An extension and axiomatic characterization of Borsuk's theory of shape

by

W. Holsztyński (Ann Arbor, Mich.)

The homotopy functor is rather a good functor for classifying compacta homotopically dominated by polyhedra and mappings into them. On the other hand, it is too delicate to be used in the classification of general compacta. Following this idea K. Borsuk [1] introduced a new classification and a new functor defined on the category of metric compact spaces into the so called shape category. This functor is, from a certain point of view, more appropriate (see § 5 of this paper). The aim of this note is a full formalization of the above ideas, i.e. axiomatization, construction and proof of uniqueness of shape category and shape functor. The startpoint is a category-functor pair satisfying some general conditions. In particular, the generalization of Borsuk's theory onto the category of all compact pairs is obtained (see Example (5.9)).

It will be convenient to use a generalize notion of category.

§ 1. Definition of category. A *category* K is a class $\mathrm{Ob}K$, together with a class $\mathrm{Mor}K$ and a partial binary operation on $\mathrm{Mor}K$, called *composition*. We postulate that $\mathrm{Mor}K$ is a union of the form

$$\mathrm{Mor}K = \bigcup_{A,B\in \mathrm{Ob}K} \mathrm{Mor}_K(A,B),$$

where each $\mathrm{Mor}_K(A,B)$ is a set. When there is no danger of confusion, we shall write $\mathrm{Mor}(A,B)$ instead of $\mathrm{Mor}_K(A,B)$. The image of the pair $(f,g)\in(\mathrm{Mor}K)^2$ under composition will be called the composition of f and g, and will be denoted by $g\circ f$. Composition is subject to three axioms.

(1.1) *The composition $g\circ f$ makes sense (is defined) iff there exist $A, B, C\in\mathrm{Ob}K$ such that $f\in\mathrm{Mor}(A,B)$ and $g\in\mathrm{Mor}(B,C)$.*

(1.2) *Whenever one of the compositions $(h\circ g)\circ f$ and $h\circ(g\circ f)$ makes a sense then the other makes a sense and $(h\circ g)\circ f = h\circ(g\circ f)$.*

(1.3) *For each $A\in\mathrm{Ob}K$ there exists an element $1_A\in\mathrm{Mor}(A,A)$ such that $1_A\circ f = f$ and $g\circ 1_A = g$ whenever the compositions make sense.*

Let K and L be categories. A *covariant* (*contravariant*) *functor* $F\colon K\to L$ is an assignement of an object $F(A) = F_{\mathrm{Ob}}(A) \in \mathrm{Ob}L$ to each object $A \in \mathrm{Ob}K$ and a morphism $F(f) = F_{\mathrm{Mor}}(f) \in \mathrm{Mor}_L(F(A), F(B))$ (respectively $F(f) \in \mathrm{Mor}_L(F(B), F(A))$) to each morphism $f \in \mathrm{Mor}_K(A, B)$, subject to the following conditions:

(1.4) *If $g\circ f$ is defined in K, then*

$$F(g\circ f) = F(g)\circ F(f) \quad (\textit{respectively } F(g\circ f) = F(f)\circ F(g))\,.$$

(1.5) $F(1_A) = 1_{F(A)}$ *for each* $A \in \mathrm{Ob}K$.

The following easy and useful theorems are not true in the classical theory of categories.

(1.6) Proposition. *Let $F\colon K\to L$ be a covariant functor. Let us put* $\mathrm{Ob}M = \mathrm{Ob}K$, $\mathrm{Mor}_M(A, B) = F(\mathrm{Mor}_K(A, B))$,

$$\mathrm{Mor}M = \bigcup_{A,B\in\mathrm{Ob}M} \mathrm{Mor}_M(A, B)\,,$$

and let the composition $g\circ f$ in $\mathrm{Mor}M$ be defined as in $\mathrm{Mor}L$ for each pair $(f, g) \in \mathrm{Mor}_M(A, B)\times\mathrm{Mor}_M(B, C)$. Then M is a category, and $F'\colon K\to M$ and $F''\colon M\to L$ given by

$$F'(A) = A, \quad F'(f) = F(f) \quad \textit{for } A\in\mathrm{Ob}K,\ f\in\mathrm{Mor}K,$$

$$F''(A) = F(A), \quad F''(f) = f \quad \textit{for } A\in\mathrm{Ob}M,\ f\in\mathrm{Mor}M$$

are the covariant functors.

(1.7) Definition (c.f. [4], p. 49). A covariant functor $E\colon C\to EC$ from C to a category EC is said to be a *projection functor* if the following three conditions hold:

(i) $\mathrm{Ob}\,C = \mathrm{Ob}\,EC$,

(ii) $E|\mathrm{Ob}\,C$ is an identity,

(iii) $E(\mathrm{Mor}_C(A, B)) = \mathrm{Mor}_{EC}(A, B)$ for every $A, B \in \mathrm{Ob}\,C$.

(1.8) Corollary. *Given a covariant functor $F\colon K\to L$, there exist a uniquely determined category EK and covariant functors $E\colon K\to EK$, $G\colon EK\to L$ such that E is a projection functor and $G(f) = f$ for each $f\in\mathrm{Mor}\,EK$.*

(1.9) Proposition. *Let K be a category, $A\in\mathrm{Ob}K$ and let $\{B\in\mathrm{Ob}K\colon f\in\mathrm{Mor}_K(A, B)\}$ be a set for each $f\in\mathrm{Mor}K$. We put*

$$\mathrm{Ob}K_A = \bigcup_{B\in\mathrm{Ob}K} \mathrm{Mor}_K(A, B)\,,$$

$$\mathrm{Mor}_{K_A}(f, g) = \{h\in\mathrm{Mor}K\colon h\circ f = g\}\,,$$

$$\mathrm{Mor}K_A = \bigcup_{f,g\in\mathrm{Ob}K_A} \mathrm{Mor}_{K_A}(f, g)\,;$$

then K_A, under the composition of morphisms which is given as in K, is a category.

Now we shall establish a relation between our definition and the classical definition of a category; a category in the sense of this paper is like a classical category with an additional structure.

(1.10) Theorem. *Let K be a category. Let us put*

$$\mathrm{Ob}L = \mathrm{Ob}K\,,$$

$$\mathrm{Mor}_L(A, B) = \mathrm{Mor}_K(A, B)\times\{(A, B)\}\,,$$

$$\mathrm{Mor}L = \bigcup_{A,B\in\mathrm{Ob}_L} \mathrm{Mor}_L(A, B)\,,$$

and

$$(g, (B, C))\circ(f, (A, B)) = (g\circ f, (A, C))$$

for every $f\in\mathrm{Mor}_K(A, B)$, $g\in\mathrm{Mor}_K(B, C)$ and $A, B, C\in\mathrm{Ob}K$. We assume also that $(g, (B', C))\circ(f, (A, B))$ is defined in $\mathrm{Mor}L$ iff $B = B'$. Then $L = L(K)$ is a classical category (i.e. L is a category such that $\mathrm{Mor}_L(A, B)$ and $\mathrm{Mor}_L(A', B')$ are disjoint sets for every different pair $(A, B), (A', B')$).

Furthermore, the equivalence relation $\underset{K}{\sim}$ defined on $\mathrm{Mor}L$ by

$$(f, (A, B))\underset{K}{\sim}(g, (C, D)) \quad \text{iff } f = g$$

is a congruence for a composition "$\circ$" such that if (*) $h\in\mathrm{Mor}_L(A, B)$, $h'\in\mathrm{Mor}_L(A', B')$ and $h\neq h'\underset{K}{\sim}h$, then $(A, B)\neq(A', B')$. On the other hand, if L is a category (classical or not) with a congruence $\sim$ defined on $\mathrm{Mor}L$, then $L/\!\sim\, = (\mathrm{Ob}L, \mathrm{Mor}L/\!\sim)$ is a category. Any category K is obtained in such a way from a classical category; this means that category K is isomorphic to $L(K)/\underset{K}{\sim}$.

The following property of a category K is trivial in the classical case.

(1.11) Proposition. *If $e\in\mathrm{Mor}(A, B)$ is a unit (i.e. $e\circ f = f$ and $g\circ e = g$ whenever the compositions make sense), then*

$$e = 1_A = 1_B \in \mathrm{Mor}(A, A)\cap\mathrm{Mor}(B, B)\,.$$

§ 2. E-objects and E-category. For any category C and object Y of C there is a contravariant functor M_C^Y from C to the category of sets and functions which assigns to an object X of C the set $M_C^Y(X) = \mathrm{Mor}(X, Y)$ and to a morphism $f\in\mathrm{Mor}(X, X')$ the function $M_C^Y f\colon \mathrm{Mor}(X', Y)\to\mathrm{Mor}(X, Y)$ defined by $(M_C^Y f)(g) = g\circ f$ for $g\in\mathrm{Mor}(X', Y)$.

A covariant (contravariant) functor $F\colon K\to L$ is called a *continuous functor for inverse systems* if $(F(X), F(p_t)\colon t\in T)$ is a representation of $F(X)$ as a limit of an inverse (direct) system $(F(X_t), F(p_t^u)\colon t < u \in T)$ in L whenever $(X, p_t\colon t\in T)$ is a representation of X as a limit of an inverse system $(X_t, p_t^u\colon t < u\in T)$ in K.

(2.1) DEFINITION. Let $E\colon C\to D$ be a projection functor. An object Y of C is said to be an *E-object* if $M_D^Y\circ E\colon C\to\mathrm{Ens}$ is a continuous contravariant functor into the category of sets.

This means that Y is an E-object iff for every representation $(X, p_t\colon t\in T)$ of $X\in\mathrm{Ob}\,C$ as a limit of an inverse system $(X_t, p_t^u\colon t<u\in T)$ the following two conditions hold:

(2.2) if $f\in\mathrm{Mor}_C(X, Y)$, then there exist a $t\in T$ and $f'\in\mathrm{Mor}_C(X_t, Y)$ such that

$$E(f')\circ E(p_t)=E(f);$$

(2.3) if $f, g\in\mathrm{Mor}_C(X_t, Y)$ and $E(f)\circ E(p_t)=E(g)\circ E(p_t)$, then

$$E(f)\circ E(p_t^u)=E(g)\circ E(p_t^u) \text{ for an } u>t.$$

Let us recall that an object Y is said to be an *r-image* of X if there exist morphisms $f\in\mathrm{Mor}(X, Y)$ and $g\in\mathrm{Mor}(Y, X)$ such that $f\circ g=1_Y$. In such a case f is called an *r-morphism* and g is called an *l-morphism*.

(2.4) PROPOSITION. *Let $E\colon C\to D$ be a projection functor. If, in category D, Y is an r-image of an E-object X, then Y is also an E-object.*

(2.5) DEFINITION. A category C is said to be an *E-category* if any object of C is an inverse limit of E-objects.

(2.6) PROPOSITION. *An object of an E-category C is an E-object if and only if it possesses property* (2.2).

Proof. If an object X of an E-category C possesses property (2.2), then

$$E(f')\circ E(p_t)=E(1_X),$$

where $(X, p_t\colon t\in T)$ is a representation of X as a limit of an inverse system of E-objects $X_t, t\in T$. Hence X is an r-image of X_t, and we can use Proposition (2.4).

Let us denote by $E-\mathrm{Ob}\,C$, or more briefly by $E-\mathrm{Ob}$, the class of all E-objects of category C.

(2.7) PROPOSITION. *Let $E\colon C\to D$ be a projection functor and let OC be a class of E-objects such that each object in C is an inverse limit of objects of OC. Then an object X of C is an E-object iff there exists a $Y\in OC$ such that X is an r-image of Y in D.*

§ 3. Axiomatic definition of Borsuk's Theory of Shape. Borsuk's Theory of Shape, briefly BTS, is a system (C, EC, SC, E, F), where C, EC, SC are categories and

$$E\colon C\to EC, \quad F\colon EC\to SC$$

are covariant functors such that the following axioms hold:

(3.1) $\mathrm{Ob}\,C=\mathrm{Ob}\,EC=\mathrm{Ob}\,SC$,

(3.2) $E|\mathrm{Ob}\,C=F|\mathrm{Ob}\,C=\text{identity}$,

(3.3) $E\big(\mathrm{Mor}_C(A, B)\big)=\mathrm{Mor}_{EC}(A, B)$ for every $A, B\in\mathrm{Ob}\,C$,

(3.4) C is an E-category,

(3.5) $F|\mathrm{Mor}_{EC}(X, Y)\colon \mathrm{Mor}_{EC}(X, Y)\to\mathrm{Mor}_{SC}(X, Y)$ is a 1-1 mapping for every object X and E-object Y of C.

The functor $S=F\circ E$ is said to be the *functor of shape*. It follows from axioms (3.1), (3.2), (3.3) that E is a projection functor.

(3.6) EXAMPLE. We obtain the trivial BTS if we put $EC=SC$ and $F=\text{identity}$.

A BTS is said to be a *full* BTS if

(3.7) for any different morphisms $f, g\in\mathrm{Mor}_{SC}(X, Y)$ there exist an E-object Z and morphism $h\colon Y\to Z$ in SC such that

$$h\circ f\neq h\circ g.$$

A full BTS is said to be a *continuous* BTS if

(3.8) for any representation $(Y, p_t\colon t\in T)$ of Y as a limit in C of an inverse system $(Y_t, p_t^u\colon t<u\in T)$ of E-objects Y_t and for the morphisms $f_t\in\mathrm{Mor}_{SC}(X, Y_t)$ such that

$$(Sp_t^u)\circ f_u=f_t \quad \text{for any } t<u\in T$$

there exists a morphism $f\in\mathrm{Mor}_{SC}(X, Y)$ such that

$$(Sp_t)\circ f=f_t \quad \text{for any } t\in T.$$

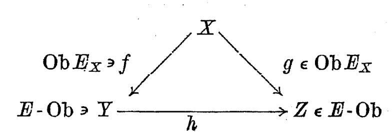
§ 4. A natural continuous BTS for a given pair (C, E). Let $E\colon C\to EC$ be a projection functor defined on an E-category C. We shall define a category E_X for each object X of C. Let us put

$$\mathrm{Ob}\,E_X=\bigcup\{\mathrm{Mor}_{EC}(X, Y)\colon\ Y\in E-\mathrm{Ob}\}$$

and

$$\mathrm{Mor}_{E_X}(f\colon X\to Y,\ g\colon X\to Z)=\mathrm{Mor}_{E_X}(f, g)=\{h\in\mathrm{Mor}_{EC}(Y, Z)\colon\ h\circ f=g\}$$

for any E-objects Y, Z. The composition of morphisms in E_X is as in EC.



(4.1) Lemma. *Let* $(X, p_t\colon t \in T)$ *be a representation of* X *as a limit of an inverse system* $(X_t, p_t^u\colon t < u \in T)$ *of* E*-objects in category* C. *If* $G, H\colon E_X \to E_Y$ *are covariant functors such that*

(i) $$G(f) = H(f) = f \quad \text{for any morphism } f \text{ in } E_X,$$

(ii) $$G\big(E(p_t)\big) = H\big(E(p_t)\big) \quad \text{for any } t \in T,$$

then $G = H$, *for arbitrary object* Y *of* C.

Proof. Let $g \in \operatorname{Ob} E_X$. Then $g \in \operatorname{Mor}_{EC}(X, Z)$ for an E-object Z. It follows from property (2.2) of E-objects that $g = g' \circ E(p_t)$ for an index $t \in T$ and $g' \in \operatorname{Mor}_{EC}(X_t, Z)$. Then $g' \in \operatorname{Mor}_{E_X}\big(E(p_t), g\big)$ and, by (i) and (ii),

$$g' \in \operatorname{Mor}_{E_Y}\Big(G\big(E(p_t)\big), G(g)\Big) \cap \operatorname{Mor}_{E_Y}\Big(G\big(E(p_t)\big), H(g)\Big).$$

Thus $G(g) = H(g) = g' \circ G\big(E(p_t)\big)$.

(4.2) Corollary. *The class of all (covariant) functors* $G\colon E_X \to E_Y$ *such that*

(*) $$G(f) = f \text{ for each morphism } g \text{ of } E_X$$

is a well-defined set.

Now we can define a category SC and a functor $F\colon EC \to SC$. We put

$$\operatorname{Ob} SC = \operatorname{Ob} C$$

and

$$\operatorname{Mor}_{SC}(X, Y) = \{G\colon E_Y \to E_X\colon G \text{ is a functor with property } (*)\}.$$

It follows from Corollary (4.2) that $\operatorname{Mor}_{SC}(X, Y)$ is a set.

The composition of a morphism in SC is the respective composition of functors (with converse succession).

From now on let us suppose about EC that

(4.3) if $\operatorname{Mor}_{EC}(A, B) \cap \operatorname{Mor}_{EC}(A', B') \neq \emptyset$ then $B = B'$.

Then we put

$$F(X) = X \quad \text{for every } X \in \operatorname{Ob} EC,$$

and for $f \in \operatorname{Mor}_{EC}(X, Y)$ we define $\big(F(f)\colon E_Y \to E_X\big) \in \operatorname{Mor}_{SC}(X, Y)$ as a functor satisfying (*) and such that

(4.4) $\big(F(f)\big)_{\mathrm{Ob}}(g) = g \circ f$ for $(g\colon Y \to Z) \in \operatorname{Ob} E_Y$, $Z \in E\text{-}\operatorname{Ob} C$.

By property (4.3) of EC, the definition of $F(f)$ does not depend on the pair (X, Y). Also the following proposition is a direct consequence of this property.

(4.5) Proposition. *Let* $G \in \operatorname{Mor}_{SC}(X, Y)$ *and* $f \in \operatorname{Mor}_{EC}(Y, Z)$ *for an* E*-object* Z *(consequently* $f \in \operatorname{Ob} E_Y$*). Then* $G_{\mathrm{Ob}}(f) \in \operatorname{Mor}_{EC}(X, Z)$.

Proof. Evidently $1_f = 1_Z$. By (*) $1_Z = G(1_f) = 1_{G(f)}$. Then the composition $1_Z \circ G_{\mathrm{Ob}}(f) = G_{\mathrm{Ob}}(f)$ is defined. Thus there exist X', Z', Z'' such that

$$G_{\mathrm{Ob}}(f) \in \operatorname{Mor}_{EC}(X', Z') \quad \text{and} \quad 1_Z \in \operatorname{Mor}_{EC}(Z', Z'').$$

Then, by (1.11), $1_Z \in \operatorname{Mor}_{EC}(Z', Z')$. Thus, by (4.3), $Z = Z'$ and $G_{\mathrm{Ob}}(f) \in \operatorname{Mor}_{EC}(X', Z)$. There exists also a Z''' such that $G_{\mathrm{Ob}}(f) \in \operatorname{Mor}_{EC}(X, Z''')$, as $G_{\mathrm{Ob}}(f) \in \operatorname{Ob} E_X$. Thus, by (4.3), $Z''' = Z$. The proof of Proposition (4.5) is finished.

Evidently, Axioms (3.1)–(3.4) are satisfied for SC and F defined above. Next, if Y is an E-object, then the mapping of $\operatorname{Mor}_{SC}(X, Y)$ onto $\operatorname{Mor}_{EC}(X, Y)$ given by

$$G \to G_{\mathrm{Ob}}(1_Y)$$

is converse to mapping $F|\operatorname{Mor}_{EC}(X, Y)\colon \operatorname{Mor}_{EC}(X, Y) \to \operatorname{Mor}_{SC}(X, Y)$.

Indeed, by Proposition (4.5), $G_{\mathrm{Ob}}(1_Y) \in \operatorname{Mor}_{EC}(X, Y)$. Let $g \in \operatorname{Ob} E_Y$. Then $g \in \operatorname{Mor}_{E_Y}(1_Y, g)$ and

$$\Big(F\big(G_{\mathrm{Ob}}(1_Y)\big)\Big)_{\mathrm{Ob}}(g) = g \circ G_{\mathrm{Ob}}(1_Y) = G_{\mathrm{Mor}}(g) \circ G_{\mathrm{Ob}}(1_Y) = G_{\mathrm{Ob}}(g),$$

as

$$g = G_{\mathrm{Mor}}(g) \in \operatorname{Mor}_{E_X}\big(G_{\mathrm{Ob}}(1_Y), G_{\mathrm{Ob}}(g)\big).$$

Thus $F\big(G_{\mathrm{Ob}}(1_Y)\big) = G$.

On the other hand, for any $f \in \operatorname{Mor}_{EC}(X, Y)$ we have

$$\big(F(f)\big)_{\mathrm{Ob}}(1_Y) = 1_Y \circ f = f.$$

Thus Axiom (3.5) is also satisfied and the system (C, EC, SC, E, F) is a BTS.

(4.6) Proposition. *Let* $G \in \operatorname{Mor}_{SC}(X, Y)$ *and* $f \in \operatorname{Mor}_{EC}(Y, Z)$ *for an* E*-object* Z. *Then*

$$F(f) \circ G = F\big(G_{\mathrm{Ob}}(f)\big).$$

Proof. $\big(F(f) \circ G\big)_{\mathrm{Ob}}(1_Z) = G_{\mathrm{Ob}}\Big(\big(F(f)\big)_{\mathrm{Ob}}(1_Z)\Big) = G_{\mathrm{Ob}}(f) = 1_Z \circ G_{\mathrm{Ob}}(f) = \Big(F\big(G_{\mathrm{Ob}}(f)\big)\Big)_{\mathrm{Ob}}(1_Z)$.

Thus Proposition (4.6) holds (see above for the proof of Axiom (3.5)).

Now let $G, H \in \operatorname{Mor}_{SC}(X, Y)$ and $G \neq H$. Then $G_{\mathrm{Ob}}(f) \neq H_{\mathrm{Ob}}(f)$ for an E-object Z and a morphism $f \in \operatorname{Mor}_{EC}(Y, Z)$.

Thus, by Propositions (4.5), (4.6), and Axiom (3.5)

$$F(f) \circ G = F\big(G_{\mathrm{Ob}}(f)\big) \neq F\big(H_{\mathrm{Ob}}(f)\big) = F(f) \circ H.$$

This means that the BTS under consideration is full, i.e. that (3.7) holds.

(4.7) THEOREM. *The described functor $S\colon C \to SC$ is continuous for inverse systems.*

Proof. Let $(Y, p_t\colon t \in T)$ be a representation of Y as a limit of an inverse system $(Y_t, p_t^u\colon t < u \in T)$ in C and let

$$f_t \in \mathrm{Mor}_{SC}(X, Y_t) \quad \text{and} \quad f_t = S(p_t^u) \circ f_u$$

for every $t < u \in T$.

Then, by Definition (2.1) of an E-object, $\big(M_{EC}^Z(Y), M_{EC}^Z \circ E(p_t)\colon t \in T\big)$ is a representation of $M_{EC}^Z \circ E(Y)$ as the limit of the direct system $\big(M_{EC}^Z(Y_t), M_{EC}^Z \circ E(p_t^u)\colon t < u \in T\big)$ in Ens for every E-object Z. Thus, for such Z there exists exactly one mapping of sets $f^Z\colon \mathrm{Mor}_{EC}(Y, Z) \to \mathrm{Mor}_{EC}(X, Z)$ such that

(4.8) $(f_t)_{\mathrm{Ob}}|\mathrm{Mor}_{EC}(Y_t, Z) = f^Z \circ \big((M_{EC}^Z \circ E)(p_t)\big)$ for every $t \in T$.

If $f \in \mathrm{Mor}_{SC}(X, Y)$ is the limit of f_t, i.e. if we have

(4.9) $f_t = S(p_t) \circ f$ for each $t \in T$,

then for every E-object Z and for $f^Z = f_{\mathrm{Ob}}|\mathrm{Mor}_{EC}(Y, Z)$ condition (4.8) holds (as $\big(S(p_t)\big)_{\mathrm{Ob}}(g) = \big(F \circ E(p_t)\big)(g) = g \circ E(p_t) = \big((M_{EC}^Z \circ E)(p_t)\big)(g)$ for $g \in \mathrm{Mor}_{EC}(Y_t, Z)$). Thus there exists at most one $f \in \mathrm{Mor}_{SC}(X, Y)$ such that condition (4.9) holds. On the other hand, such a morphism f exists. It is a functor $f\colon E_Y \to E_X^*$ defined by

(*′) $f_{\mathrm{Mor}}(g) = g$ for every $g \in \mathrm{Mor}\,E_Y$ (see the definition of $\mathrm{Mor}_{SC}(X, Y)$)

and by the condition

(*″) $f_{\mathrm{Ob}}|\mathrm{Mor}_{EC}(Y, Z) = f^Z$ for every E-object Z, where f^Z is the mapping of sets uniquely defined by (4.8).

We have only to verify that if $g \in \mathrm{Mor}_{E_Y}(h, h')$, then

$$f_{\mathrm{Mor}}(g) = g \in \mathrm{Mor}_{E_X}\big(f_{\mathrm{Ob}}(h), f_{\mathrm{Ob}}(h')\big).$$

Indeed, let $g \in \mathrm{Mor}_{E_Y}(h, h')$ for $h \in \mathrm{Mor}_{EC}(Y, Z)$, $h' \in \mathrm{Mor}_{EC}(Y, Z')$, where Z, Z' are E-objects. Then $g \in \mathrm{Mor}_{EC}(Z, Z')$ and $g \circ h = h'$ and by (2.2), $h = h_t \circ E(p_t)$ for a $t \in T$ and an $h_t \in \mathrm{Mor}_{EC}(Y_t, Z)$. Since $f_t \in \mathrm{Mor}_{SC}(X, Y_t)$ is a functor from E_{Y_t} to E_X with property (*), we have

$$g = (f_t)_{\mathrm{Mor}}(g) \in \mathrm{Mor}_{E_X}\big((f_t)_{\mathrm{Ob}}(h_t), (f_t)_{\mathrm{Ob}}(g \circ h_t)\big)$$

(as $g \in \mathrm{Mor}_{E_{Y_t}}(h_t, g \circ h_t)$) and consequently

(4.10) $$g \circ \big((f_t)_{\mathrm{Ob}}(h_t)\big) = (f_t)_{\mathrm{Ob}}(g \circ h_t).$$

Thus, by (4.9) and (4.10)

$$\begin{aligned} g \circ f_{\mathrm{Ob}}(h) &= g \circ f^Z(h) \\ &= g \circ f^Z \circ \big((M_{EC}^Z \circ E)(p_t)\big)(h_t) \\ &= g \circ \big((f_t)_{\mathrm{Ob}}(h_t)\big) = (f_t)_{\mathrm{Ob}}(g \circ h_t) \\ &= f^Z \circ \big((M_{EC}^Z \circ E)(p_t)\big)(g \circ h_t) \\ &= f^Z\big(g \circ h_t \circ E(p_t)\big) = f^Z(h') = f_{\mathrm{Ob}}(h'). \end{aligned}$$

The theorem is proved.

(4.11) COROLLARY. *The described system (C, EC, SC, E, S) is a continuous* BTS *for every category C and EC with property* (4.3) *and functor $E\colon C \to EC$ such that C is an E-category.*

§ 5. Uniqueness and quotient properties of a continuous BTS and a continuous shape functor. In this paragraph $E\colon C \to CE$ denotes a projection functor defined on an E-category C into a category EC with property (4.3). We will call such a category a *semi-classical category*.

(5.1) THEOREM. *Let (C, EC, SC, E, F) be a* BTS *and let $G\colon EC \to D$ be a functor such that $G' = G \circ E\colon C \to D$ is a continuous functor. Then there is exactly one functor $H\colon SC \to D$ such that $G' = H \circ S$.*

Proof. First, let us remark that if $f, g \in \mathrm{Mor}_C(X, Y)$ and Y is an E-object and $S(f) = S(g)$, then $G'(f) = G'(g)$. Next, for an arbitrary object Y from C, if $S(f) = S(g)$ then

$$E(h \circ f) = E(h \circ g) \quad \text{and} \quad G'(h \circ f) = G'(h \circ g)$$

for each E-object Z and $h \in \mathrm{Mor}_C(Y, Z)$. Since Y is a limit of an inverse system of E-objects and G' is a continuous functor, we have $G'(f) = G'(g)$.

It follows from the above considerations that we can give the following definition of $H\colon SC \to D$:

$$H(X) = G'(X) \quad \text{for every } X \in \mathrm{Ob}\,C = \mathrm{Ob}\,SC$$

and

$$H(f) = \lim\mathrm{inv}\big(G'(g_t)\colon G'(X) \to G'(Y_t), t \in T\big)$$

for every $f \in \mathrm{Mor}_{SC}(X, Y)$, X and $Y \in \mathrm{Ob}\,C$, where

$$E(g_t) = F^{-1}\big(S(p_t) \circ f\big), \quad t \in T,$$

for a representation $(Y, p_t\colon t \in T)$ of Y as a limit of an inverse system $(Y_t, p_t^u\colon t < u \in T)$. Let us remark that such morphisms g_t exist and

$$E(g_t) = E(p_t^u) \circ E(g_u)$$

and consequently

$$G'(g_t) = G'(p_t^u) \circ G'(g_u) \quad \text{for } t < u ,$$

so that $H(f) = \lim \operatorname{inv} G'(g_t)$ is a well defined morphism. In the case of $Y \in E\text{-Ob}$ we can put $p_t = 1_Y$, so that $H(f) = G \circ F^{-1}(f)$ (see Axiom (3.5)). If $f = S(f_0)$, then

$$H \circ S(f_0) = G'(f_0) .$$

Thus $H \circ S = G'$, as G' is continuous. We have only to prove that H preserves composition.

Let $h \in \operatorname{Mor}_{SC}(Y, Z)$ be another morphism and $Z \in E\text{-Ob}$. Then $h = h_t \circ S(p_t)$ for a $t \in T$ and an $h_t \in \operatorname{Mor}_{SC}(Y_t, Z)$. Hence (for $f \in \operatorname{Mor}_{SC}(X, Y)$)

$$\begin{aligned} H(h \circ f) &= H\big(h_t \circ S(p_t) \circ f\big) = H\big(h_t \circ S(g_t)\big) = G \circ F^{-1}\big(h_t \circ S(g_t)\big) \\ &= \big(G \circ F^{-1}(h_t)\big) \circ (G \circ F^{-1})\big(S(g_t)\big) = \big(G \circ F^{-1}(h_t)\big) \circ G'(g_t) \\ &= \big(G \circ F^{-1}(h_t)\big) \circ G'(p_t) \circ H(f) = G\big(F^{-1}(h_t) \circ E(p_t)\big) \circ H(f) \\ &= G \circ F^{-1}\big(h_t \circ S(p_t)\big) \circ H(f) = H(h) \circ H(f) , \end{aligned}$$

i.e.

$$H(h \circ f) = H(h) \circ H(f) \quad \text{if } Z \in E\text{-Ob} .$$

Next, let Z be an arbitrary object of C, $Z_t \in E\text{-Ob}$ and $g_t \in \operatorname{Mor}_C(Z, Z_t)$. Then

$$\begin{aligned} G'(g_t) \circ H(h \circ f) &= H\big(S(g_t)\big) \circ H(h \circ f) = H\big(S(g_t) \circ h \circ f\big) \\ &= H\big(S(g_t) \circ h\big) \circ H(f) = H\big(S(g_t)\big) \circ H(h) \circ H(f) \\ &= G'(g_t) \circ \big(H(h) \circ H(f)\big) , \end{aligned}$$

so that by the continuity of G'

$$H(h \circ f) = H(h) \circ H(f) .$$

Thus we have constructed a functor H such that $G' = H \circ S$. Such a functor H is the unique one since for such H

(5.2) $H(f) = G \circ F^{-1}(f)$ *for every* $f \in \operatorname{Mor}_{SC}(X, Y)$, $Y \in E\text{-Ob}$,

and G' being a continuous functor.

H is also the unique functor such that $G = H \circ F$.

Indeed, since E is a projection functor, we have $G = H \circ F$.

The uniqueness is a consequence of the continuity of G'. The theorem is proved.

Now we are going to prove that the shape functor of a continuous BTS is continuous.

(5.3) LEMMA. *Let* $S\colon C \to SC$ *be a continuous functor of shape and* $(C, EC, S'C, E, F')$ *be a full* BTS *(which is based on the same pair* $(C, E\colon C \to EC)$*). We put* $S' = F' \circ E$. *Then the functor* H *from Theorem* (5.1), *such that* $S = H \circ S'$, *has the following property:*

$$\textit{if } H(f) = H(g) \textit{ then } f = g \textit{ for every } f, g \in \operatorname{Mor} S'C .$$

Proof. If $H(f) = H(g)\colon X \to Y$ and $h \in \operatorname{Mor}_{EC}(Y, Z)$, $Z \in E\text{-Ob}$, then $H\big(S'(h) \circ f\big) = S(h) \circ H(g) = H\big(S'(h) \circ g\big)$ and, by (5.2),

$$F \circ (F')^{-1}\big(S'(h) \circ f\big) = F \circ (F')^{-1}\big(S'(h) \circ g\big)\colon X \to Z,\ Z \in E\text{-Ob} .$$

Thus

$$S'(h) \circ f = S'(h) \circ g$$

and $f = g$, as the second BTS is full.

(5.4) THEOREM. *A* BTS *is continuous if and only if its functor of shape is continuous.*

Proof. Let $(C, EC, S'C, E, F')$ be a continuous BTS and (C, EC, SC, E, F) be a BTS with the continuous functor $S = F \circ E$. Then for $S' = F' \circ E$ the functor $H\colon S'C \to SC$ given by Theorem (5.1) (such that $H \circ S' = S$) has the property from Lemma (5.3). We have only to prove that $H\big(\operatorname{Mor}_{S'C}(X, Y)\big) = \operatorname{Mor}_{SC}(X, Y)$ for every $X, Y \in \operatorname{Ob} C$.

Let $f \in \operatorname{Mor}_{SC}(X, Y)$. Then for any $Z \in E\text{-Ob}$, $g \in \operatorname{Mor}_{S'C}(Y, Z)$ there exists exactly one $f_g \in \operatorname{Mor}_{S'C}(X, Z)$ such that

$$H(f_g) = H(g) \circ f .$$

Now, let $(Y, p_t\colon t \in T)$ be a representation of Y as a limit of an inverse system $(Y_t, p_t^u\colon t < u \in T)$ of E-objects Y_t in C. We put $f_t = S'(p_t) \circ f$. Then for $t < u$

$$H\big(S'(p_t^u) \circ f_u\big) = H\big(S'(p_t^u)\big) \circ H\big(S'(p_u)\big) \circ f = H\big(S'(p_t)\big) \circ f = H(f_t)$$

and

$$S'(p_t^u) \circ f_u = f_t .$$

Thus, by continuity Axiom (3.8), there exists a morphism $f' \in \operatorname{Mor}_{S'C}(X, Y)$ such that

$$S'(p_t) \circ f' = f_t \quad \text{for each } t \in T .$$

Hence

$$S(p_t) \circ H(f') = H\big(S'(p_t) \circ f'\big) = H(f_t) = H\big(S'(p_t)\big) \circ f = S(p_t) \circ f$$

for each $t \in T$. Thus $H(f') = f$. The proof is finished.

The following theorem on the uniqueness of BTS is a consequence of the above results:

(5.5) THEOREM. *For any pair* (C, E) *where* $E\colon C \to EC$ *is a projection functor from an* E*-category* C *into a semi-classical category* EC *there exists a unique continuous* BTS (C, EC, SC, E, F). *It is unique in the following sense: If* $(C, EC, S'C, E, F')$ *is another continuous* BTS *(and* $S' = F' \circ E$*), then there exist functors (which are uniquely determined)* $H\colon SC \to S'C$ *and* $H'\colon S'C \to SC$ *such that*

$$H' \circ H\colon SC \to SC \quad \text{and} \quad H \circ H'\colon S'C \to S'C$$

are identity functors and

$$F' = H \circ F \quad \text{and} \quad F = H' \circ F'.$$

(5.6) Remark. We can say that a continuous functor of shape is a Dedekind section between the functors of shape and the continuous functors.

(5.7) Remark. It is clear that Theorem (5.1) holds for the contravariant functors G, G' also.

(5.8) EXAMPLE. Given an arbitrary BTS (C, EC, SC, E, F), let $Y \in E\text{-Ob}\,C$. Then, by Definition (2.1), $M^Y_{EC} \circ E\colon C \to \mathrm{Ens}$ is a continuous contravariant functor. Then, by Theorem (5.1), there exists a contravariant functor H such that $M^Y_{EC} = H \circ F$. It is easy to see that it must be $H = M^Y_{SC}$.

(5.9) EXAMPLE. Let $H\colon C \to HC$ be the homotopy functor from the topological category of compact pairs C to the homotopy category of compact pairs HC. Then H is a projection functor and C is an H-category. The Čech homology and cohomology functors and the cohomotopy functors π^n are H-invariant and continuous on C. Thus they are shape-invariant in the sense of Theorem (5.1) (see [2] and compare [3] and Example (5.8)). H-objects are precisely the pairs homotopically dominated by polyhedral pairs.

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.

[2] K. Borsuk and W. Holsztyński, *Concerning the ordering of shapes of compacta*, Fund. Math. 68 (1970), pp. 107-115.

[3] S. Godlewski, *Homomorphisms of cohomotopy groups induced by fundamental classes*, Bull. Acad. Polon. Sci. 17 (1969), pp. 277–283.

[4] B. Mitchel, *Theory of Category*, New York and London 1965.

Reçu par la Rédaction le 7. 10. 1969

Some results on fixed points — III

by

R. Kannan (Calcutta, Ind.)

Recently many authors have proved fixed point theorems (see for example [1], [4], [5], [8]) for operators mapping a Banach space X into itself. In each of these theorems it has been assumed that the mapping is non-expansive i.e., if φ maps the Banach space X into itself, then

(a) $$\|\varphi(x) - \varphi(y)\| \leqslant \|x - y\|, \quad \text{for } x, y \in X.$$

The main purpose of the present paper is to prove some fixed point theorems for operators mapping a Banach space into itself which, instead of the non-expansive property, possess the following: if φ is a mapping of a Banach space X into itself, then

(b) $$\|\varphi(x) - \varphi(y)\| \leqslant \tfrac{1}{2}\{\|x - \varphi(x)\| + \|y - \varphi(y)\|\} \quad \text{for } x, y \in X.$$

It may be noted that condition (a) implies the continuity of the operator in the whole space while condition (b) has no such implications. Moreover, it is known [6] that (a) and (b) are independent. For relevant works on fixed point theorems for operators mapping a metric space M into itself which satisfy condition (b) on M, one may refer to [6] and [7].

Before going into the theorems, we state the following well-known definitions and results.

DEFINITION ([2], p. 27). A norm in a normed linear space X is *uniformly convex* if

$$\|x_n\| = \|y_n\| = 1 \ (n = 1, 2, \ldots), \quad \lim_{n\to\infty} \|x_n + y_n\| = 2$$

imply

$$\lim_{n\to\infty} \|x_n - y_n\| = 0 \quad \text{for } x_n, y_n \in X.$$

THEOREM A ([2], p. 28). *Let* X *be a uniformly convex normed linear space and let* ε, M *be positive constants. Then there exists a constant* δ *with* $0 < \delta < 1$ *such that*

$$\|x\| \leqslant M, \quad \|y\| \leqslant M, \quad \|x - y\| \geqslant \varepsilon$$

imply

$$\|x + y\| \leqslant 2\delta \max(\|x\|, \|y\|).$$