Refinements of Lebesgue covers

by

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1. Introduction. Let (X, ϱ) be a metric space and $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be a cover of X. The cover $\mathcal{G}$ is called *Lebesgue* if there exists some real number $\delta > 0$ such that for every $x \in X$, $S(x, \delta) \subseteq G_\alpha$ for some $\alpha \in A$. Here $S(x, \delta) = \{y \in X: \varrho(x, y) < \delta\}$. In [3], Theorem 3.3 the author has proved the following.

THEOREM 1.1. *Let (X, ϱ) be a metric space. If $\mathcal{G}$ is a countable Lebesgue cover of X, then $\mathcal{G}$ has a locally finite Lebesgue refinement.*

A natural question now arises as to what other types of Lebesgue covers of X have locally finite Lebesgue refinements. In § 2 we answer this question for point finite and star-countable Lebesgue covers and characterize such covers in terms of a "uniformly" locally finite property. In § 3 we show that all Lebesgue covers having a locally finite Lebesgue refinement also have the somewhat strong property of being "Lebesgue normal". In § 4 we generalize these results for uniform spaces.

2. Locally finite Lebesgue refinements.

DEFINITION 2.1. Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be a cover of a metric space (X, ϱ). Then $\mathcal{G}$ is *uniformly shrinkable* if there exists a real number $\delta > 0$ and a cover $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ such that

(1) $F_\alpha \subset G_\alpha$ for all $\alpha \in A$,

(2) $\varrho(F_\alpha, X - G_\alpha) > \delta$ for all $\alpha \in A$.

In [3], Theorem 2.2 the author proved the following

THEOREM 2.2. *Let $\mathcal{G}$ be a cover of a metric space (X, ϱ). Then $\mathcal{G}$ is Lebesgue if and only if $\mathcal{G}$ is uniformly shrinkable. Also the uniform shrink of $\mathcal{G}$ can be made Lebesgue.*

THEOREM 2.3. *Let $\mathcal{G}$ be a Lebesgue cover of a metric space (X, ϱ). If $\mathcal{G}$ is point finite, then $\mathcal{G}$ has a locally finite Lebesgue refinement.*

Proof. Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be a point finite Lebesgue cover of X. By Theorem 2.1 $\mathcal{G}$ is uniformly shrinkable. Hence there exists $\delta > 0$ and a Lebesgue cover $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ such that

(1) $F_\alpha \subset G_\alpha$ for all $\alpha \in A$,

(2) $\varrho(F_\alpha, X - G_\alpha) > \delta$ for all $\alpha \in A$.

Since x belongs to only finitely many G_α, then $S(x, \delta)$ intersects only finitely many F_α. Hence $\mathcal{F}$ is locally finite.

COROLLARY. *Every σ-point finite Lebesgue cover of a metric space (X, ϱ) has a locally finite Lebesgue refinement.*

Proof. Let $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ be a Lebesgue cover of (X, ϱ) such that $\mathcal{G}_i = \{G_\alpha\colon \alpha \in A_i\}$ is point finite. For each i define $G_i = \bigcup_{\alpha \in A_i} G_\alpha$ so that $\{G_i\colon i = 1, 2, \ldots\}$ is a countable Lebesgue cover of X. By Theorem 1.1 there exists a locally finite Lebesgue refinement $\{H_i\colon i = 1, 2, \ldots\}$ such that $H_i \subset G_i$ for all i. Clearly $\{\{H_i\} \wedge \mathcal{G}_i\colon i = 1, 2, \ldots\}$ is a point finite Lebesgue cover of X which refines $\mathcal{G}$. Hence by Theorem 2.3, $\mathcal{G}$ has a locally finite Lebesgue refinement.

Remark. It should be noted at this point that the neighborhoods used in both the proofs of Theorem 1.1 and Theorem 2.3 to establish local finiteness were of uniform size. This immediately brings up the question as to whether this type of uniformness plays an important role in finding locally finite Lebesgue refinements.

DEFINITION 2.4. Let $\mathcal{S} = \{S_\lambda\colon \lambda \in \Lambda\}$ be a collection of subsets of a metric space (X, ϱ). Then $\mathcal{S}$ is called *uniformly locally finite* if there exists a real number $\delta > 0$ such that for every $x \in X$, $S(x, \delta)$ intersects only finitely many members of $\mathcal{S}$.

Remark. This definition of "uniform local finiteness" should not be confused with that of Katětov [1]. Katětov's "uniform local finiteness" is stated for normal spaces and means that there exists an integer $n > 0$ such that each $x \in X$ has a neighborhood N_x which intersects at most n members of the given collection.

THEOREM 2.5. *Let $\mathcal{G}$ be a Lebesgue cover of a metric space (X, ϱ). Then $\mathcal{G}$ has a locally finite Lebesgue refinement if and only if $\mathcal{G}$ has a uniformly locally finite refinement.*

Proof. (i) Let $\mathcal{G} = \{G_\alpha\colon \alpha \in A\}$ have a locally finite Lebesgue refinement. We may assume that this refinement is one-to-one and hence denote it by $\mathcal{F} = \{F_\alpha\colon \alpha \in A\}$ so that $F_\alpha \subset G_\alpha$ for all $\alpha \in A$. Since $\mathcal{F}$ is Lebesgue, by Theorem 2.2 above there exists a uniform shrink $\mathcal{H} = \{H_\alpha\colon \alpha \in A\}$ such that $H_\alpha \subset F_\alpha$ and $\varrho(H_\alpha, X - F_\alpha) > \delta > 0$ for all $\alpha \in A$. Also since $\mathcal{F}$ is locally finite, for each $x \in X$, there exists a neighborhood N_x such that N_x intersects only finitely many members of $\mathcal{F}$. Therefore $S(N_x, \delta)$ intersects only finitely many members of $\mathcal{H}$, and hence $\mathcal{H}$ is uniformly locally finite and refines $\mathcal{G}$.

(ii) Let $\mathcal{G} = \{G_\alpha\colon \alpha \in A\}$ be a Lebesgue cover of X which has a uniformly locally finite refinement. Then there exists a $\delta > 0$ and a refinement $\mathcal{F}$ of $\mathcal{G}$ such that $S(x, \delta)$ intersects only finitely many members of $\mathcal{F}$ for all $x \in X$. Again we may assume $\mathcal{F} = \{F_\alpha\colon \alpha \in A\}$. Define $\mathcal{H} = \{H_\alpha\colon \alpha \in A\}$ where $H_\alpha = S(F_\alpha, \delta/2)$ for all $\alpha \in A$. Clearly $\mathcal{H}$ is Lebesgue and $S(x, \delta/2)$ intersects only finitely many members of $\mathcal{H}$ for all $x \in X$. Thus $\mathcal{G} \wedge \mathcal{H} = \{G_\alpha \cap H_\alpha\colon \alpha \in A\}$ refines $\mathcal{G}$ and is a locally finite Lebesgue cover.

COROLLARY. *Let $\mathcal{S} = \{S_\lambda\colon \lambda \in \Lambda\}$ be a uniformly locally finite collection in a metric space (X, ϱ). Then each S_λ can be enlarged uniformly such that the resulting collection is still uniformly locally finite.*

THEOREM 2.6. *Let $\mathcal{G} = \{G_\alpha\colon \alpha \in A\}$ be a star-countable open cover of a T_1 space X. We divide the index set A into subsets $\{A_\beta\colon \beta \in B\}$ such that α and γ belong to A_β if and only if there exists a positive integer n such that $G_\alpha \subset \mathrm{St}^n(G_\gamma, \mathcal{G})$. Define $X_\beta = \bigcup_{\alpha \in A_\beta} G_\alpha$. Then we have the following:*

(1) $X = \bigcup_{\beta \in B} X_\beta$,

(2) $X_\beta \cap X_{\beta'} = \emptyset$ *for* $\beta \neq \beta'$,

(3) X_β *is open and closed in* X *for each* $\beta \in B$,

(4) $\{G_\alpha\colon \alpha \in A_\beta\}$ *is a countable open cover of* X_β *for each* $\beta \in B$.

Proof. See [2], Theorem 2.

THEOREM 2.7. *Every star-countable Lebesgue cover of a metric space (X, ϱ) has a locally finite Lebesgue refinement.*

Proof. Let $\mathcal{G} = \{G_\alpha\colon \alpha \in A\}$ be a star-countable Lebesgue cover of (X, ϱ). By Theorem 2.6 above we partition X as follows:

(1) $X = \bigcup_{\beta \in B} X_\beta$,

(2) $X_\beta \cap X_{\beta'} = \emptyset$ if $\beta \neq \beta'$,

(3) X_β is open and closed in X for each $\beta \in B$,

(4) $\mathcal{G}_\beta = \{G_\alpha\colon \alpha \in A_\beta\}$ is a countable Lebesgue cover of X_β for each $\beta \in B$.

By Theorem 1.1 $\mathcal{G}_\beta$ has a locally finite Lebesgue refinement $\mathcal{F}_\beta$ for each $\beta \in B$. Since $\{X_\beta\colon \beta \in B\}$ is discrete, $\mathcal{F} = \bigcup_{\beta \in B} \mathcal{F}_\beta$ is a locally finite Lebesgue refinement of $\mathcal{G}$.

COROLLARY. *Every σ-star-countable Lebesgue cover of a metric space (X, ϱ) has a locally finite Lebesgue refinement.*

DEFINITION 2.8. Let $\mathcal{S} = \{S_\lambda\colon \lambda \in \Lambda\}$ be a collection of subsets of a metric space (X, ϱ). Then $\mathcal{S}$ is called *uniformly discrete* if there exists a real number $\delta > 0$ such that for each $x \in X$, $S(x, \delta)$ intersects at most one member of $\mathcal{S}$.

THEOREM 2.9. *Every star-countable Lebesgue cover of a metric space (X, ϱ) has a σ-uniformly discrete Lebesgue refinement.*

Proof. This follows immediately from Theorem 1.1 and Theorem 2.7 above.

3. Lebesgue normal covers.

Definition 3.1. Let X be a set and $\mathcal{G} = \{\mathcal{G}_\lambda: \lambda \in \Lambda\}$ be a collection of collections of subsets of X. For each $\lambda \in \Lambda$, let $\mathcal{G}_\lambda = \{G_\alpha: \alpha \in A_\lambda\}$. Then

$$\bigwedge_{\lambda \in \Lambda} \{\mathcal{G}_\lambda\} = \{\bigcap G_{\alpha(\lambda)}: \alpha(\lambda) \in A_\lambda, \lambda \in \Lambda\}.$$

Theorem 3.2. *Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be a locally finite Lebesgue cover of a metric space (X, ϱ). Then $\mathcal{G}$ has a Δ-refinement which is locally finite and Lebesgue.*

Proof. By Theorem 2.2 above $\mathcal{G}$ is uniformly shrinkable to a Lebesgue covers $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ such that

(1) $F_\alpha \subset G_\alpha$ for all $\alpha \in A$,

(2) $\varrho(F_\alpha, X - G_\alpha) > \delta > 0$ for all $\alpha \in A$.

Define $\mathcal{H} = \bigwedge_{\alpha \in A} \{G_\alpha, X - F_\alpha\}$. By [3], Lemma 1, $\mathcal{H}$ is a locally finite Lebesgue cover of X. It is well known [2] that if $H \in \mathcal{H}$ and $H \cap F_\alpha \neq \emptyset$, then $H \subset G_\alpha$. Therefore $\mathrm{St}(x, \mathcal{H}) \subset \mathrm{St}(F_\alpha, \mathcal{H}) \subset G_\alpha$ for some $\alpha \in A$, since $\mathcal{F}$ covers X. Hence $\mathcal{H}$ Δ-refines $\mathcal{G}$.

Definition 3.3. Let $\mathcal{G}$ be a Lebesgue cover of a metric space (X, ϱ). Then $\mathcal{G}$ is called *Lebesgue normal* if there exists a sequence of Lebesgue covers $\{\mathcal{G}_i\}_{i=1}^\infty$ such that $\mathcal{G} = \mathcal{G}_1$ and $\mathcal{G}_{n+1}$ is a $*$-refinement of $\mathcal{G}_n$ for $n = 1, 2, 3, \ldots$

Theorem 3.4. *Every locally finite Lebesgue cover of a metric space (X, ϱ) is Lebesgue normal.*

Proof. By Theorem 3.2 there exists a sequence of Lebesgue covers $\{\mathcal{G}_i\}_{i=1}^\infty$ such that $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_{n+1}$ is a Δ-refinement of $\mathcal{G}_n$ for $n = 1, 2, \ldots$ Define $\mathcal{G}_n^* = \mathcal{G}_{2n}$ for $n = 1, 2, \ldots$, so that now $\mathcal{G}_{n+1}^*$ is a $*$-refinement of $\mathcal{G}_n^*$. Hence $\mathcal{G}$ is Lebesgue normal.

Corollary. *The following types of Lebesgue covers are Lebesgue normal:*

(1) *point finite,*

(2) *σ-point finite,*

(3) *star-countable,*

(4) *σ-star-countable.*

Proof. All of the above have locally finite Lebesgue refinements.

4. Uniform spaces.

In [4] the concepts of Lebesgue covers, uniformly shrinkable, and uniformly separated are extended to uniform spaces. The reader is referred to this paper for these definitions. The following theorems are also proved.

Theorem 4.1. *A cover $\mathcal{G}$ of a uniform space $(X, \mathfrak{U})$ is Lebesgue if and only if $\mathcal{G}$ is $\mathfrak{U}$-shrinkable.*

Theorem 4.2. *Let $(X, \mathfrak{U})$ be a uniform space. Then every countable Lebesgue cover has a locally finite Lebesgue refinement.*

We now obtain analogous results for uniform spaces to those proved in the previous sections.

Theorem 4.3. *Let $\mathcal{G}$ be a Lebesgue cover of a uniform space $(X, \mathfrak{U})$. If $\mathcal{G}$ is point finite, then $\mathcal{G}$ has a locally finite Lebesgue refinement.*

Proof. Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be a point finite Lebesgue cover of X. Then there exists $U \in \mathfrak{U}$ such that $\{U(x): x \in X\}$ refines $\mathcal{G}$. Choose $V \in \mathfrak{U}$ such that V is symmetric and $V^2 \subset U$. Define $F_\alpha = \{x: V(x) \subset G_\alpha\}$ for all $\alpha \in A$ and $\mathcal{F} = \{F_\alpha: \alpha \in A\}$. Then by [4], Theorem 2.4, $\mathcal{F}$ is a Lebesgue cover which refines $\mathcal{G}$. Since $\mathcal{G}$ is point finite, $x \in X$ implies x belongs to only finitely many members of $\mathcal{G}$. We claim that $V(x)$ intersects only finitely many members of $\mathcal{F}$. For if $y \in V(x) \cap F_\alpha$, then $x \in V(y) \subset G_\alpha$. Hence $\mathcal{F}$ is locally finite.

Corollary. *Every σ-point finite Lebesgue cover of a uniform space $(X, \mathfrak{U})$ has a locally finite Lebesgue refinement.*

Definition 4.4. Let $\mathcal{S} = \{S_\lambda: \lambda \in \Lambda\}$ be a collection of subsets of a uniform space $(X, \mathfrak{U})$. Then $\mathcal{S}$ is called *uniformly locally finite* if there exists $U \in \mathfrak{U}$ such that for each $x \in X$, $U(x)$ intersects only finitely many members of $\mathcal{S}$.

Theorem 4.5. *Let $\mathcal{G}$ be a Lebesgue cover of a uniform space $(X, \mathfrak{U})$. Then $\mathcal{G}$ has a locally finite Lebesgue refinement if and only if $\mathcal{G}$ has a uniformly locally finite refinement.*

Proof. (1) Let $\mathcal{G}$ be any locally finite Lebesgue cover of X. Then any uniform shrink of $\mathcal{G}$ is a uniformly locally finite refinement as seen by Theorem 4.3.

(2) Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be a Lebesgue cover of X and $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ be a uniformly locally finite Lebesgue refinement. Then there exists $U \in \mathfrak{U}$ such that for each $x \in X$, $U(x)$ intersects only finitely many members of $\mathcal{F}$. Choose $V \in \mathfrak{U}$ such that V is symmetric and $V^2 \subset U$. Define $H_\alpha = \mathrm{St}(F_\alpha, \mathcal{V})$ and $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ where $\mathcal{V} = \{V(x): x \in X\}$. Thus $\mathcal{H}$ is a locally finite Lebesgue cover of X. Clearly $\mathcal{G} \wedge \mathcal{H}$ is the desired refinement as in Theorem 2.5 above.

Theorem 4.6. *Every star-countable Lebesgue cover of a uniform space $(X, \mathfrak{U})$ has a locally finite Lebesgue refinement.*

Proof. Same as Theorem 2.7.

Definition 4.7. Let $\mathcal{S} = \{S_\lambda: \lambda \in \Lambda\}$ be a collection of subsets of a uniform space $(X, \mathfrak{U})$. Then $\mathcal{S}$ is called *uniformly discrete* if there exists $U \in \mathfrak{U}$ such that for $x \in X$, $U(x)$ intersects at most one member of $\mathcal{S}$.

Theorem 4.8. *Every star-countable Lebesgue cover of a uniform space $(X, \mathfrak{U})$ has a σ-uniformly discrete Lebesgue refinement.*

Proof. Same as Theorem 2.9.

THEOREM 4.9. *Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be a locally finite Lebesgue cover of a uniform space $(X, \mathcal{U})$. Then $\mathcal{G}$ has a Δ-refinement which is locally finite and Lebesgue.*

Proof. By the proof of Theorem 3.2 above, it suffices to show that if $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ and $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ are Lebesgue covers of X and $\mathcal{F}$ is a uniform shrink of $\mathcal{G}$, then $\mathcal{H} = \bigwedge_{\alpha \in A} \{G_\alpha, X - F_\alpha\}$ is Lebesgue in the uniform sense. As before we may assume that there exists $U \in \mathcal{U}$ such that $F_\alpha = \{x: U(x) \subset G_\alpha\}$ for all $\alpha \in A$. Choose $V \in \mathcal{U}$ such that V is symmetric and $V^2 \subset U$. Let $x \in X$ and define $A_x = \{\alpha \in A: V(x) \subset G_\alpha\}$. Note that $\beta \notin A_x$ implies that $V(x) \cap (X - G_\beta) \neq \emptyset$, so let $z \in V(x) \cap (X - G_\beta)$. Then for $y \in V(x)$ we have $(x, y) \in V$ and $(x, z) \in V$, so that $(y, z) \in V^2 \subset U$. Thus $z \in U(y)$, and hence $y \notin F_\beta$. Therefore $V(x) \cap F_\beta = \emptyset$ for all $\beta \in A - A_x$. Finally we have $V(x) \subset [\bigcap_{\alpha \in A_x} G_\alpha] \cap [\bigcap_{\beta \in A - A_x} (X - F_\beta)]$, so that $\mathcal{H}$ is Lebesgue.

THEOREM 4.10. *Every locally finite Lebesgue cover of a uniform space $(X, \mathcal{U})$ is Lebesgue normal.*

5. Concluding remarks. It is still unknown whether an arbitrary Lebesgue cover of a metric space (X, ϱ) has a locally finite Lebesgue refinement. This problems seems very difficult. An affirmative answer to this question would answer a number of unsolved problems in Dimension Theory as well as give the extremely strong property that every Lebesgue cover is Lebesgue normal.

References

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On compactifications with continua as remainders

by

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1. Introduction. A *compactification* of a space X is a compact Hausdorff space $\hat{X}$ with a dense subspace X' homeomorphic to X. The set $\hat{X} - X'$ is called a remainder of X in $\hat{X}$. We are concerned here with spaces that have every continuum (compact connected metric space) as a remainder in some compactification. Aarts and van Emde Boas have shown [1] that every locally compact, non-compact, separable metric space is such a space. Earlier, in [4], K. D. Magill had given an argument that, as observed in [2], shows that every Peano continuum is a remainder in some compactification of any locally compact, non-pseudocompact Hausdorff space. (A space is *pseudocompact* if and only if there is no unbounded real-valued continuous function on it.) More recently, Steiner and Steiner have observed [5] that the methods of Aarts and van Emde Boas are also applicable to Magill's theorem. We show here that their methods can in fact be used to generalize both their theorem and Magill's, i.e. we show in Theorem 2 that non-pseudocompactness is a necessary and sufficient condition on a locally compact Hausdorff space X in order that every continuum be a remainder of a *certain type* of X in some compactification of X.

It would be of interest to characterize the spaces which have every continuum as a remainder in some compactification, without any added conditions on the remainder. We give in Section 3 an example to show that there is a pseudocompact space with this property.

2. Theorems.

DEFINITION. A collection G of subsets of a space X is *discrete* (in X) if and only if each point of X lies in an open subset of X which does not intersect two elements of G.

THEOREM 1. *A completely regular space is pseudocompact if and only if there is no infinite discrete collection of open subsets of it.*

Proof. If a space X is not pseudocompact, it is not difficult to get a map f from X into the non-negative real numbers such that $f(X)$ contains