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Concerning groups of dyadic relations of an arbitrary field [1]).

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§ 1. Introductory.

Upon the threshold of the logic of relations it is found that abstract relations present a variety of aspects which serve as principles of classification. Thus a relation may be dyadic or triadic or n-cornered; it may be finite or infinite; it may be homogeneous or non-homogeneous; it may be one-one, one-some, some-one, one-many, many-one, some-some, some-many, many-some, or many-many; it may be symmetric, asymmetric or non-symmetric; it may be transitive, intransitive, or non-transitive; and so on. Logical combination of such cardinal classes and the use of less generic principles of discrimination yield additional classes in endless number and variety. Equally familiar is the fact that there exist rules of combination — such as logical addition, logical multiplication, relative multiplication — by which any two relations of a class may be combined so as to produce a relation. If by a system of relations we mean a class of relations together with a rule of combination, it is evident that abstract relations offer for consideration many systems respecting which it is natural to enquire whether they do or do not constitute groups.

It seems needless to insist upon the interest or the importance of the general problem thus indicated. The following enquiry does not deal with the general problem but only with one aspect or fragment of it. With a few exceptions indicated in course of the

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discussion the relations herein studied are such as satisfy the following requirements: (1) the relations are dyadic and homogeneous; (2) they have a common field; (3) the field is infinite; (4) the domain, the co-domain and the field of a relation coincide.

It is to be added that the investigation is confined to systems in which the rule of combination is relative multiplication.

In accordance with requirement (4) it is to be noted that, whenever a relation is spoken of as „a relation of or belonging to such and such a field“, it is to be understood that the relation has the field for its domain and also for its co-domain.

It will be advantageous to employ certain technical terms and symbols borrowed from the literature of symbolic logic. Unless otherwise indicated these will be used in accordance with the definitions of them to be found in volume I of the *Principia Mathematica* of Messrs. Whitehead and Russell.

It should be carefully noted once for all that relations are to be taken in their extensional as distinguished from their intensional sense. A word reminding the reader of the nature of the distinction may be in place. If a dyadic relation R be defined, or determined, by a propositional function $F(x, y)$ containing two variables, the extension of R is the class of couples, or pairs of values of x and y, that satisfy the function. If R' be defined by another propositional function $\varphi(x, y)$, it may happen that the extension of R' is the same as that of R. In such case the only difference between R and R' is an intensional difference, a difference due solily to the distinction between two equivalent but nonidentical definitions Troughout this article intensional differences will be disregarded: two relations having the same extensions will be regarded as identical, the principle being that the extension of a relation *is* the relation.

§ 2. Certain Auxiliary Theorems.

The theorems of this section will be of frequent use in subsequent developments.

It will often be convenient to denote a field under consideration by the capital F. Unless the contrary is indicated, it will be understood that the field so denoted is infinite. The elements of a field — the terms of the relations — will be denoted by x, y, z, etc.

The proofs of theorems 1, 2 and 3, here set down for future reference, are omitted, these theorems being respectively $*$ 31·33, $*$ 34·2, $*$ 34·21 of the *Principia* above cited.

Theorem 1. — $\mathrm{Cnv}'\,\mathrm{Cnv}'\,R = R$: *the converse of the converse of a relation R is identical with R*

Theorem 2. $\mathrm{Cnv}'\,(R_1 \mid R_2) = \breve{R}_2 \mid \breve{R}_1$: *the converse of the relative product of R_1 by R_2 is the relative product of the converse of R_2 by the converse of R_1.*

Theorem 3. — $(R_1 \mid R_2) \mid R_3 = R_1 \mid (R_2 \mid R_3)$: *relative multiplication is associative.*

Theorem 4. — *If R be a relation of a field F (finite or infinite,) the converse $\breve{R}$ belongs to F.*

It is necessary and sufficient to prove that $D'\,\breve{R} = \text{Г}'\,\breve{R} = F$. By definition of $\breve{R}$ and Theorem 1, $D'\,R = \text{Г}'\,\breve{R}$, $\text{Г}'\,R = D'\,\breve{R}$; by hypothesis, $D'\,R = \text{Г}'\,R = F$; hence $D'\,\breve{R} = \text{Г}'\,\breve{R} = F$.

Theorem 5. *The relative product of any two relations, R_1, R_2, of a field (finite or infinite) is a relation of the field.*

Denote the field by F. It is to be proved that $D'\,(R_1 \mid R_2) = \text{Г}'\,(R_1 \mid R_2) = F$. Let x be any given element of F; by hypothesis F contains an element y and an element z such that $x\,R_1\,y \,.\, y\,R_2\,z$; $\therefore x\,(R_1 \mid R_2)\,z$; $\therefore$ (1) $F \subset D'\,(R_1 \mid R_2)$. If $x'\,(R_1 \mid R_2)\,z'$, then there exists a y' such that $x'\,R_1\,y'$; since R_1 is a relation of F, x' is an element of F; $\therefore$ (2) $D'\,(R_1 \mid R_2) \subset F$; from (1) and (2) it follows that (3) $D'\,(R_1 \mid R_2) = F$. Next let z be any given element of F; F contains a y and an x such that $y\,R_2\,z \,.\, x\,R_1\,y$; $\therefore x\,(R_1 \mid R_2)\,z$; $\therefore$ (4) $F \subset \text{Г}'\,(R_1 \mid R_2)$. If $x'\,(R_1 \mid R_2)\,z$, there exists a y' such that $y'\,R_2\,z'$; R_2 being a relation of F, z' is an element of F; $\therefore$ (5) $\text{Г}'\,(R_1 \mid R_2) \subset F$; (4) and (5) imply (6) $\text{Г}'\,(R_1 \mid R_2) = F$. From (3) and (6) follows the theorem.

Theorem 6. — *If two relations R_1 and R_2 of a field F (finite or infinite) be such that $R_1 \mid R_2 = R_2 \mid R_1 = R_i$, R_i being the identity relation of F, R_1 and R_2 are mutually converse one-one relations.*

For let x be any given element of F; then if $x\,(R_1 \mid R_2)\,z$, we have $x\,R_i\,z$, and hence $x = z$; hence, if α be the class of relata of x asto R_1 and β be the class of referents of x as to R_2, (1) $\alpha \subset \beta$; let y be any element of β; then for some element x', we have $x'\,R_1\,y \,.\, y\,R_2\,x$ and hence $x'\,(R_1 \mid R_2)\,x$; hence $x' = x$ and therefore y belongs to α; $\therefore$ (2) $\beta \subset \alpha$; by (1) and (2), $\alpha = \beta$. Now let y be any given element of α; then $y R_2 x . x R_1 y'$ and therefore also $y (R_2 \mid R_1) y'$, where y' is any element of α; since $y R_\iota y'$, $y' = y$; $\therefore \alpha$ contains but one element; hence R_1 relates a given referent to only one relatum and R_2 relates a given relatum to only one referent. Similarly it may be proved that R_2 relates a given referent to only one relatum and R_1 a given relatum to only one referent. Therefore R_1 and R_2 are one one relations.

It remains to prove that R_1 and R_2 arc mutually converse relations. Suppose $y \breve{R}_1 x$; then also $x R_1 y$; but, if $x R_1 y$, then $y R_2 x$ since $x (R_1 \mid R_2) x$ and since, if x be given, there is but one y for which $x R_1 y$. Hence $y \breve{R}_1 x$ implies $y R_2 x$, and $\therefore \breve{R}_1 \subset R_2$; in like manner, $R_2 \subset \breve{R}_1$; $\therefore R_2 = \breve{R}_1$, and by theorem 1, $R_1 = \breve{R}_2$.

We may now establish the converse of the preceding theorem.

Theorem 7. — *If R_1 and R_2 be mutually converse ono-one relations of a field F (finite or infinite), $R_1 \mid R_2 = R_2 \mid R_1 = R_\iota$, R_ι being the identity relation of F.*

Let x be any given element of F; we have $x R_1 y$ for some element y of F; hence $y \breve{R}_1 x$ and hence $y R_2 x$; $\therefore x (R_1 \mid R_2) x$ for every element of F; $\therefore (1)\ R_\iota \subset R_1 \mid R_2$. Next suppose $x (R_1 \mid R_1) z$; then there is a y for which $x R_1 y . y R_2 z$; for that y we have also $y R_2 x$, since $R_2 = \breve{R}_1$; but R_2 is a one-one relation; hence $z = x$; $\therefore$ (2) $R_1 \mid R_2 \subset R_\iota$; $\therefore$, by (1) and (2), $R_\iota = R_1 \mid R_2$; similarly, $R_\iota = R_2 \mid R_1$; hence also $R_1 \mid R_2 = R_2 \mid R_1$.

§ 3. Nine Types of Relations.

The discussion will henceforth deal with certain relation types including every variety of relation that may belong to an abstract field finite or infinite. The types in question, nine in number, are defined and symbolized as follows.

A *one one* (1—1) relation is a relation such that each of its referents has but one relatum and each relatum but one referent.

A *one-some* (1—s) relation: one such that at least one of its referents has but one relatum, at least one referent has two or more relata. and each relatum has but one referent.

A *some-one* (s—1) relation: one such that at least one of its relata has but one referent, at least one relatum has more than one referent, and each referent has but one relatum.

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A *one-many* (1—*m*) relation: one such that each referent has more than one relatum and each relatum but one referent.

A *many-one* (*m*—1) relation: one such that each of its relata has two or more referents and each of its referents but one relatum.

A *some-some* (*s*—*s*) relation: one such that at least one of its referents (relata) hat but one relatum (referent), and at least one referent (relatum) has two or more relata (referents).

A *some-any* (*s*—*m*) relation: one such that each referent has more than one relatum, at least one relatum has but one referent, and at least one relatum has two or more referents.

A *many-some* (*m*—*s*) relation: one such that each relatum has two or more referents, at least one referent has only one relatum, and at least one referent has more than one relatum.

A *many-many* (*m*—*m*) relation: one such that each of its referents has more than one relatum and each relatum has more than one referent.

Theorem 1. — (α) *No relation of a given field (finite or infinite) belongs to more than one of the nine types*; (β) *every relation of any given field (finite or infinite) belongs to one or another of the types*; — (γ) *relations representing all of the types may coexist in a single field.*

The truth of (α) is evident on comparing the definitions of the types.

To prove (β), let *R* denote any given relation of the given field. If *R* be not a 1—1 relation, then (1) some referent of *R* has more than one relatum or (2) some relatum has more than one referent or (3) both (1) and (2) hold. If (1) holds and (2) does not, then *R* is a 1—*m* or a 1—*s* relation according as the „some“ of (1) includes all the referents of *R* or only a proper part of then. If (2) holds and (1) does not, then *R* is a *m*—1 or a *s*—1 relation according as the „some“ of (2) includes all the relata or only a proper part of them. Suppose both (1) and (2) to hold; if (1) holds for only a proper part of the referents and (2) for only a proper part of the relata, then *R* is plainly of type *s*—*s*; if (1) holds for all referents end (2) for only part of the relata, *R's* type in *s*—*m*; if (1) holds for only a part of the referents and (2) for all relata, the type of *R* is *m*—*s*; finally, to make the only remaining hypothesis, if (1) holds for all referents and (2) for all relata. *R* is evidently of the type *m*—*m*.

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For proposition (γ) it will suffice to present a system of the required relations in a specified field. Let F be any given denumerably infinite ensemble of elements and suppose these to be denoted by the positive integers. It is obvions that the following nine relations $R_1, \dots, R_9$ representing the nine relation types coexist in F:

R_1 (type 1—1): (1, 2), (2, 1), (3, 4), (4, 3), ...
R_2 („ 1—s): (1, 1), (2, 2), (2, 3), (3, 4), (4, 5), (5, 6), ...
R_3 („ s—1): (1, 1), (2, 2), (3, 2), (4, 3), (5, 4), (6, 5), ...
R_4 („ 1—m): (1, 1), (1, 2), (2, 3), (2, 4), (3, 5), (3, 6), ...
R_5 („ m—1): (1, 1), (2, 1), (3, 2), (4, 2), (5, 3), (6, 3), ...
R_6 („ s—s): (1, 1), (2, 3), (2, 4), (3, 2), (4, 2), (5, 5), (6, 6), ...
R_7 („ s—m): (1, 1), (1, 2), (2, 2), (2, 3), (3, 4), (3, 5), (4, 6), (4, 7)....
R_8 („ m—s): the couples of s—m inverted
R_9 (type m—m): (1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 3), (4, 3), (4, 4),...

Theorem 2. *The nenessary and sufficient condition that the nine relation types be represented by the relations of a given field is that the field be infinite.*

The condition is necessary. For suppose F to be finite and to consist of the n elements: $l_1, l_2, \dots, l_n$. We prove that F has no relation of type 1—m. If F have such a relation R, each l is a referent of R having at least two of the l's for relata, and each l is a relatum having but one referent. Suppose the relata of l_1 to be l_{a_1} and l_{α_1}, those of l_2 to be l_{a_2} and l_{α_2} and so on, the relata of l_n being l_{a_n} and l_{α_n}. As no two of the l's with subscripts a or α can be idetical, their number is $2n$. Being the elements of F, their number is n; hence $n = 2n$, which is impossible since n is finite.

It remains to show that the condition of the theorem is sufficient. Let F be any given infinite field.

It is evident that any field has an identity relation R_i and that R_i is of 1—1 type.

[In the following argument the symbols $R_2, \dots, R_8$ will denote the relations denoted by them in the foregoing table].

Being infinite, F contains denumerably infinite classes of elements. Suppose the elements of one such class F_1 (a proper part of F) to be designated by the cardinal integers. Denote the class of the remaining elements of F by F_2. R_2, we have seen, is a relation of F_1; let R_i be the identity relation of F_2; it is evident that the logical sum, $R_2 \cup R_i$, of R_2 and R_i is a relation of F, the logical sum of F_1 and F_2, and that the former sum is of type $1-s$.

To show that F contains a relation of type $s-1$, it suffices to replace R_2 of the preceding paragraph by R_3 or one may proceed as follows: $R_3 = \breve{R}_2$, hence $R_3 \cup R_i = \mathrm{Cnv}'(R_2 \cup R_i)$; hence, by § 2, theorem 4, the relation $R_3 \cup R_i$ belongs to F; but this relation is of type $s-1$, being the converse of a relation of type $1-s$.

For the case $1-m$, let the denumerable part F_1 (of F) be such that the Cantor power of the remainder $F_2 =$ the prover of F. Suppose F_3 and F_4 to be non-intersecting parts of F_2 such that $F_3 \cup F_4 = F_2$ and that the powers of F_2, F_3, and F_4 are equal. Let R be a one-one relation between F_2 and F_3, and R' a one-one relation between F_2 and F_4. Any given element x of F, its R correspondent y and its R' correspondent z together yield a pair of couples (x, y) and (x, z). The ensemble of all such couples is evidently a $1-m$ relation belonging to F_2. Denote it by R''. The sum $R_4 \cup R''$ is a relation of F and is elearly of type $1-m$.

The converse of the last relation is $R_5 \cup \breve{R}''$, which belongs to F and is of type $m-1$.

As before let F_1 be a denumerably infinite proper part of F, and denote the remainder by F_2; then, if R_i be the indently relation of F_2, the relation $R_6 \cup R_i$, which belongs to F, is of type $s-s$.

Let R be the universal relation of the remainder F_2 (here permissibly supposed to contain more than one element). Then the relations $R_7 \cup R$ and $R_8 \cup R$ belong to F, their types being respectively $s-m$ and $m-s$.

Finally, if R and F_2 be taken as in the preceding paragraph, the relation $R_9 \cup R$, belongs to F and is of type $m-m$. Of course the universal relation of F is of $m-m$ type, „all" being, as Aristotle pointed out, „a species of many".

Theorem 3. — (α) *Relations of the types* $1-s$, $1-m$, $s-1$, $m-1$ *exist only in infinite fields*; (β) $s-s$ *and* $m-m$ *relations exist in all fields of more than one element;* (γ) $s-m$ *and* $m-s$ *relations exist in all fields of more than two elements.*

Proposition (α) has been proved for type $1-m$ in connection with the preceding theorem. By help of theorem 4, § 2, (α) is seen to hold for type $m-1$. Very similar argument, here omitted, will avail to show the validity of (α) for the types $1-s$ and $s-1$.

In prof (β) et F consist of a finite number n of elements:

$l_1, l_2, \ldots, l_n$. If $n=1$, it is evident that F has neither a $s-s$ nor a $m-m$ relation. If $n \gtreqless 2$, the relation consisting of the couples (l_1, l_1), (l_2, l_1), (l_2, l_2), (l_3, l_3), $(l_4, l_4), \ldots, (l_n, l_n)$ is of type $s-s$; and the relation,

$$(l_1, l_1), (l_1, l_2), (l_2, l_1), (l_2, l_2), (l_3, l_1), (l_3, l_2), (l_1, l_3), (l_2, l_3), \ldots,$$
$$(l_k, l_1), (l_k, l_2), (l_1, l_k), (l_2, l_k), \ldots$$

where $3 \lesseqgtr k \lesseqgtr n$, is of type $m-m$.

In respect to (γ), take F as above. If $n=1$, it is obvious that F has neither a $s-m$ nor a $m-s$ relation. If $n=2$, then, if F have a $s-m$ relation, either l_1 or l_2 is relatum to a single referent (l_1 or l_2) and the other (l_2 or l_1) is a relatum to both l_1 and l_2. Suppose l_1 has l_1 for sole referent, then l_1 cannot occur again as relatum, and l_2 must occur as relatum to both l_1 and l_2; and so arise the couples (l_1, l_1), (l_1, l_2), (l_2, l_2); thus for l_2 has but one relatum; it requires two and thus we get (l_2, l_1) which is inadmissible as giving l_1 a second referent; a like contradiction results from supposing l_1 to have l_2 for sole referent or l_2 to have l_1 or l_2 for sole referent; hence, if $n=2$, F has no $s-m$ relation and therefore, by § 2, theorem 4, no $m-s$ relation.

If, however, $n \gtreqless 3$, the relation, (l_1, l_1), (l_1, l_2), (l_2, l_2), (l_2, l_3), (l_3, l_2), (l_3, l_3), (l_4, l_3), $(l_4, l_4), \ldots, (l_n, l_{n-1})$, (l_n, l_n), is of type $s-m$, and the converse relation is of type $m-s$.

Cor. — *Of the nine types the* 1—1 *type is the only oni represented in every field (finite or infinite).*

§ 4. Types of Relative Products of Relations of Given Types.

By § 2, theorem 4, and § 3, theorem 1, it is seen that the relative product of two relations of any given field (finite or infinite) belongs to one of such of the nine types as are represented in the field.

The present section is devoted to the solution of the following

Problem: *To determine the relation types that may be represented by the relative product of two relations of given type or types.*

Inasmuch as the type of the relative product of two relations depends on both the order and the type of the factors, the solution of the problem involves finding a series of about four score theorems of the general from: The relative product of two relations belonging respectively to this and that given type may represent any one of

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such types and no other. Instead of a long series of formal theorematic stalements, each followed by its demonstration, it seems better, in the interest of economy, and quite sufficient (1) to describe the metod of finding the required theorems, (2) to exemplify its use, and (3) finally to state all of the theorems compactly in the form of a kind of relative-multiplication table exhibiting to the aye the possible types of product of relations of assigned type.

The method employed involves the use of four principles, which may be respectively called (A) Exclusion, (B) Inclusion, (C) Conversion, and (D) Construction, and which have the following meanings: (A), by direct use of the type definitions and the definition of relative product, serves to show that, among the types represented by the product of two relations of given type or types, certain types cannot occur, — are *excluded*; (B) asserts that, if the occurrence of a product type is not excluded by (A) in a given case, the type in question occurs in that case, it is *included*; (C) asserts that, if certain types may be represented by the product of two relations of given type (or types), the converses of the former types may be represented by the product of two relations whose type (or types) is the converse (converses) of the given type (or types); principle (D) consists in actually construing (or exhibiting) relations to show the occurrence of types whose occurrence is not excluded by (A).

It is evident that (A) and (B) are alone sufficient; (C) has been employed merely to abbreviate the work, and (D) for the purpose of confirmation, in the interest of certainty.

To exemplify the use of the principles, suppose R_1 and R_2 are of type $1—s$. We show that $R_1 \mid R_2$ may be of type $1—s$ or $1—m$ but of no other type. Let z be any given element of the field F; there is in F one and but one x, and one and but one y, such that $x R_1 y . y R_2 z$; $\therefore R_1 \mid R_2$ can represent none of the types $s-1$, $m—1$, $s—s$, $s—m$, $m-s$, $m—m$. For some y, $y R_2 z . y R_2 z'$, where $z \neq z'$; hence the x for which $x R_1 y$ has at least two relata as to $R_1 \mid R_2$, which is, therefore, not a $1 \mid 1$ relation. Hence $R_1 \mid R_2$ must be of type $1—s$ or $1-m$: it will be of type $1—s$ if, as obviously may happen. y is sole relatum of x asto R_1 and z is sole relatum asto R_2; and it will be of type $1—m$ if, as may happen, y be the only element that is a sole relatum as to F_1 and have two or more relata as to F_2.

By (C) it follows that, if R_1 and R_2 are of type $s—1$, $R_1 \mid R_2$ is of type $s—1$ or $m—1$ but of no other type.

For another example, let R_1 be of type $1—s$ and R_2 of type $m—1$. We show that $R_1 \mid R_2$ may represent any of the types $s—1$, $m—s$, $m—1$, $s—s$, but no other. It is obvious that (A) excludes the types $1—1$, $1-s$, $s—m$, $m—m$. By (B) the theorem follows. By (D) we confirm the conclusion. Let F be any denumerably infinite ensemble, whose elements are designated by the cardinal integers. $R_1 \mid R_2$ will be of the type indicated below if the R's denote the indicated relations

$s—1$ { R_1: (1, 2), (2, 3), (3, 1), (2, 4), (3, 5), (4, 6),...
R_2: (2, 3), (3, 1), (1, 2), (4, 1), (5, 2), (6, 3), (7, 4), (8, 4), (9, 5), (10, 5),...

$m—s$ { R_1: (1, 3), (2, 4), (3, 1), (2, 5), (4, 2), (5, 6), (1, 7), (6, 8), (7, 9), (8, 10),...
R_2: (1, 2), (2, 3), (3, 1), (4, 1), (5, 2), (6, 3), (7, 4), (8, 4),...

$m—1$ { R_1: same as preceding
R_2: (1, 2), (2, 3), (3, 1), (4, 1), (5, 1), (6, 2), (7, 1), (8, 3), (9, 3), (10, 4), (11, 4),..,

$s—s$ { R_1: (1, 2), (1, 3), (2, 1), (2, 4), (3, 5), (4, 6), (5, 7),...
R_2: (1. 2), (2, 3), (3, 4), (4, 2), (5, 1), (6, 1), (7, 3), (8, 4), (9, 5), (10, 5), (11, 6), (12, 6),...

By (C) it follows that, if R_1 be of type $1—m$ and R_2 of type $s—1$, $R_1 \mid R_2$ will represent any of the types $1—s$, $s—m$, $1—m$, $s—s$ but no other.

The theorems thens found are embodied in the following table. Each line, except the top line (which merely contains the symbols of the nine relation types under consideration), is to be regarded as stating a theorem. For exemple, the 11[th] line means: the relative product of a relation of type $1—s$ by a relation of type $s-1$ may belong to any of the types $1—1$, $1-s$, $s—1$, $s-s$ but to no other type.

There is no restriction upon the field of the relations [except (4), § 1], provided we agree to say that, in case the factors of a relative product do not exist in a certain field, a theorem respecting the product is satisfied „vacuously“; otherwise, the validity of a theorem regarding a product requires that the field contain the factors.

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Relative-Multiplication Table

Showing

Types Represented by Relative Products of Relations of Assigned Types

		1—1	1—s	s—1	1—m	m—1	s—s	s—m	m—s	m—m
1—1	1—1	1—1								
	1—s		1—s							
	s—1			s—1						
	1—m				1—m					
	m—1					m—1				
	s—s						s—s			
	s—m							s—m		
	m—s								m—s	
	m—m									m—m
1—s	1—s		1—s		1—m					
	s—1	1—1	1—s	s—1			s—s			
	1—m				1—m					
	m—1			s—1		m—1	s—s		m—s	
	s—s		1—s		1—m		s—s	s—m		
	s—m				1—m			s—m		
	m—s						s—s	s—m	m—s	m—m
	m—m							s—m		m—m
	1—1		1—s							
s—1	s—1			s—1		m—1				
	1—m							s—m		
	m—1					m—1				
	s—s						s—s		m—s	
	s—m							m—s		m—m
	m—s								m—s	
	m—m									m—m
	1—1			s—1						
	1—s						s—s			

	1—1	1—s	s—1	1—m	m—1	s—s	s—m	m—s	m—m
1—m | 1—m	1—1			1—m					
m—1		1—s	s—1	1—m	m—1	s—s	s—m	m—s	m—m
s—s		1—s		1—m		s—s	s—m		
s—m				1—m			s—m		
m—s		1—s		1—m		s—s	s—m	m—s	m—m
m—m				1—m			s—m		m—m
1—1				1—m					
1—s				1—m					
s—1		1—s		1—m		s—s	s—m		
m—1 | m—1					m—1				
s—s								m—s	
s—m									m—m
m—s								m—s	
m—m									m—m
1—1					m—1				
1—s								m—s	
s—1					m—1				
1—m									m—m
s—s | s—s						s—s	s—m	m—s	m—m
s—m							s—m		m—m
m—s						s—s	s—m	m—s	m—m
m—m							s—m		m—m
1—1						s—s			
1—s						s—s	s—m		
s—1			s—1		m—1	s—s		m—s	
1—m							s—m		
m—1			s—1		m—1	s—s		m—s	
s—m | s—m							s—m		m—m
m—s						s—s	s—m	m—s	m—m
m—m							s—m		m—m
1—1							s—m		
1—s							s—m		
s—1						s—s	s—m	m—s	m—m
1—m							s—m		
m—1			s—1		m—1	s—s	s—m	m—s	m—m
s—s						s—s	s—m	m—s	m—m

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	1—1	1—*s*	*s*—1	1—*m*	*m*—1	*s*—*s*	*s*—*m*	*m*—*s*	*m*—*m*
m—*s* | *m*—*s*								*m*—*s*	*m*—*m*
m—*m*									*m*—*m*
1—1								*m*—*s*	
1—*s*								*m*—*s*	*m*—*m*
s—1								*m*—*s*	
1—*m*					*m*—1				*m*—*m*
m—1					*m*—1			*m*—*s*	
s—*s*								*m*—*s*	*m*—*m*
s—*m*									*m*—*m*
m—*m* | *m*—*m*									*m*—*m*
1—1									*m*—*m*
1—*s*									*m*—*m*
s—1								*m*—*s*	*m*—*m*
1—*m*									*m*—*m*
m—1					*m*—1			*m*—*s*	—*m*
s—*s*								*m*—*s*	*m*—*m*
s—*m*									*m*—*m*
m—*s*								*m*—*s*	*m*—*m*

§ 5. Group and Non-group Systems Determined by the Nine Relation Types.

The relations of given field that belong to a given type constitute a class (of relations). A field containing relations of all the types thus furnishes nine *primary* classes. Logical addition of these, k at a time, $k \leqq q$, yields a large variety of additional classes. A class may be said to be of such and such a type: if a class be primary, its type will be that of the relations composing it; if a class be the logical sum of two or more primary classes its type will be indicated by the symbol for the logical sum of the symbols of the types of the component classes; thus, for example, the type of the class composed of the class $s—s$, the class $m—s$ and the class $m—m$ is $s—s \cup m—s \cup m—m$. Each of these classes together with a rule of combination (relative multiplication) is a system.

It will be convenient bo say that the type of a system is the same as that of the class contained in it; and a primary system is one whose class is primary. In this section such systems are examined with a view to ascertaining the presence or absence of group properties.

For that purpose I shall employ the following definition of a group. A system Σ (composed of a class C and a rule of combination) is a group if and only if it satisfies the following conditions:

(*a*) If a and b are elements of C, $a \circ b$ is an element of C.

(*b*) If a, b, c are elements of C, $(a \circ b) \circ C = a \circ (b \circ c)$.

(*c*) C contains an element i such that, if a be an element of C, $a \circ i = i \circ a = a$.

(*d*) C contains an element ε such that, if a be an element of C, there is in C an element a' such that $a \circ a' = a' \circ a = \varepsilon$.

(*e*) If i and ε are in C, $i = \varepsilon$.

Theorem 1. — *Condition* (*b*) *is satisfied by a system of any type.*

The theorem follows from theorem 3, § 2.

Theorem 2. — *Of the nine primary systems that of type* 1—1 *is the only one satisfying condition* (*c*).

That (*c*) is satisfied by the system 1—1 is obvions, the element i being the identity relation of the field.

That none of the other systems satisfies (*c*) will be shown by proving that the contrary supposition involves a contradiction.

System of type 1—*s*. Suppose it to satisfy (*c*); let i be denoted by R'; R' has at least one referent, say y, such that $y\,R'\,z$, where $z \neq y$; there is in the system a relation R such that $x\,R\,y$, where $x \neq y$, and that $x\,R\,y'$ only when $y' = y$; since $x\,R\,y\,.\,y\,R'\,z$, $x\,(R \mid R')\,z$; hence $x\,R\,z$, which is impossible, since $z \neq y$.

System of type 1—*m*. Suppose (*c*) tobe satisfied and denote i by R'; as to R' any given element, say x, has at least two relata $z_1, \dots$; suppose $z_1 \neq x$; for any R of the system, $z_1\,R\,z$ for some element z; since $x\,R'\,z_1\,.\,z_1\,R\,z$, $x\,(R' \mid R)\,z$; and, since $R' \mid R = R$ by hypothesis, we have $x\,R\,z$, but this is impossible since R is of type 1—*m* and $x \neq z_1$.

For the systems *s*—*s* and *m*—*s* the argument is the same as for the system 1—*s*.

System of type *m*—*m*. Suppose (*c*) to be satisfied and denote i by R'; any given referent of R', say y, has two or more relata $z_1, z_2, \dots$; suppose $z_1 \neq y$; the system contains a relation R such that $z_1\,R\,y$

but not $y R y$; we have $y R' z_1 . z_1 R y$; hence, $y (R' \mid R) y$, and, by hypothesis, $y R y$, which is impossible.

Systems s—1, m—1, s—m. Evidently the relations of these are respectively the converses of the relations of the systems 1—s, 1—m, m—s. Suppose the system s—1 to satisfy (c) and denote i by R'; let R be any relation of the system; then $R \mid R' = R' \mid R = R$, whence, by conversion, $\breve{R}' \mid \breve{R} = \breve{R} \mid \breve{R}' = \breve{R}$; since R is any relation of the system s—1, $\breve{R}$ is any relation of the system 1—s; hence, if the former system satifies (c), so does the latter; but we have proved that the latter does not. In exactly like manner the theorem may be proved for the remaining systems m—1 and s—m.

Theorem 3. — *Of the primary systems, those of types* 1—1 *and* m—m *are the only ones that satisfy condition* (d).

It is obvious that (d) is satisfied by the 1—1 system, ε being the identity relation of the field, and by the m—m system, ε being the universal (all-all) relation of the field.

System 1—s. Suppose (d) to be satisfied, denote ε by R'; R' has a referent x having but one relatum z; the system contains a relation R_1 such that $x R_1 y . x R_1 y'$, $y' \neq y$; hence, by hypothesis, the system contains a relation R_2 such that $x (R_1 \mid R_2) z$; hence $y R_2 z . y' R_2 z$; which is impossible since R_2 is a 1—s relation and $y \neq y'$.

System 1—m. If there be a ε, denote it by R'. We first prove $R' \mid R' \neq R'$. Asto R' any given referent, say x, has two or more relata (1) $z_1, z_2, \ldots$ or (2) $x, z_2, \ldots$, where no $z = x$. In case (1), $x (R' \mid R') z_1$; hence $x R' y . y R' z_1$, where $y = z_1$ or $z_2, \ldots$; say $y = z_2$, then $z_2 R' z_1$, which is impossible. In case (2), $x (R' \mid R') x, z_2, \ldots$; hence $x R' y . y R' x, z_2, \ldots$; if we take $y = z_2$, as we may, then $z_2 R' x$ or $z_2 R' z_2$ or $z' R' w$, where w is some relatum of x asto R'; but all these alternatives are impossible. As $R' \mid R' \neq R'$, there must be an $R (\neq R')$ such that $R' \mid R = R \mid R' = R'$. This we show to be impossible. For a given x we have (1) or (2) as above. If (1), then $x (R' \mid R) z_1, z_2, \ldots$; hence $x R' y . y R z_1$; as $y =$ one of the z's, say z_2, we have $z_2 R z_1$, which is impossible. If (2), then $x (R \mid R') x, z_1, \ldots$; hence $x R y . y R' z_2$; $y = x$, for else, we should not have $y R' z_2$; $\therefore R$ cannot be of type 1—m.

The argument for the systems s—s, s—m is very like the foregoing. For the systems s—1, m—1, m—s, the method of conversion is available as in the preceding theorem.

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Theorem 4. — *Condition (e) is satisfied by all the primary systems.*

This theorem, in which the term satisfied means „vacuously" satisfied, except asto the system of 1—1 type (where the term has its usual signification), is an obvious corollary to the two preceding theorems.

The two following theorems embrace the chief results to which the foregoing discussion has led. They are immediate consequences of the three preceding theorems together with those embodied in the Table of § 4.

Theorem 5. — *Among the* 511 *systems that arise on taking relative multiplication as a rule of combination with each of the primary classes of relations of an infinite field and with each of the classes formed of the primary classes by logical addition, there are* 30 *and only* 30 *systems that have the group property; one of them satisfies all of the five group conditions; all of them satisfy condition (b) — the associative law;* 16 *of them satisfy condition (c);* 21 *of them satisfy condition (d); and* 19 *of them satisfy condition (e). These systems, of which* 4 *are primary, are of the following types:*

1—1: G_{abcde} [1])
1—m: G_{abe}
m—1: „
m—m: G_{abde}
1—1 ∪ 1—m: G_{abce}
1 –1 ∪ m—1: „
1—1 ∪ m—m: G_{abcd}
1—s ∪ 1—m: G_{abe}
s—1 ∪ m—1: „
s—m ∪ m—m: G_{abde}
m—s ∪ m—m: „
1—1 ∪ 1—s ∪ 1—m: G_{abce}
1—1 ∪ s—1 ∪ m—1: „
1—1 ∪ s m ∪ m—m: G_{abcd}
1—1 ∪ m—s ∪ m—m: „
1—m ∪ s—m ∪ m—m: G_{abde}
m—1 ∪ m—s ∪ m—m: „

[1]) Each type symbol is followed by G with subscripts. These indicate that the corresponding group conditions are satisfied.

$1-s \cup 1-m \cup s-m \cup m-m$: G_{abde}
$s-1 \cup m-1 \cup m-s \cup m-m$: „
$s-s \cup s-m \cup m-s \cup m-m$: G_{abde}
$1-1 \cup 1-m \cup s-m \cup m-m$: G_{abcd}
$1-1 \cup m-1 \cup m-s \cup m-m$: „
$1-1 \cup 1-s \cup 1-m \cup s-m \cup m-m$: G_{abcd}
$1-1 \cup s-1 \cup m-1 \cup m-s \cup m-m$: „
$1-1 \cup s-s \cup s-m \cup m-s \cup m-m$: „
$1-s \cup 1-m \cup s-s \cup s-m \cup m-s \cup m-m$: G_{abde}
$s-1 \cup m-1 \cup s-s \cup m-s \cup s-m \cup m-m$: „
$1-1 \cup 1-s \cup 1-m \cup s-s \cup s-m \cup m-s \cup m-m$: G_{abcd}
$1-1 \cup s-1 \cup m-1 \cup s-s \cup m-s \cup s-m \cup m-m$: „
$1-1 \cup 1-s \cup 1-m \cup s-s \cup s-m \cup s-1 \cup m-1 \cup m-s \cup m-m$: G_{abcd}

Theorem 5. — (α) *If the field be finite and contain more than three elements there are* 8 *and but* 8 *systems having the group property: namely, those whose types are those which remain on deleting in the above list the type symbols involving* $1-s$, $1-m$, $s-1$ *or* $m-1$; *and* (β) *if the field contain but two elements, there are* 2 *and but* 2 *systems having the group property: namely the systems of types* $1-1$ *and* $m-m$.

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