathfrak{Y}\times\mathfrak{Z})+\mathfrak{Z}^2)$. But if $\mathfrak{X}|(\mathfrak{Y}^2+(\mathfrak{Y}\times\mathfrak{Z})+\mathfrak{Z}^2)$, then since $\mathfrak{X} \cong \mathfrak{Y}+\mathfrak{Z}$ and $\mathfrak{Y}^2+(\mathfrak{Y}\times\mathfrak{Z}) \cong \mathfrak{Y}\times(\mathfrak{Y}+\mathfrak{Z})$, we get $\mathfrak{X}|((\mathfrak{Y}\times\mathfrak{X})+\mathfrak{Z}^2)$; hence $\mathfrak{X}|\mathfrak{Z}^2$. But this is impossible since none of the ideals of $\mathfrak{Z}$ have $\mathfrak{W}$ as a factor, while at least one of the ideals of $\mathfrak{X}$ does. Therefore $\mathfrak{X}|(\mathfrak{Y}^3+\mathfrak{Z}^3)$, or $\mathfrak{X}\times\mathfrak{P} \cong \mathfrak{Y}^3+\mathfrak{Z}^3$ for some $\mathfrak{P} \in \mathfrak{R}^f$.

Now let $\mathfrak{x}_1, \mathfrak{x}_2, \ldots$ be a sequence of mutually non-isomorphic $\mathfrak{R}_C^f$-structures such that every $\mathfrak{x}_i$ is indecomposable and every indecomposable $\mathfrak{R}_C^f$-structure is isomorphic to some $\mathfrak{x}_i$. Consider the domain of polynomials, with integer coefficients, in the indeterminates $\mathfrak{x}_1, \mathfrak{x}_2, \ldots$ If the positive integer n is interpreted as the structure $\mathfrak{I}_n$, then every such polynomial with *positive* integer coefficients corresponds to a *structure* in the obvious way; e.g., $\mathfrak{x}_1^2+3\mathfrak{x}_2$ corresponds to $\mathfrak{x}_1^2+(\mathfrak{I}_3\times\mathfrak{x}_2)$. Furthermore, since $\mathfrak{R}_C^f$-structures have the unique factorization property, every $\mathfrak{R}^f$-structure is isomorphic to a unique cardinal sum of cardinal products of the structures $\mathfrak{x}_i$; and these two correspondences are mutual inverses, so we have a sum- and product-preserving one-to-one correspondence between the isomorphism types of $\mathfrak{R}^f$-structures and those polynomials

in $\mathfrak{x}_1, \mathfrak{x}_2, \ldots$ having only positive integral coefficients. Suppose $\mathfrak{X}$, $\mathfrak{P}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ correspond to positive-coefficient polynomials X, P, Y, and Z, respectively. Now of course

$$X\cdot P = Y^3+Z^3 = (Y+Z)\cdot(Y^2-(Y\cdot Z)+Z^2) = X\cdot(Y^2-(Y\cdot Z)+Z^2).$$

Now it is well-known that the domain of polynomials in $\mathfrak{x}_1, \mathfrak{x}_2, \ldots$, with integer coefficients is a unique factorization domain; hence $P = Y^2-(Y\cdot Z)+Z^2$. That is, $Y^2-(Y\cdot Z)+Z^2$ is equal to some polynomial having only positive coefficients. Since the polynomials Y and Z have only positive coefficients, this means that every term of the polynomial $Y\cdot Z$ must be "cancelled out" by terms appearing in Y^2 or Z^2. Let $\mathfrak{w}$ be the indeterminate corresponding to $\mathfrak{W}$; then every term of Y has $\mathfrak{w}$ as a factor. Let $\mathfrak{t}$ be a term of Y which has $\mathfrak{w}$ as a factor the *least* number of times, say k times. Then, since *no* term of Z has $\mathfrak{w}$ as a factor, there is a term $\mathfrak{t}'$ of $Y\cdot Z$ which has $\mathfrak{w}$ as a factor exactly k times; but every term of Y^2 has $\mathfrak{w}$ as a factor at least $2k$ times, and no term of Z^2 has $\mathfrak{w}$ as a factor at all. So $\mathfrak{t}'$ is *not* cancelled out, contradicting our earlier conclusion. This shows that our assumption $\mathfrak{X} \notin \mathfrak{I}$ was wrong, so $\mathfrak{X} \in \mathfrak{I}$. This proves $\mathrm{Pr}(\mathfrak{R}^f) \cup \mathrm{In}(\mathfrak{R}_C^f) \subseteq \mathfrak{I}$.

COROLLARY 3.9. $\mathrm{Pr}(\mathfrak{E}^f) = \mathfrak{F}^f \cup \mathfrak{I}$ *and* $\mathrm{Pr}(\mathfrak{E}_C^f) = \mathfrak{F}^f$.

Added in proof: The "forthcoming paper by R. McKenzie" mentioned on pages 188 and 200 above will appear in Fund. Math. this volume, pp. 59-101, its title is *Cardinal multiplication of structures with a reflexive relation.*

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