where

$$c \in V \quad \text{and} \quad x \in p^{-1}(V).$$

We claim that $\hat{s}_V$ is continuous.

Let $\mathcal{N} = \{(f, x) | \; f: p^{-1}(b) \to p^{-1}(b') \text{ and } x \in p^{-1}(b)\}$. $\mathcal{N} \subseteq \mathcal{M} \times E$. Define $e: \mathcal{N} \to E$ by $e(f, x) = f(x)$. We will show that e is continuous. Suppose that W is open in E, and consider $e^{-1}(W) \subseteq \mathcal{N}$. Let $(f, x) \in e^{-1}(W)$, and choose $\varepsilon > 0$ such that $N_\varepsilon(f(x)) \subseteq W$. Since F is compact, f is uniformly continuous, so that given $\alpha > 0$, there exists $\mu(\alpha) > 0$ such that if $d(y, y') < \mu(\alpha)$, then $d(f(y), f(y')) < \alpha$. We can assume that $\mu(\alpha) \leqslant \alpha$. Put $\delta(\varepsilon, f) = \frac{1}{2}\mu(\frac{1}{2}\varepsilon)$. We claim that $(S_{\delta(\varepsilon,f)}(f) \times N_{\delta(\varepsilon,f)}(x)) \cap \mathcal{N} \subseteq e^{-1}(W)$.

To see this, let $(g, x') \in (S_{\delta(\varepsilon,f)}(f) \times N_{\delta(\varepsilon,f)}(x)) \cap \mathcal{N}$. Then $\varrho(\mathrm{Gr}(f), \mathrm{Gr}(g)) < \delta(\varepsilon, f)$, so that there must exist $x'' \in p^{-1}(b)$, with $d(x', x'') < \delta(\varepsilon, f)$ and $d(f(x''), g(x')) < \delta(\varepsilon, f)$. But then $d(x, x'') < 2\delta(\varepsilon, f) = \mu(\frac{1}{2}\varepsilon)$. Hence $d(f(x), f(x'')) < \frac{1}{2}\varepsilon$. Finally, we have $d(f(x), g(x')) < \frac{1}{2}\varepsilon + \delta(\varepsilon, f) < \varepsilon$, so that $g(x') \in N_\varepsilon(f(x)) \subseteq W$. Thus e is continuous.

Now $\hat{s}_V$ is obtained as the composition

$$(c, x) \to ((p(x), c), x) \to (\bar{s}(p(x), c), x) \xrightarrow{e} (\bar{s}(p(x), c))(x),$$

so that $\hat{s}_V$ is continuous. We also observe that $\hat{s}_V$ satisfies $p\hat{s}_V(c, x) = c$ and $\hat{s}_V(p(x), x) = x$, for all $(c, x) \in V \times p^{-1}(V)$.

Hence, in the language of ([1], p. 404), $\hat{s}_V$ is a slicing map for V, and V is a slicing neighborhood. But every point of B has such a neighborhood and map, so that $p: E \to B$ is a sliced fiber space ([3], p. 97). We then conclude, since B is metric, that p is a Hurewicz fibration ([1], p. 405).

References

[1] J. Dugundji, *Topology*, Boston 1966.

[2] E. Dyer and M. E. Hamstrom, *Completely regular mappings*, Fund. Math. 45 (1958), pp. 103–118.

[3] S. T. Hu, *Homotopy Theory*, New York 1959.

[4] — *Theory of Retracts*, Detroit 1965.

[5] K. Kuratowski, *Topologie I*, Warszawa-Lwów 1933.

[6] L. McAuley, *Existence of a complete metric for a special mapping space*, Ann. of Math. Studies, 60 (1966), pp. 135–139.

[7] E. Michael, *Continuous selections* I, Ann. of Math. 63 (1956), pp. 361–382.

[8] — *Continuous selections* II, Ann. of Math. 64 (1956), pp. 562–580.

[9] — *Continuous selections* III, Ann. of Math. 65 (1957), pp. 375–390.

[10] S. Seidman, *Completely regular mappings with locally compact fiber*, Trans. Amer. Math. Soc. 147 (1970), pp. 461–471.

NEW YORK UNIVERSITY

Reçu par la Rédaction le 16. 7. 1969

A characterization of strong inductive dimension*

by

J. M. Aarts (Delft)

§ 1. Introduction. In [7] Nishiura has presented a theory for (weak) inductive invariants of separable metrizable spaces. (Weak) inductive invariants (first introduced by Lelek [5]) are obtained by replacing the empty set in the definition of (weak) inductive dimension by members of some family of topological spaces (see § 3 for precise definition).

By studying inductive invariants it is determined what part of dimension theory is due to the inductive nature of the definition of dimension and what part is due to the special role of the empty set.

In the paper of Nishiura, this has resulted in a characterization of weak inductive dimension on separable metrizable spaces by means of seven (independent) conditions.

Essentially, by weakening one of these conditions, a characterization of the strong inductive dimension on the class of all metrizable spaces is obtained (see § 2). In § 3 a theory for strong inductive invariants is developed in order to prove the independence of the conditions by which dimension is characterized (see § 4).

Throughout, $B(U)$ denotes the boundary of U. $\dim X$ stands for the strong inductive dimension of X. All spaces under discussion are *metrizable*.

§ 2. A characterization theorem. An extended real-valued function f defined on the class of metrizable spaces, is said to be *topological* (*monotone*) if $f(X) = f(Y)$ ($f(X) \leqslant f(Y)$) whenever X is homeomorphic to (is a subset of) Y. f is called *pseudo-inductive* if in each space X every non empty closed set has arbitrarily small neighborhoods U such that $f(B(U)) \leqslant f(X) - 1$ (we agree that $\infty - 1 = \infty$). f is *weakly subadditive* if for all X and Y we have

$$f(X \cup Y) \leqslant f(X) + f(Y) + 1$$

(cf. [7] inductively subadditive).

Now we state a theorem which characterizes dimension.

* Research supported by National Science Foundation Grants GP-6867 and GP-8637.

THEOREM 1. *Suppose f is an extended real-valued function defined on the class of all metrizable spaces. Then* $f(X) = \dim X$ *if and only if f satisfies the following conditions* D1–D7. *Furthermore, the seven conditions are independent.*

D1. *f is topological.*

D2. *f is monotone.*

D3. *If X is the union of a σ-locally finite family* $\mathcal{A}$ *of closed subsets of itself, then* $f(X) \leqslant \sup\{f(A)|\ A \in \mathcal{A}\}$.

D4. *For every space X there exists a complete space Y such that Y is an extension of X and* $f(Y) \leqslant f(X)$.

D5. *f is pseudo-inductive.*

D6. *f is weakly subadditive.*

D7. $f(\{\emptyset\}) = 0$.

COROLLARY. *Suppose f is an extended real-valued function defined on the class of all separable metrizable spaces. Then* $f(X) = \dim X$ *if and only if f satisfies the following (independent) conditions*: D1, D2, D3 *with* $\mathcal{A}$ *countable and* D4 *through* D7.

Proof. The independence of the conditions will be discussed in § 4. Clearly, the dimension function satisfies the conditions D1 through D7. In order to prove the converse we first assume

A1. $f(X) \leqslant 0$ *if and only if* $\dim X \leqslant 0$.

Then, by D5 and D6 we have A2 and A3 below.

A2. *For each integer n* $(n \geqslant 0)$, *if* $\dim X \leqslant n$, *then* $f(X) \leqslant n$.

A3. *For each real number a* $(a \geqslant 0)$, *if* $f(X) \leqslant a$, *then* $\dim X \leqslant a$.

The proof of A2 is as follows. Suppose $\dim X \leqslant n$. By the decomposition theorem ([6], Theorem II.4) $X = \bigcup \{X_i|\ i = 0, \ldots, n\}$, where $\dim X_i \leqslant 0$. Then $f(X) \leqslant \sum_{i=0}^{n} f(X_i) + n \leqslant n$ (cf. [7]).

The proof of A3 is by induction. Let n be an integer $\geqslant 0$. Suppose A3 holds for all a with $0 \leqslant a < n$. Let a be a real number with $n \leqslant a < n+1$ and $f(X) \leqslant a$. By D5 each closed subset of X has arbitrarily small open neighborhoods U with $f(B(U)) \leqslant a-1$. By A1 or by the induction hypothesis it follows that $\dim(B(U)) \leqslant a-1$. Hence $\dim X \leqslant a$.

A2 and A3 imply the theorem. It remains to prove A1.

In a routine way by D1, D2, D6 and D7 it is proved that $f(X) = -1$ if and only if $X = \emptyset$. Furthermore, $f(X) \geqslant 0$ whenever $X \neq \emptyset$ (cf. [7]).

Suppose $f(X) \leqslant 0$. Then either $f(X) = -1$ and $\dim X = -1$ or $f(X) = 0$ and $\dim X = 0$ by D5.

It remains to be proved that if $\dim X \leqslant 0$, then $f(X) \leqslant 0$. Let $\dim X \leqslant 0$. Suppose the weight of X is $\mathfrak{m}$ (the weight of a space is the minimal cardinal number of an open base). Let D be a discrete space consisting of $\mathfrak{m}$ points. P denotes the countable product $\prod\{D_i|\ D_i = D;\ i = 1, 2, \ldots\}$ and π_i the natural projection onto D_i. Let p be a point of D, $G = \{x|\ x \in P;$ $\pi_i(x) = p$ except for at most finitely many $i\}$ and $G_i = \{x|\ x \in P;\ \pi_k(x) = p$ if $k > i\}$. Obviously, $G = \bigcup\{G_i|\ i = 1, 2, \ldots\}$ and G_i is discrete for each i. By D3, it follows that $f(G_i) = 0$ and $f(G) = 0$. Observe that G is not complete because each G_i is a nowhere dense closed subset of G. By D4, there exists a complete extension $\widetilde{G}$ of G such that $f(\widetilde{G}) = 0$. By the extension theorem of Lavrentiev ([4], p. 335) the identity map of G onto itself can be extended to a homeomorphism of a G_δ-subset of $\widetilde{G}$ which contains G onto a G_δ-subset H of P which contains G. By D1 and D2 we have $f(H) = 0$.

In the following lemma it will be proved that every zero-dimensional space X with weight $\mathfrak{m}$ can be embedded in H. Then from D2 it follows that $f(X) = 0$ and the proof is completed. The proof of the corollary is obvious.

LEMMA 1. *Suppose D is a discrete space with* $\mathfrak{m}$ *points and* $p \in D$. *Let*

$$P = \prod\{D_i|\ D_i = D;\ i = 1, 2, \ldots\}$$

and

$$G = \{x|\ x \in P;\ \pi_i(x) = p \text{ except for at most finitely many } i\}.$$

Suppose H is a G_δ-*subset of P which contains G. Then every zero-dimensional space the weight of which does not exceed* $\mathfrak{m}$, *can be embedded in H.*

Proof. Let $P \backslash H = \bigcup\{F_i|\ i = 1, 2, \ldots\}$ where each F_i is a closed subset of P. Observe that H is dense in P, since G is dense in P. Hence each F_i is a nowhere dense subset of P.

Let $B = D \backslash \{p\}$ and $Q = \prod\{B_i|\ B_i = B;\ i = 1, 2, \ldots\}$. Q is a closed subset which is disjoint from G.

Suppose $\dim X = 0$ and the weight of X does not exceed $\mathfrak{m}$. Because $\dim X = 0$, for each i there exists a cover $\mathfrak{U}_i$ of X such that

(i) each member of $\mathfrak{U}_i$ is closed and open,

(ii) any two distinct members of $\mathfrak{U}_i$ are disjoint,

(iii) mesh $\mathfrak{U}_i \leqslant 1/i$,

(iv) $\mathfrak{U}_{i+1}$ is a refinement of $\mathfrak{U}_i$.

Because the weight of X does not exceed $\mathfrak{m}$, the potency of $\mathfrak{U}_i$ does not exceed $\mathfrak{m}$ $(i = 1, 2, \ldots)$ (cf. [6], Theorem II.9).

For each i $(i = 1, 2, \ldots)$ let $\widetilde{g}_i$ be a one-to-one correspondence between $\mathfrak{U}_i$ and a subset of B. $g_i\colon X \to B$ is defined by $g_i(x) = \widetilde{g}_i(U)$ where $x \in U \in \mathfrak{U}_i$. We shall define $f_i\colon X \to D$ such that $f\colon X \to P$ defined by $\pi_i \circ f(x) = f_i(x)$ is an embedding and $f(X) \subset H$.

Let $f_1 = g_1$. Observe that f_1 is constant on each member of $\mathfrak{U}_1$.

We define f_2 on each member of $\mathfrak{U}_1$.

(i) If $\pi_1^{-1}(f_1(x)) \cap F_1 = \emptyset$, then we define $f_2(x) = g_2(x)$.

(ii) If $\pi_1^{-1}(f_1(x)) \cap F_1 \neq \emptyset$ and $x \in U \in \mathfrak{U}_1$, then an open set

$$\{f_1(x)\} \times \{p\} \times \ldots \times \{p\} \times \prod \{D_i \mid i \geqslant k\}$$

containing $(f_1(x), p, p, \ldots)$ is selected which is disjoint from $Q \cup F_1$. Then for each $y \in U$ we define $f_2(y) = p$. Clearly, f_2 is continuous on each member U of $\mathfrak{U}_1$ and therefore f_2 is continuous on X.

Suppose $f_1, \ldots, f_n$ have been defined in this way. We shall define f_{n+1}. Let φ_n denote the natural projection of D onto $D_1 \times \ldots \times D_n$. f_{n+1} will be defined on each member of $\mathfrak{U}_n$.

We consider two cases.

(i) $f_n(x) = g_n(x)$.

Let $K = \varphi_n^{-1}((f_1(x), \ldots, f_n(x))) \cap (F_1 \cup \ldots \cup F_n)$. If $K = \emptyset$, then we define $f_{n+1}(x) = g_{n+1}(x)$.

If $K \neq \emptyset$ and $x \in U \in \mathfrak{U}_n$, then an open set

$$\{f_1(x)\} \times \ldots \times \{f_n(x)\} \times \{p\} \times \ldots \times \{p\} \times \prod \{D_i \mid i \geqslant k > n+1\}$$

is selected which is disjoint from $Q \cup F_1 \cup \ldots \cup F_n$. For each $y \in U$ we define $f_{n+1}(y) = p$.

(ii) $f_n(x) = p$.

Then for some $k < n$ $\varphi_k^{-1}((f_1(x), \ldots, f_k(x))) \cap (F_1 \cup \ldots \cup F_k) \neq \emptyset$ and an open set

$$\{f_1(x)\} \times \ldots \times \{f_k(x)\} \times \{p\} \times \ldots \times \{p\} \times \prod \{D_i \mid i \geqslant l \geqslant n+1\}$$

has already been defined which is disjoint from $Q \cup F_1 \cup \ldots \cup F_k$. If $l = n+1$, then $f_{n+1}(x) = g_{n+1}(x)$ is defined. If $l > n+1$, then $f_{n+1}(x) = p$.

Clearly, $\{f_1, f_2, \ldots\}$ is a collection of continuous functions. In order to show that $f\colon X \to P$ defined by $\pi_i \circ f = f_i$ is an embedding, it is proved that $\{f_1, f_2, \ldots\}$ separates points and closed sets ([3], p. 116). Let L be a closed set and $x \notin L$. Choose i such that if $x \in V \in \mathfrak{U}_i$, then $V \cap L = \emptyset$. Let $k > i$ be the least integer such that $f_k(x) = g_k(x)$. Then $f_k(x) = g_k(x) = \tilde{g}_k(U) \in B$, where $x \in U \in \mathfrak{U}_k$. For each $y \in L$ we have $f_k(y) = p$ or $f_k(y) = \tilde{g}(V)$ where $y \in V \in \mathfrak{U}_k$. Because $\tilde{g}$ is one-to-one and D is discrete, we have

$$\overline{f_k(L)} = f_k(L) \subset D \setminus \{f_k(x)\}.$$

Finally, we show that $f(X) \subset H$. Suppose $x \in X$ and k $(k \geqslant 1)$ is an integer. We shall show that $f(x) \notin F_k$. Let $n > k$ be the least integer with $f_n(x) = g_n(x)$.

Let $K = \varphi_n^{-1}((f_1(x), \ldots, f_n(x))) \cap (F_1 \cup \ldots \cup F_n)$.

If $K = \emptyset$, then $\varphi_n^{-1}((f_1(x), \ldots, f_n(x)))$ is a neighborhood of $f(x)$ which is disjoint from $F_1 \cup \ldots \cup F_n$. It follows that $x \notin F_k$. If $K \neq \emptyset$, then a neighborhood

$$W = \{f_1(x)\} \times \ldots \times \{f_n(x)\} \times \{p\} \times \ldots \times \{p\} \times \prod \{D_i \mid i \geqslant l > n+1\}$$

of $f(x)$ has been selected which is disjoint from $Q \cup F_1 \cup \ldots \cup F_n$. It follows that $f(x) \notin F_k$.

§ 3. Strong inductive invariants. In this section some of the results of [7] are generalized to general metric spaces. Let P be a class of topological spaces which is closed for topological mappings i.e. if X and Y are homeomorphic and $X \in P$, then $Y \in P$.

The strong (weak) inductive invariant $P\text{-Ind}\,X$ ($P\text{-ind}\,X$) induced by the class P is defined for every space X as follows.

(i) $P\text{-Ind}\,X = P\text{-ind}\,X = -1$ if and only if $X \in P$.

(ii) For each $n \geqslant 0$, $P\text{-Ind}\,X \leqslant n$ ($P\text{-ind}\,X \leqslant n$) provided that each non-empty closed subset (each point) of X has arbitrarily small neighborhoods U of X such that

$$P\text{-Ind}\,B(U) \leqslant n-1 \quad (P\text{-ind}\,B(U) \leqslant n-1).$$

For each integer $n \geqslant 0$, $P\text{-Ind}\,X = n$ if $P\text{-Ind}\,X \leqslant n$ and $P\text{-Ind}\,X \nleqslant n-1$. If $P\text{-Ind}\,X \nleqslant n$ for each n, then $P\text{-Ind}\,X = \infty$.

Inductive dimension ($P = \{\emptyset\}$) is a well-known example of an inductive invariant. Other inductive invariants are compactness degree [2] and completeness degree [1].

First, we state some theorems which are straightforward generalizations of theorems in [7]. Proofs are omitted.

THEOREM 2. *$P\text{-Ind}\,X$ is a topological invariant.*

THEOREM 3. *For all X and all P*

$$P\text{-Ind}\,X \leqslant \dim X + 1.$$

Moreover, if P is empty, then $P\text{-Ind}\,X = \dim X + 1$. If $\emptyset \in P$, then $P\text{-Ind}\,X \leqslant \dim X$.

A class P is (*c*-)*monotone* if whenever $X \in P$ and Y is a (closed) subset of X, then $Y \in P$.

An extended real-valued function f on the class of metrizable spaces is *c-monotone* if whenever X is a closed subset of Y, then $f(X) \leqslant f(Y)$.

THEOREM 4. *A class P is c-monotone if and only if P-Ind is c-monotone.*

From now on the procedure is somewhat different from that in [7]. We have the following sum theorem.

THEOREM 5. *Suppose P is c-monotone. The following conditions are equivalent.*

B1. *If X is the union of a σ-locally finite collection $\mathcal{A}$ of closed subsets of itself and each member of $\mathcal{A}$ belongs to P, then $X \in P$.*

B2. *For each $n \geqslant -1$, if $X = \bigcup \mathcal{A}$ where $\mathcal{A}$ is a σ-locally finite collection of F_σ-subsets of X with $\sup\{P\text{-Ind} A \mid A \in \mathcal{A}\} \leqslant n$, then $P\text{-Ind} X \leqslant n$.*

We need the following theorem.

THEOREM 6. *Suppose P is c-monotone and satisfies* B1. *For each $n \geqslant 0$ and all spaces X the following conditions are equivalent.*

C1. $P\text{-Ind} X \leqslant n$.

C2. *There exists a σ-locally finite open base $\mathcal{B}$ of X such that $P\text{-Ind} B(V) \leqslant n-1$ for all $V \in \mathcal{B}$.*

C3. *There exist subspaces A and B of X such that $P\text{-Ind} A \leqslant n-1$ and* $\dim B \leqslant 0$.

We also need the following lemma.

LEMMA 2. *Let $n \geqslant -1$. Suppose P is c-monotone and satisfies* B1. *If $P\text{-Ind} X \leqslant n$ and Y is a σ-locally finite union of closed subsets of X, then $P\text{-Ind} Y \leqslant n$.*

Theorems 5 and 6 and Lemma 2 are proved simultaneously.

Proof. We may assume that $P \neq \emptyset$. Otherwise, by Theorem 3 we have $P\text{-Ind} X = \dim X + 1$. Theorem 5 then follows from [6], II.2, C and D. (Observe that if X is a σ-locally finite union of F_σ-subsets of itself, then X also is the σ-locally finite union of closed subsets of itself.) Theorem 6 follows from [6], Theorem II.2 and Theorem II.4. Lemma 2 is obvious, since dimension is monotone.

Clearly, B1 is equivalent to B2 for $n = -1$. By induction it is proved that B1 implies B2. We assume that B2 has been proved for each integer $\leqslant n-1$ $(n \geqslant 0)$ and show that B2 holds for n.

First, we prove the equivalence of C1, C2 and C3 for n.

C1 implies C2: This is a straightforward generalization of [6], II.1. D.

C2 implies C3: Let $\mathcal{B}$ be a σ-locally finite base of X such that $P\text{-Ind} B(V) \leqslant n-1$ for all $V \in \mathcal{B}$.

Let $A = \bigcup\{B(V) \mid V \in \mathcal{B}\}$ and $B = X \setminus A$. $\dim B \leqslant 0$ by [6], II.1, E and $P\text{-Ind} A \leqslant n-1$ by B2 and the induction hypothesis.

C3 implies C1: Let $X = A \cup B$ with $P\text{-Ind} A \leqslant n-1$ and $\dim B = 0$. Now, the proof is almost identical to the proof of [6], II.1. C.

Now, we prove Lemma 2 for n. (Observe that the case $n = -1$ is a direct consequence of the definitions.) We assume that Lemma 2 has been proved for each integer $\leqslant n-1$. Suppose $P\text{-Ind} X \leqslant n$ and Y is a σ-locally finite union of closed subsets of X. Let $X = A \cup B$ with $P\text{-Ind} A \leqslant n-1$ and $\dim B \leqslant 0$. $Y \cap A$ is a σ-locally finite union of closed subsets of A and $P\text{-Ind}(Y \cap A) \leqslant n-1$ by the induction hypothesis. Obviously, $\dim(Y \cap B) \leqslant 0$.

The lemma now follows from Theorem 6.

Finally, we prove that B2 holds for n.

Suppose $X = \bigcup\{A_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\}$ and for each $i = 1, 2, \ldots$ we have $\{A_\gamma \mid \gamma \in \Gamma_i\}$ is a locally finite collection of F_σ-subsets of X. Suppose $P\text{-Ind} A_\gamma \leqslant n$. Let $A_\gamma = \bigcup\{A_\gamma^k \mid k = 1, 2, \ldots\}$ where each A_γ^k is closed in X. By Theorem 4 we have $P\text{-Ind} A_\gamma^k \leqslant n$ for each k and γ. It follows that X is the union of a σ-locally finite collection of closed subsets of itself each of which has $P\text{-Ind} \leqslant n$. Thus we assume $X = \bigcup\{A_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\}$, for each i the collection $\{A_\gamma \mid \gamma \in \Gamma_i\}$ is locally finite, each A_γ is closed and $P\text{-Ind} A_\gamma \leqslant n$.

Moreover, we assume that each Γ_i is well ordered. Suppose $\gamma \in \Gamma_i$. Let $B_\gamma = \bigcup\{A_\delta \mid \delta \in \bigcup_{k=1}^{i-1} \Gamma_k$ or $\delta \in \Gamma_i$ and $\delta < \gamma\}$. B_γ is a locally finite union of closed sets. Thus B_γ is closed. $C_\gamma = A_\gamma \setminus B_\gamma$ is an F_σ-subset of X.

By Lemma 2 we have $P\text{-Ind} C_\gamma \leqslant n$.

Obviously, $X = \bigcup\{C_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\}$, the C_γ's are pairwise disjoint and C_γ is an F_σ-subset of X. Applying C3 to each C_γ we have $C_\gamma = D_\gamma \cup E_\gamma$ where $P\text{-Ind} D_\gamma \leqslant n-1$ and $\dim E_\gamma \leqslant 0$. Let $D = \bigcup\{D_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\}$ and $E = \bigcup\{E_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\}$. Then $X = D \cup E$. Each D_γ is an F_σ-subset of D and $P\text{-Ind} D \leqslant n-1$ by B2 for $n-1$. Each E_γ is an F_σ-subset of E and $\dim E \leqslant 0$ (see the observations made at the beginning of the proof). Then by Theorem 6 we have $P\text{-Ind} X \leqslant n$.

As a consequence of Theorem 6 we have

THEOREM 7. *Suppose P is c-monotone and satisfies* B1. *For every separable metrizable space X we have $P\text{-Ind} X = P\text{-ind} X$.*

Proof. The proof is a straightforward generalization of [6], Theorem IV.1 using Theorem 6 above.

A family P is called *additive* if $X \cup Y \in P$ whenever $X \in P$ and $Y \in P$. In the same way as Lemma 2 has been proved by means of Theorem 6 the following theorems can be proved. The "if" parts are trivial.

THEOREM 8. *Suppose P is c-monotone and satisfies* B1. *Then P is additive if and only if P-Ind is weakly subadditive.*

THEOREM 9. *Suppose P is c-monotone and satisfies* B1. *Then P is monotone if and only if P-Ind is monotone.*

§ 4. Independence of the conditions D1–D7. In this section we show that each of the conditions D1 through D7 is independent of the other conditions by means of examples. We only indicate proofs of non-trivial facts (cf. [7] for 1, 3, 6 and 7).

1. Independence of D1. Let $f(\emptyset) = -1$, $f(\{\emptyset\}) = 0$ and $f(X) = \dim X + 1$ if $X \neq \emptyset$ and $X \neq \{\emptyset\}$.

2. Independence of D2. Let C be the collection of all spaces X such that $\dim X \leqslant 0$ and X is the union of a σ-locally finite collection of (topologically) complete subsets of itself. Obviously, C is c-monotone.

Suppose $X = \bigcup \mathcal{A}$, where $\mathcal{A}$ is a σ-locally finite collection of closed subsets of X and each member of $\mathcal{A}$ belongs to C. Then $\dim X \leqslant 0$ by [6], II.2, C and D. Moreover, X is the union of a σ-locally finite collection of complete subsets of itself. This follows from the fact that if $\{A_\gamma |\ \gamma \in \Gamma\}$ is a locally finite collection of *closed* sets such that each A_γ is the union of a locally finite collection $\{A_\gamma^\delta |\ \delta \in \Delta_\gamma\}$ of subsets of A_γ, then $\{A_\gamma^\delta |\ \gamma \in \Gamma, \delta \in \Delta_\gamma\}$ is locally finite.

Hence, $X \in C$ and C satisfies condition B1.

Now, let

$$f(X) = C\text{-Ind}\,X + 1 \quad \text{if } X \neq \emptyset \quad \text{and} \quad f(\emptyset) = -1 .$$

We shall show that f satisfies all conditions D1 through D7 except for D2.

Clearly, f satisfies D1. D3 is a consequence of Theorem 5. Observe that if $f(X) \leqslant 0$ then $\dim X \leqslant 0$. It follows that f is pseudo-inductive (D5). Moreover, by induction it can be proved that if $f(X) \leqslant n$, then $\dim X \leqslant n$. Hence $\dim X \leqslant f(X)$. By Theorem 3 we have either $f(X) = \dim X$ or $f(X) = \dim X + 1$. Observe that if X is the union of a σ-locally finite collection of complete subsets of itself, then $f(X) = \dim X$. Now, it is clear that f satisfies condition D4. Indeed, for any dimension preserving complete extension Y of X we have $f(Y) = f(X)$ or $f(Y) = f(X) - 1$. The existence of such an extension easily follows from [6], Theorem II.10.

Now, we show that f is weakly subadditive (D6). Let $X = A \cup B$ and $f(A) = m$ and $f(B) = n$. By simultaneous induction it is proved that $f(X) \leqslant m + n$. This obviously holds for $m = -1$ or $n = -1$. If both $m = n = 0$ then X is the union of a σ-locally finite collection of complete subsets of itself. It follows that $f(X) = \dim X \leqslant 1$ by [6], Theorem II.4. Now, let $m = r$ and $n = s$, and suppose the weak subadditivity has been proved for $m \leqslant r$, $n \leqslant s-1$ and $m \leqslant r-1$, $n \leqslant s$. We may assume $f(A) = r \geqslant 1$. Then C-Ind $A \geqslant 0$. By Theorem 6, C3, there exist subspaces D and E of A such that $f(D) \leqslant r-1$ and $\dim E \leqslant 0$. Then $f(D \cup B) \leqslant (r-1) + s + 1 = r + s$ by the induction hypothesis and $f(X) = f\big((D \cup B) \cup E\big) \leqslant r + s + 1$ by Theorem 6.

Obviously, f satisfies D7. f does not satisfy D2. Indeed, if $f(X) = 0$, then $\dim X = 0$ and X is a Borel set in every complete space which contains X (cf. [4], p. 27). However, there exist zero-dimensional subsets of the irrationals which are not Borel sets.

Thus, f satisfies the conditions D1 through D7 except for D2.

3. Independence of D3. Let $f(X) = \dim X$ if X is finite and $(X) = \dim X + 1$ otherwise.

4. Independence of D4. Let Q be the collection of all spaces X such that X is the union of a σ-locally finite collection of singletons. As in 2 it is proved that Q satisfies B1. Moreover, Q is additive and monotone. From Theorems 2, 5, 8 and 9 it follows that Q-Ind satisfies D1, D2, D3 and D6. Let $f(X) = Q\text{-Ind}\,X + 1$ if $X \neq \emptyset$ and $f(\emptyset) = -1$.

Then f satisfies D1, D2, D3, D5, D6 and D7.

If X is separable and $f(X) = 0$, then X is countable. It follows that f does not satisfy D4.

5. Independence of D5. Let $f(\emptyset) = -1$ and $f(X) = 0$ if $X \neq \emptyset$.

6. Independence of D6. $f(X) = \dim X$ if and only if $\dim X \leqslant 0$. $f(X) = \dim X + 1$ if and only if $\dim X > 0$.

7. Independence of D7. $f(X) = \dim X + 1$.

References

[1] J. M. Aarts, *Completeness degree, A generalization of dimension*, Fund. Math. 63 (1968), pp. 27–41.

[2] J. de Groot and T. Nishiura, *Inductive compactness as a generalization of semi-compactness*, Fund. Math. 58 (1966), pp. 201–218.

[3] J. L. Kelley, *General Topology*, Princeton (N. J.) 1955.

[4] C. Kuratowski, *Topologie* I (4-th ed.), Warszawa 1958.

[5] A. Lelek, *Dimension and mappings of spaces with finite deficiency*, Colloq. Math. 12 (1964), pp. 221–227.

[6] J. Nagata, *Modern dimension theory*, Groningen 1965.

[7] T. Nishiura, *Inductive invariants and dimension theory*, Fund. Math. 59 (1966). pp. 243–262.

LOUISIANA STATE UNIVERSITY
Baton Rouge, Louisiana

DELFT INSTITUTE OF TECHNOLOGY
Delft, Netherlands

Reçu par la Rédaction le 3. 9. 1969