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Completely regular mappings with compact ANR fiber

by

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1. Introduction. Completely regular mappings were first considered by Dyer and Hamstrom in 1958 [2]. They are discrete analogues of locally trivial projections. If the homeomorphism group of the fiber is locally path-connected, and if the fiber is locally compact and separable, then a completely regular mapping is a Serre fibration [10].

In this paper, we will prove that if $p: E\to B$ is completely regular with fiber F, where F is a compact ANR and B is finite-dimensional and locally compact, then p is a Hurewicz fibration. We have thus improved a result of Michael ([9], p. 381) which would imply that p is a Serre fibration.

2. Definitions and notation.

(2.1) A continuous surjection $p: E\to B$ is *completely regular* if E and B are metric, and if for each $b\in B$ and $\varepsilon>0$, there exists $\delta(b,\varepsilon)>0$ such that if $d(b,b')<\delta(b,\varepsilon)$, there exists a homeomorphism $h: p^{-1}(b)\to p^{-1}(b')$, such that $d\big(x,h(x)\big)<\varepsilon$ for all $x\in p^{-1}(b)$.

The space B will always be assumed to be connected. Thus all fibers are homeomorphic, and we will denote this common fiber by F.

A topological space X is *locally n-connected* ($\mathbf{LC}^n$) if, given $x\in X$ and an open neighborhood U of x, there exists an open set V with $x\in V\subseteq U$, such that if $f: S^m\to V$ is a map ($m\leqslant n$), then f extends to $F: B^{m+1}\to U$.

Let $\{S_\alpha\}(\alpha\in A)$ be a collection of subsets of X. $\{S_\alpha\}$ is *equi*-$\mathbf{LC}^n$ [8] if, given $x\in X$ and an open neighborhood U of x, there exists an open set V with $x\in V\subseteq U$, such that if $f: S^m\to V\cap S_\alpha$ is a map ($m\leqslant n$ and $\alpha\in A$), then f extends to $F: B^{m+1}\to U\cap S_\alpha$.

Let Y be a topological space, and let $\mathcal{S}(Y)$ denote the collection of subspaces of Y. A function $\varphi: X\to\mathcal{S}(Y)$ is called a *lower semi-continuous carrier* (*l.s.c. carrier*) [7] if, given $x\in X$ and an open subset U of Y with $\varphi(x)\cap U\neq\emptyset$, then there exists an open neighborhood V of x, such that if $x'\in V$, then $\varphi(x')\cap U\neq\emptyset$.

(2.2) Note that if $p: E\to B$ is continuous and open, then the function taking b to $p^{-1}(b)$ is a l.s.c. carrier from B to $\mathcal{S}(E)$ ([7], p. 382).

We now quote a version of a result of Michael ([8], Theorem 1.2, p. 563):

(2.3) PROPOSITION. *Suppose that X is metric and that Y is complete metric. Let $\varphi\colon X\to\mathfrak{F}(Y)$ be a l.s.c. carrier, where $\varphi(x)$ is closed for each $x\in X$, and the collection $\{\varphi(x)\}$ $(x\in X)$ is equi-*$\mathbf{LC}^n$. *Suppose also that $\dim Y\leqslant n+1$. Finally let $A\subseteq X$ be closed, and let $f\colon A\to Y$ be continuous, with $f(a)\in\varphi(a)$ for all $a\in A$. Then f can be extended to a continuous $g\colon U\to Y$, where $U\supseteq A$ is open, and $g(x)\in\varphi(x)$ for all $x\in U$.*

A map $p\colon E\to B$ is a *Hurewicz fibration* if it has the ACHP in the sense of Hu ([3], p.62).

If A and B are closed, bounded subsets of a metric space X, then the *Hausdorff metric* ϱ *is defined by*

$$\varrho(A,B)=\max\{\sup_{x\in A}d(x,B),\ \sup_{y\in B}d(y,A)\}\,.$$

If $f\colon X\to Y$ is a map, then $\mathrm{Gr}(f)\subseteq X\times Y$ will denote the graph of f.

If X and Y are metric spaces, then a metric on $X\times Y$ is given by $d\big((x_1,y_1),(x_2,y_2)\big)=\max\{d(x_1,x_2),\,d(y_1,y_2)\}$. We will use this metric for $X\times Y$ without further explicit mention.

3. Generalities on complete regularity. We first recall some results from [10].

(3.1) *If $p\colon E\to B$ is completely regular with fiber F, and B and F are locally compact, then E is locally compact.*

(3.2) *If the fiber is compact, then complete regularity of $p\colon E\to B$ does not depend on the choice of metric for E.*

(3.3) *If $p\colon E\to B$ is completely regular, then p is an open map.*

Proofs of these statements will be found in ([10], § 3).

As a simple consequence of the proof of (3.1), we have

(3.4) PROPOSITION. *If $p\colon E\to B$ is completely regular with fiber F, where B and F are compact, then E is compact.*

We next have a result that is useful in the application of (2.3) to completely regular mappings.

(3.5) PROPOSITION. *Let $p\colon E\to B$ be completely regular with fiber F, where F is* $\mathbf{LC}^n$. *Then the collection $\{p^{-1}(b)\}$ $(b\in B)$ is equi-*$\mathbf{LC}^n$.

Proof. Let $x\in E$, and let U be an open neighborhood of x in E. Choose $\varepsilon>0$ so that $N_\varepsilon(x)\subseteq U$. Now $N_{\frac12\varepsilon}(x)\cap p^{-1}\big(p(x)\big)$ is open in $p^{-1}\big(p(x)\big)$, and F is $\mathbf{LC}^n$, so that we can find μ with $0<\mu<\frac12\varepsilon$, such that if $g\colon S^m\to N_\mu(x)\cap p^{-1}\big(p(x)\big)$ is a map $(m\leqslant n)$, then g extends to a map

$$G\colon B^{m+1}\to N_{\frac12\varepsilon}(x)\cap p^{-1}\big(p(x)\big)\,.$$

Put $V=N_{\frac12\mu}(x)\cap p^{-1}\big(N_{\delta(\frac12\mu,p(x))}(p(x))\big)$. Clearly $x\in V\subseteq U$. Suppose now that $f\colon S^m\to V\cap p^{-1}(b)$ $(b\in B$ and $m\leqslant n)$. Then there exists a homeomorphism $h\colon p^{-1}(b)\to p^{-1}\big(p(x)\big)$, such that $d\big(y,h(y)\big)<\frac12\mu$, for all $y\in p^{-1}(b)$. But then $h\circ f\colon S^m\to N_\mu(x)\cap p^{-1}\big(p(x)\big)$, so that $h\circ f$ extends to $H\colon B^{m+1}\to N_{\frac12\varepsilon}(x)\cap p^{-1}\big(p(x)\big)$. Now define $F=h^{-1}\circ H\colon B^{m+1}\to N_\varepsilon(x)\cap p^{-1}(b)$. F is clearly the desired map extending f.

4. Proof of the main theorem. We will now prove

(4.1) THEOREM. *Let $p\colon E\to B$ be completely regular with fiber F, where F is a compact* ANR *and B is finite-dimensional and locally compact. Then p is a Hurewicz fibration.*

Proof. Define $\mathcal{M}=\{f\colon p^{-1}(b)\to p^{-1}(b')\,|\,\{b,b'\}\subseteq B\}$, where all the maps f under consideration are required to be continuous. By (3.1) we see that E is locally compact. Hence we can find a bounded complete metric for E, and by (3.2) p is completely regular with respect to this metric.

Now if $f\in\mathcal{M}$, $\mathrm{Gr}(f)$ is a closed, bounded subset of $E\times E$. For $f,g\in\mathcal{M}$, define $d(f,g)=\varrho\big(\mathrm{Gr}(f),\mathrm{Gr}(g)\big)$, where ϱ is the Hausdorff metric defined above. Thus we have defined a metric on $\mathcal{M}$. A similar metric in a slightly different situation is used by McAuley in [6]. Let $\pi\colon\mathcal{M}\to B\times B$ be defined by $\pi(f)=(b;b')$ if $f\colon p^{-1}(b)\to p^{-1}(b')$.

In order to apply (2.3), we must have more information about $\mathcal{M}$ and π.

(4.2) *π is continuous.*

Proof. Let $(b,b')\in B\times B$, and let $U\times V$ be a neighborhood of (b,b') in $B\times B$, where U and V are open in B. We claim that $\pi^{-1}(U\times V)$ is open in $\mathcal{M}$. Let $g\in\pi^{-1}(U\times V)$. Thus $g\colon p^{-1}(c)\to p^{-1}(c')$, with $c\in U$ and $c'\in V$.

Let $\varepsilon>0$ be so small that $N_\varepsilon(c)\subseteq U$ and $N_\varepsilon(c')\subseteq V$, and let $x\in p^{-1}(c)$. Since $x\in p^{-1}\big(N_\varepsilon(c)\big)$ and $g(x)\in p^{-1}\big(N_\varepsilon(c')\big)$, we can find $\tau(\varepsilon)>0$, such that $N_\tau(x)\subseteq p^{-1}\big(N_\varepsilon(c)\big)$ and $N_\tau\big(g(x)\big)\subseteq p^{-1}\big(N_\varepsilon(c')\big)$. We claim that if $\varrho\big(\mathrm{Gr}(g),\mathrm{Gr}(\bar g)\big)<\tau$, then $\bar g\in\pi^{-1}(U\times V)$.

Suppose that $\bar g\colon p^{-1}(\bar c)\to p^{-1}(\bar c')$. Then there must exist $y\in p^{-1}(\bar c)$, with $d(x,y)<\tau$ and $d\big(g(x),\bar g(y)\big)<\tau$. Thus $y\in N_\tau(x)\subseteq p^{-1}\big(N_\varepsilon(c)\big)$, and $\bar g(y)\in N_\tau\big(g(x)\big)\subseteq p^{-1}\big(N_\varepsilon(c')\big)$. Hence $\bar c\in N_\varepsilon(c)\subseteq U$ and $\bar c'\in N_\varepsilon(c')\subseteq V$, so that $\pi(\bar g)\in U\times V$. Thus $\pi^{-1}(U\times V)$ is open in $\mathcal{M}$, as claimed.

(4.3) *π is completely regular.*

Proof. Let $(b, b') \in B\times B$ and $\varepsilon > 0$ be given. Put

$$\eta = \min\{\delta(b, \varepsilon), \delta(b', \varepsilon)\}.$$

Clearly $\eta > 0$. We claim that η satisfies the condition of (2.1).

Thus suppose that $(c, c') \in B\times B$ is such that $d\big((b, b'), (c, c')\big) < \eta$. Then $d(b, c) < \delta(b, \varepsilon)$ and $d(b', c) < \delta(b', \varepsilon)$. Now we have homeomorphisms $h\colon p^{-1}(b)\to p^{-1}(c)$ and $g\colon p^{-1}(b')\to p^{-1}(c')$, such that $d\big(x, h(x)\big) < \varepsilon$ for all $x \in p^{-1}(b)$ and $d\big(y, g(y)\big) < \varepsilon$ for all $y \in p^{-1}(b')$. Define

$$\Phi\colon \pi^{-1}(b, b')\to\pi^{-1}(c, c') \quad\text{by}\quad \Phi(f) = g\circ f\circ h^{-1}.$$

We will show that Φ is continuous. An ε-neighborhood of k in $\mathcal{M}$ will be denoted by $S_\varepsilon(k)$.

Thus let $S_\varepsilon(k) \cap \pi^{-1}(c, c')$ be an open set in $\pi^{-1}(c, c')$. We will show that

$$\Phi^{-1}\big(S_\varepsilon(k) \cap \pi^{-1}(c, c')\big)$$

is open in $\pi^{-1}(b, b')$. Suppose that $f \in \Phi^{-1}\big(S_\varepsilon(k) \cap \pi^{-1}(c, c')\big)$. Then $g\circ f\circ h^{-1} \in S_\varepsilon(k)$. Now $0 \leqslant d\big(\Phi(f), k\big) < \varepsilon$. Since F is compact, $g\colon p^{-1}(b')\to p^{-1}(c')$ and $h\colon p^{-1}(b)\to p^{-1}(c)$ are uniformly continuous. Hence, if $a > 0$ is given, there exists $\mu(a) > 0$ such that if $\{z, z'\}\subseteq p^{-1}(b')$ with $d(z, z') < \mu(a)$, then $d\big(g(z), g(z')\big) < a$, while if $\{y, y'\}\subseteq p^{-1}(b)$ with $d(y, y') < \mu(a)$, then $d\big(h(y), h(y')\big) < a$. Put

$$\tau = \varepsilon - d\big(\Phi(f), k\big) > 0.$$

Now consider $S_{\mu(\tau)}(f) \cap \pi^{-1}(b, b')$. If $f' \in S_{\mu(\tau)}(f) \cap \pi^{-1}(b, b')$, then $d(f, f') < \mu(\tau)$, so that $\varrho\big(\mathrm{Gr}(f), \mathrm{Gr}(f')\big) < \mu(\tau)$. Now let $x \in p^{-1}(c)$. Then $\big(x, (\Phi(f))(x)\big) \in \mathrm{Gr}\big(\Phi(f)\big)$. But $\big(\Phi(f)\big)(x) = (g\circ f\circ h^{-1})(x)$. The inequality $\varrho\big(\mathrm{Gr}(f), \mathrm{Gr}(f')\big) < \mu(\tau)$, implies that there exists

$$y \in p^{-1}(b), \quad \text{with } d\Big(\big((h^{-1})(x), (f\circ h^{-1})(x)\big), \big(y, f'(y)\big)\Big) < \mu(\tau).$$

Write $y = h^{-1}(z)$. Then $d\big((f\circ h^{-1})(x), (f'\circ h^{-1})(z)\big) < \mu(\tau)$, so that

$$d\big((g\circ f\circ h^{-1})(x), (g\circ f'\circ h^{-1})(z)\big) < \tau.$$

But since $d\big(h^{-1}(x), h^{-1}(z)\big) < \mu(\tau)$, we have $d(x, z) < \tau$. Thus we see that

$$d\Big(\big(x, (\Phi(f))(x)\big), \mathrm{Gr}\big(\Phi(f')\big)\Big) < \tau.$$

Similarly, we can conclude that $d\Big(\big(x, (\Phi(f'))(x)\big), \mathrm{Gr}\big(\Phi(f)\big)\Big) < \tau$. Hence we have shown that $\varrho\Big(\mathrm{Gr}\big(\Phi(f)\big), \mathrm{Gr}\big(\Phi(f')\big)\Big) < \tau$, and consequently $d\big(\Phi(f), \Phi(f')\big) < \tau$. Thus $d\big(\Phi(f'), k\big) < \varepsilon$, and so we have shown that

$$S_{\mu(\tau)}(f) \cap \pi^{-1}(b, b') \subseteq \Phi^{-1}\big(S_\varepsilon(k) \cap \pi^{-1}(c, c')\big).$$

Therefore Φ is continuous, as asserted. It easily follows that Φ is a homeomorphism.

Let $f \in \pi^{-1}(b, b')$. If $x \in p^{-1}(b)$, then $d\big(f(x), (g\circ f)(x)\big) < \varepsilon$, so that $d\Big(f(x), \big(\Phi(f)\big)\big(h(x)\big)\Big) < \varepsilon$. Since $\Big(h(x), \big(\Phi(f)\big)\big(h(x)\big)\Big) \in \mathrm{Gr}\big(\Phi(f)\big)$, we see that $d\Big(\big(x, f(x)\big), \mathrm{Gr}\big(\Phi(f)\big)\Big) < \varepsilon$. As above, we infer that $\varrho\Big(\mathrm{Gr}(f), \mathrm{Gr}\big(\Phi(f)\big)\Big) < \varepsilon$, so that $d\big(f, \Phi(f)\big) < \varepsilon$. But then η satisfies the condition of (2.1), as claimed. Thus π is completely regular.

We can certainly conclude that π is an open map, and hence that the function $\varphi\colon B\times B\to\mathfrak{F}(\mathcal{M})$, defined by $\varphi(b, b') = \pi^{-1}(b, b')$, is a l.s.c. carrier.

The metric of $\mathcal{M}$ induces a metric on each fiber, and we will now show

(4.4) *The metric induced by $\mathcal{M}$ on $\pi^{-1}(b, b') = \big(p^{-1}(b')\big)^{p^{-1}(b)}$ is equivalent to the metric given by*

$$\bar{d}(f, g) = \sup_{x\in p^{-1}(b)} \big\{d\big(f(x), g(x)\big)\big\}.$$

Proof. To be consistent with the notation used above, we will denote an ε-neighborhood of $f\colon p^{-1}(b)\to p^{-1}(b')$, in the metric induced from $\mathcal{M}$, by $S_\varepsilon(f)$. $N_\varepsilon(f)$ will denote an ε-neighborhood of f in the metric $\bar{d}$.

But $N_\varepsilon(f) \subseteq S_\varepsilon(f)$, for if $g \in N_\varepsilon(f)$, then $d\Big(\big(x, f(x)\big), \big(x, g(x)\big)\Big) < \varepsilon$ for all $x \in p^{-1}(b)$, so that $\varrho\big(\mathrm{Gr}(f), \mathrm{Gr}(g)\big) < \varepsilon$ and thus $g \in S_\varepsilon(f)$.

On the other hand, consider $N_\varepsilon(f)$. Since F is compact, f is uniformly continuous. Thus, if $a > 0$ is given, there exists $\mu(a) > 0$ such that if $d(x, y) < \mu(a)$, then $d\big(f(x), f(y)\big) < a$. We may assume that $\mu(a) \leqslant a$. We claim that $S_{\mu(\frac{1}{2}\varepsilon)}(f) \subseteq N_\varepsilon(f)$. In order to see this, let $g \in S_{\mu(\frac{1}{2}\varepsilon)}(f)$. Then $\varrho\big(\mathrm{Gr}(f), \mathrm{Gr}(g)\big) < \mu(\tfrac{1}{2}\varepsilon)$. Let $x \in p^{-1}(b)$. There must exist $x' \in p^{-1}(b)$ with $d\Big(\big(x, g(x)\big), \big(x', f(x')\big)\Big) < \mu(\tfrac{1}{2}\varepsilon)$. But then $d(x, x') < \mu(\tfrac{1}{2}\varepsilon)$, so that $d\big(f(x), f(x')\big) < \tfrac{1}{2}\varepsilon$. Thus $d\big(f(x), g(x)\big) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$, so that $g \in N_\varepsilon(f)$. Hence the metrics are equivalent, as asserted.

Since F is compact, we now see that the metric of $\mathcal{M}$ induces the compact-open topology on $\pi^{-1}(b, b')$. But F is a compact ANR, so that ([4], p. 186) F^F is an ANR. But then ([4], p. 96) F^F is $\mathbf{LC}^n$ for all n. Finally, we conclude from (4.3) and (3.5) that the collection $\{\pi^{-1}(b, b')\}$ $(b, b' \in B)$ is equi-$\mathbf{LC}^n$ for all n. Since $\dim B < \infty$, we see that $\dim(B\times B) < \infty$. Thus we have enough equi-local connectedness to satisfy the hypothesis of (2.3) for the l.s.c. carrier φ. It only remains to investigate the completeness of $\mathcal{M}$. Thus we have

(4.5) *$\mathcal{M}$ is topologically complete.*

Proof. We will first show that if U and V are open sets in B such that $\overline{U}$ and $\overline{V}$ are compact, then $\pi^{-1}(\overline{U}\times\overline{V})$ is topologically complete.

We will denote $\pi^{-1}(\overline{U}\times\overline{V})$ by $\mathcal{M}_{\overline{U}\times\overline{V}}$. Let the collection of graphs of functions in $\mathcal{M}_{\overline{U}\times\overline{V}}$ be denoted by $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}$. Let the collection of closed and bounded subsets of $E\times E$ be denoted by $\mathcal{F}_0(E\times E)$; then $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}\subseteq\mathcal{F}_0(E\times E)$. With the Hausdorff metric, $\mathcal{F}_0(E\times E)$ is a metric space. Since E is complete, a classical result ([5], p. 198) allows us to conclude that $\mathcal{F}_0(E\times E)$ is complete. Thus $\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})$, the closure of $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}$ in $\mathcal{F}_0(E\times E)$, is also complete. We will show that $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}$ is a G_δ-set in $\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})$.

First, let $A\in\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})$. Suppose that $A=\lim\{\mathrm{Gr}(f_i)\}$, where $f_i\colon p^{-1}(b_i)\to p^{-1}(b_i')$, for $b_i\in\overline{U}$ and $b_i'\in\overline{V}$. By taking a subsequence, we may assume that $\varrho\big(\mathrm{Gr}(f_i), A\big)<1/i$ for each i. But then if $(x,y)\in A$, we can find $x_i\in p^{-1}(b_i)$ with $d(x_i,x)<1/i$ and $d\big(f_i(x_i),y\big)<1/i$. Thus $\{x_i\}$ converges to x, and $\{f_i(x_i)\}$ converges to y, so that $\{b_i\}$ converges to $p(x)$, and $\{b_i'\}$ converges to $p(y)$. If we put $b=\lim\{b_i\}$, and $b'=\lim\{b_i'\}$, then $(x,y)\in p^{-1}(b)\times p^{-1}(b')$. Thus we see that $A\subseteq p^{-1}(b)\times p^{-1}(b')$, where $b\in\overline{U}$ and $b'\in\overline{V}$.

Now define

$$\mathcal{A}_i=\left\{A\in\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})\;\middle|\;\begin{array}{l}A\subseteq p^{-1}(b)\times p^{-1}(b')\quad\text{and}\\ \sup\limits_{x\in p^{-1}(b)}\big\{\mathrm{diam}\big((\{x\}\times p^{-1}(b'))\cap A\big)\big\}\geqslant 1/i\end{array}\right\}.$$

We assert that $\mathcal{A}_i$ is closed for each i. To see this, let $B=\lim\{B_j\}$, where $B_j\in\mathcal{A}_i$ for $j=1,2,\ldots$, and $B\in\mathcal{F}_0(E\times E)$. Since $B_j\in\mathcal{A}_i$, then

$$\sup_{x\in p^{-1}(b_j)}\big\{\mathrm{diam}\big((\{x\}\times p^{-1}(b_j))\cap B_j\big)\big\}\geqslant 1/i,$$

where $B_j\subseteq p^{-1}(b_j)\times p^{-1}(b_j')$, and $(b_j,b_j')\in\overline{U}\times\overline{V}$ for $j=1,2,\ldots$

Let η be given with $0<\eta<1/3i$. Then we can find (x_j,t_j) and (x_j,t_j') in B_j, with $d(t_j,t_j')>1/i-\eta$, where $x_j\in p^{-1}(b_j)$ and $\{t_j,t_j'\}\subseteq p^{-1}(b_j')$ Now, by (3.4), $p^{-1}(\overline{U})$ and $p^{-1}(\overline{V})$ are compact. Hence we can assume (by taking subsequences, if necessary) that $\{x_j\}$ converges to $x\in p^{-1}(\overline{U})$, and that $\{t_j\},\{t_j'\}$ converge to $t,t'\in p^{-1}(\overline{V})$ respectively. Clearly $(x,t)\in B$ and $(x,t')\in B$.

Now since $\{t_j\}$ converges to t and $\{t_j'\}$ converges to t', we can choose j so large that $d(t_j,t)<\eta$ and $d(t_j',t')<\eta$. Then $d(t,t')+d(t,t_j)+d(t',t_j')\geqslant d(t_j,t_j')$, so that $d(t,t')\geqslant 1/i-\eta-\eta-\eta=1/i-3\eta$. Hence $d\big((x,t),(x,t')\big)\geqslant 1/i-3\eta$, so that

$$\mathrm{diam}\big((\{x\}\times p^{-1}(b'))\cap B\big)\geqslant 1/i-3\eta\qquad\big(\text{where } B\subseteq p^{-1}(b)\times p^{-1}(b')\big).$$

But η can be chosen arbitrarily small, so that we must have

$$\sup_{x\in p^{-1}(b)}\big\{\mathrm{diam}\big((\{x\}\times p^{-1}(b'))\cap B\big)\big\}\geqslant 1/i.$$

Hence $B\in\mathcal{A}_i$, so that $\mathcal{A}_i$ is closed, as claimed.

We will now show that $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}=\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})-\bigcup_{i=1}^{\infty}\mathcal{A}_i$. First, if $f\colon p^{-1}(b)\to p^{-1}(b')$, then $\mathrm{Gr}(f)\notin\mathcal{A}_i$ for all i, since for $x\in p^{-1}(b)$, $\big(\{x\}\times p^{-1}(b')\big)\cap\mathrm{Gr}(f)=\big\{\big(x,f(x)\big)\big\}$, which has diameter 0.

Thus it will suffice to show that

$$\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})-\bigcup_{i=1}^{\infty}\mathcal{A}_i\subseteq\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}.$$

To see this, let $A\in\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})-\bigcup_{i=1}^{\infty}\mathcal{A}_i$. Suppose that $A\subseteq p^{-1}(b)\times p^{-1}(b')$ and that $A=\lim\{\mathrm{Gr}(f_j)\}$, where $f_j\colon p^{-1}(b_j)\to p^{-1}(b_j')$, with $b_j\in\overline{U}$ and $b_j'\in\overline{V}$. It follows easily that $b=\lim\{b_j\}$ and $b'=\lim\{b_j'\}$, while $b\in\overline{U}$ and $b'\in\overline{V}$. If $x\in p^{-1}(b)$, we can find $x_j\in p^{-1}(b_j)$, with $\{x_j\}$ converging to x, and also such that $\big(x_j,f_j(x_j)\big)\in N_{1/j}(A)$ (perhaps after taking subsequences).

Then there exists $(y_j,z_j)\in A$, with $d(x_j,y_j)<1/j$ and $d\big(f_j(x_j),z_j\big)<1/j$. We may assume that the $\{z_j\}$ converge to $z\in p^{-1}(b)$. Clearly the $\{y_j\}$ converge to x, so that the $\{(y_j,z_j)\}$ converge to (x,z). But each $(y_j,z_j)\in A$, and A is closed, so that $(x,z)\in A$.

Thus we have that $\big(\{x\}\times p^{-1}(b)\big)\cap A\neq\emptyset$. Now if $(x,y)\in A$ and $(x,y')\in A$, then we must have $y=y'$, for if $d(y,y')>0$, then $A\in\mathcal{A}_i$ for some i, which is impossible. We can therefore define a function $f\colon p^{-1}(b)\to p^{-1}(b')$ by $f(x)=y$, where y is such that $(x,y)\in A$. Clearly $A=\mathrm{Gr}(f)$. It also follows that f is continuous, since $p^{-1}(b')$ is compact ([1], p. 228). But then $A\in\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}$, so that $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}=\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})-\bigcup_{i=1}^{\infty}\mathcal{A}_i$.

Hence $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}$ is a G_δ-set in the complete metric space $\mathbf{CL}(\hat{\mathcal{M}}_{\overline{U}\times\overline{V}})$, so that ([1], p. 308), $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}$ is topologically complete. Since the metric on $\mathcal{M}_{\overline{U}\times\overline{V}}$ was obtained from that on $\hat{\mathcal{M}}_{\overline{U}\times\overline{V}}$, we can evidently find a complete metric for $\mathcal{M}_{\overline{U}\times\overline{V}}$. Since π is continuous, $\pi^{-1}(U\times V)$ is open in $\mathcal{M}_{\overline{U}\times\overline{V}}$, so that $\pi^{-1}(U\times V)$ is topologically complete. Thus, since B is locally compact, we see that $\mathcal{M}$ is locally topologically complete. Finally, by ([1], p. 314), $\mathcal{M}$ is topologically complete.

Thus $\mathcal{M}$ has a complete metric, so that we have verified all the hypotheses of (2.3).

Now let $\Delta(B)=\{(b,b)\mid b\in B\}\subseteq B\times B$. We can define $s\colon\Delta(B)\to\mathcal{M}$ by $s(b,b)=\mathrm{id}_{p^{-1}(b)}$. s is evidently continuous, and $s(b,b)\in\varphi(b,b)=\pi^{-1}(b,b)$ for all $b\in B$. Since $\Delta(B)$ is closed in $B\times B$, we can apply (2.3) to find an open set U, where $U\supseteq\Delta(B)$, and a continuous extension $\bar{s}\colon U\to\mathcal{M}$ of s, where $\pi\bar{s}(b,b')=(b,b')$ for all $(b,b')\in U$.

Let $b\in B$. Then $(b,b)\in U$, so that we can find an open neighborhood V of b, with $(b,b)\in V\times V\subseteq U$. Define

$$\hat{s}_V\colon V\times p^{-1}(V)\to V\quad\text{by}\quad\hat{s}_V(c,x)=\big(\bar{s}\big(p(x),c\big)\big)(x),$$

where

$$c \in V \quad \text{and} \quad x \in p^{-1}(V).$$

We claim that $\hat{s}_V$ is continuous.

Let $\mathcal{N} = \{(f, x) | f: p^{-1}(b) \to p^{-1}(b') \text{ and } x \in p^{-1}(b)\}$. $\mathcal{N} \subseteq \mathcal{M} \times E$. Define $e: \mathcal{N} \to E$ by $e(f, x) = f(x)$. We will show that e is continuous. Suppose that W is open in E, and consider $e^{-1}(W) \subseteq \mathcal{N}$. Let $(f, x) \in e^{-1}(W)$, and choose $\varepsilon > 0$ such that $N_\varepsilon(f(x)) \subseteq W$. Since F is compact, f is uniformly continuous, so that given $a > 0$, there exists $\mu(a) > 0$ such that if $d(y, y') < \mu(a)$, then $d(f(y), f(y')) < a$. We can assume that $\mu(a) \leqslant a$. Put $\delta(\varepsilon, f) = \frac{1}{2}\mu(\frac{1}{2}\varepsilon)$. We claim that $(S_{\delta(\varepsilon,f)}(f) \times N_{\delta(\varepsilon,f)}(x)) \cap \mathcal{N} \subseteq e^{-1}(W)$.

To see this, let $(g, x') \in (S_{\delta(\varepsilon,f)}(f) \times N_{\delta(\varepsilon,f)}(x)) \cap \mathcal{N}$. Then $\varrho(\mathrm{Gr}(f), \mathrm{Gr}(g)) < \delta(\varepsilon, f)$, so that there must exist $x'' \in p^{-1}(b)$, with $d(x', x'') < \delta(\varepsilon, f)$ and $d(f(x''), g(x')) < \delta(\varepsilon, f)$. But then $d(x, x'') < 2\delta(\varepsilon, f) = \mu(\frac{1}{2}\varepsilon)$. Hence $d(f(x), f(x'')) < \frac{1}{2}\varepsilon$. Finally, we have $d(f(x), g(x')) < \frac{1}{2}\varepsilon + \delta(\varepsilon, f) < \varepsilon$, so that $g(x') \in N_\varepsilon(f(x)) \subseteq W$. Thus e is continuous.

Now $\hat{s}_V$ is obtained as the composition

$$(c, x) \to ((p(x), c), x) \to (\bar{s}(p(x), c), x) \overset{e}{\to} (\bar{s}(p(x), c))(x),$$

so that $\hat{s}_V$ is continuous. We also observe that $\hat{s}_V$ satisfies $p\hat{s}_V(c, x) = c$ and $\hat{s}_V(p(x), x) = x$, for all $(c, x) \in V \times p^{-1}(V)$.

Hence, in the language of ([1], p. 404), $\hat{s}_V$ is a slicing map for V, and V is a slicing neighborhood. But every point of B has such a neighborhood and map, so that $p: E \to B$ is a sliced fiber space ([3], p. 97). We then conclude, since B is metric, that p is a Hurewicz fibration ([1], p. 405).

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A characterization of strong inductive dimension*

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§ 1. Introduction. In [7] Nishiura has presented a theory for (weak) inductive invariants of separable metrizable spaces. (Weak) inductive invariants (first introduced by Lelek [5]) are obtained by replacing the empty set in the definition of (weak) inductive dimension by members of some family of topological spaces (see § 3 for precise definition).

By studying inductive invariants it is determined what part of dimension theory is due to the inductive nature of the definition of dimension and what part is due to the special role of the empty set.

In the paper of Nishiura, this has resulted in a characterization of weak inductive dimension on separable metrizable spaces by means of seven (independent) conditions.

Essentially, by weakening one of these conditions, a characterization of the strong inductive dimension on the class of all metrizable spaces is obtained (see § 2). In § 3 a theory for strong inductive invariants is developed in order to prove the independence of the conditions by which dimension is characterized (see § 4).

Throughout, $B(U)$ denotes the boundary of U. $\dim X$ stands for the strong inductive dimension of X. All spaces under discussion are *metrizable*.

§ 2. A characterization theorem. An extended real-valued function f defined on the class of metrizable spaces, is said to be *topological* (*monotone*) if $f(X) = f(Y)$ ($f(X) \leqslant f(Y)$) whenever X is homeomorphic to (is a subset of) Y. f is called *pseudo-inductive* if in each space X every non empty closed set has arbitrarily small neighborhoods U such that $f(B(U)) \leqslant f(X) - 1$ (we agree that $\infty - 1 = \infty$). f is *weakly subadditive* if for all X and Y we have

$$f(X \cup Y) \leqslant f(X) + f(Y) + 1$$

(cf. [7] inductively subadditive).

Now we state a theorem which characterizes dimension.

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