Concerning the sum of two continua each irreducible between the same pair of points.

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In considering a question concerning the division of a plane by an irreducible continu [1]), the question arose as to whether two different continua, each irreducible between the same pair of points could have a sum irreducible between two of its points. In § 1 of the present paper I shall show that this question may be answered in the affirmative while in § 2 we shall show certain properties of continua, irreducible between the same pair of points and having the above mentioned property with respect to their sum.

§ 1.

Let us first prove the following theorem:

Theorem A. *Suppose M and N are two different bounded continua, then the necessary and sufficient condition that their sum S be irreducible between two of its points C_1 and C_2 is that S be expressible as the sum of two sets M_1 and M_2 such that*

(1) *M_1 and M_2 have points in common*

(2) *M_i $(i=1, 2)$ is an irreducible continu between C_i and each point common to M_1 and M_2.*

Proof: *The conditions are necessary.* Let us suppose that M and N are two different bounded continua whose sum S is irreducible

[1]) A set of points is said to be *connected* if, however it be divided into two mutually exclusive sets, one of them contains a limit point of the other one. A *continu*, is a set that is both closed and connected. A continu is said to be *indecomposable* whenever it is not the sum of two continua each different from itself.

between C_1 and C_2. It is clear that C_1 and C_2 are neither both in M nor both in N. Suppose C_1 is in M and C_2 is in N. It follows that $(S-M)'$ [1]) is a continu [2]) containing C_2 and lying entirely in N while $[S-(S-M)']'$ is a continu containing C_1 and lying in M. Let $[S-(S-M']'$ be designated by M_1 while $(S-M)'$ is designated by M_2.

As S is connected, M_1 and M_2 have at least one point in common. Now as C_1 belongs to $[S-(S-M)']'$, the continu $[S-(S-M)']'$ is irreducible between C_1 and each point common to $(S-M)'$ and $[S-(S-M)']'$ [3]). In other words M_1 is irreducible between C_1 and every point common to M_2 and M_1.

As $(S-[S-M]')'$ is a subset of M, it does not contain C_2 which is therefore a point of $\{S-(S-[S-M]')'\}'$. But $\{S-(S-[S-M]')'\}' \equiv$ $\equiv (S-M)'$ [4]), i. e. $[S-(S-M_2)']' \equiv M_2$. Then by the theorem of Kuratowski quoted before as C_2 belongs to $[S-(S-M_2)']'$, the continu $[S-(S-M_2)']'$ is irreducible between C_2 and each point common to $[S-(S-M_2)']'$ and $(S-M_2)'$, which is $[S-(S-M)']'$. But as $[S-(S-M_2)']' \equiv M_2$ and $[S-(S-M)']' \equiv M_1$, then M_2 is irreducible between C_2 and every point common to M_2 and M_1.

The conditions are sufficient. Let us suppose that M and N are two different bounded continua satisfying the conditions of our theorem. Then S, the sum of M and N is the sum of two continua M_1 and M_2, containing points C_1 and C_2, respectively, such that (1) M_i is irreducible between C_i and every point common to M_1 and M_2 and (2) M_1 and M_2 have points in common. Let us suppose S is not irreducible between C_1 and C_2. Then as S is a bounded continu, there is an irreducible continu K from C_1 to C_2 such that K is a proper subset of S [5]). Let H denote the set of

[1]) If K is a point set, K' shall denote the point set composed of K plus its limit points.

[2]) Cf. C. Kuratowski, *Théorie des continus irréductibles entre deux points*, Fundamenta Mathematicae, vol. III, (1922) theorem III p. 203.

[3]) Cf. C. Kuratowski, loc. cit. Theorem VI p. 205.

[4]) Cf. C. Kuratowski, *L'Opération $\overline{A}$ d'Analysis Situs.* Fundamenta Mathematicae, vol. III theorem 6, p. 183. See also remark under demonstration of Theorem 6 of Théorie des continus irréductibles loc. cit. p. 205.

[5]) Cf. S. Janiszewski, *Sur les continus irréductibles entre deux points* Journal de l'Ecole Polytechnique ser. 2, vol. 16, Theorem 1, p. 109.

points common to K and M_1. Let g the component[1]) of H which contains C_1. Let us suppose g contains no limit point of M_2 and hence no point of M_2. The set g is closed. Then there exists an ε such that for any point P of g and any circle T having P as centre and ε as radius, there is no point of M_2 within or on T. Hence by a theorem due to Zoretti[2]) there is a simple closed curve $\bar{g}$ containing g in its interior, having on it no point of H and such that all points on $\bar{g}$ are at a distance less than $\varepsilon/4$ from g. Thus there are no point of M_2 on $\bar{g}$. Now as K is a connected set having a point C_1 within and a point C_2 without $\bar{g}$, then K must have at least one point Q on $\bar{g}$. Now, as Q cannot belong to M_2, it is a point to M_1. But this contradicts the fact that no point of H is on $\bar{g}$. Hence we are led to a contradiction if we suppose g contains no point of M_2. Hence as g is a closed connected set containing C_1 and a point of M_2 and as g is also a subset of M_1, then g is identical with M_1, which is irreducible between C_1 and every point common to M_1 and M_2. In like manner the set K must contain all points of M_2. Hence K is identical with S and hence S is irreducible between C_1 and C_2. Thus the conditions in Theorem A are both necessary and sufficient.

By a theorem due to Mazurkiewicz[3]), if M is an indecomposable continu there are on M three points A, B, and C such that M is irreducible between any two of the three points A, B and C. Let M_1 and M_2 designate two bounded indecomposable continua such that there exist four distinct points A, B, C_1 and C_2 such that

(1) M_i $(i = 1, 2)$ is irreducible between any two of the three points A, B and C_i.

(2) M_1 and M_2 have only A and B in common[4]). It follows with the use of theorem A that $S = M_1 + M_2$ is irreducible between C_1 and C_2.

[1]) Cf. Hausdorff, Grundzüge der Mengenlehre, Leipzig 1914 p. 245. If A is a non vacuous set, then Hausdorff defines as a *component of A* a connected subset of A, which is not a proper subset of another connected subset of A.

[2]) Cf. L. Zoretti, *Sur les fonctions analytiques uniformes*, Journal de Mathématiques pures et appliquées 6 series vol. 1 (1905) p. 10—11.

[3]) Cf. S. Mazurkiewicz, *Un théorème sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1. pp. 35—39.

[4]) That it is possible to find indecomposable continua having these properties may be shown as follows: — Let g denote the simple closed curve composed

§ 2.

Theorem B. *Suppose M and N are two different bounded continua each irreducible between the same pair of points A and B and such that their sum S is irreducible between C_1 and C_2. Under these conditions the continua M_1 and M_2 (of theorem A) may be supposed such that*

(*a*) *M_1 and M_2 are each irreducible between the same pair of points $\bar{A}$ and $\bar{B}$.*

(*b*) *M_1 and M_2 are both indecomposable.*

(*c*) $(S-M_i)' = M_{i+1}$ [1])

Proof. — Suppose M and N are two different continua satisfying the conditions of our theorem. Clearly C_1 and C_2 are neither both in M nor both in N. Suppose C_1 is in M and C_2 is in N. It follows that $(S-N)'$ and $(S-M)'$ are continua [2]) containing C_1 and C_2, respectively. Let $(S-N)' = M_1$ and $(S-M)' = M_2$.

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of the arc of $x^2+y^2=25$ for which $y \geqslant 0$ plus the arc of $x^2+y^2-8y=25$ for which $y \geqslant 0$. Let K_1 denote the indecomposable continu of Dr. Knaster described on pp. 269—70 of volume III of the Fundamenta Mathematicae. Let Π_{11} denote a continuous (1—1) correspondence between the arc of $x^2+y^2=25$ for which $y \geqslant 0$ and the arc from (0, 1) to (1, 0) which is made up of the straight line interval from (0, 1) to (0, 0) plus the straight line interval from (0. 0) to (1, 0) such that $\Pi_{11}\,(0, 1) = (-5, 0)$ and $\Pi_{11}\,(1, 0) = (5, 0)$. Let Π_{12} be a continuous (1—1) correspondence between the arc of $x^2+y^2-8y=25$ for which $y \geqslant 0$ and the arc from (0, 1) to (1, 0) composed of the straight line interval from (0, 1) to (1, 1) plus the straight line interval from (1, 1) to (1, 0) such that $\Pi_{12}\,(0, 1) = (-5, 0)$ and $\Pi_{12}\,(1, 0) = (5, 0)$. Then the correspondence $\Pi = \Pi_{11} + \Pi_{12}$ is a continuous (1—1) correspondence between the simple closed curve g and the square S of Knaster's example. Then by a theorem due to Schoenflies published on page 324 of Volume 62 of the Mathematische Annalen, there exists a contin-ious (1—1) correspondence between the square plus its interior and the simple closed curve g plus its interior of such a nature that points on S and g correspond as fixed by the correspondence Π. It is clear that the set K_1 of Knaster's example will be such that $\Sigma(K_1)$ which we shall designate by M_1 will be indecomposable and that it will be irreducible between any two of the three points $\Sigma(0, 1) = (-5, 0)$, $\Sigma(\frac{1}{2}, \frac{1}{2})$ and $\Sigma(1, 0) = (5, 0)$. Call $(-5, 0)$, A; $\Sigma(\frac{1}{2}, \frac{1}{2})$, C_1 and $(5, 0)$, B.

Let M_2 and C_2 designate the sets symmetric to M_1 and C_1, respectively, with respect to the X axis. Clearly M_2 will be such that (1) M_1 and M_2 have only the points A and B in common. (2) M_2 is irreducible between any two of the three points A, B and C_2.

[1]) It is understood throughout this argument that subscripts are reduced modulo 2.

[2]) Cf. C. Kuratowski, loc. cit. theorem III p. 203.

We shall now show that $S = M_1 + M_2$. Several cases may arise: —

Case a I $(S-N)' \neq M$ and $[M-(S-N)']'$ is a continu. Now, one of the points A and B must lie while the other is in $[M-(S-N)']'$. Suppose $(S-N)'$ contains A and $[M-(S-N)']'$ contains B. Now let us suppose $S \neq (S-N)' + (S-M)'$. Then it follows that $(S-N)'$ and $(S-M)'$ are continua having no point in common for otherwise S would not be irreducible between C_1 and C_2.

Two cases may arise: —

(α) $[N-(S-M)']' = H_1 + H_2$, two mutually separated continua [1]). Then as $(S-M)' + H_i$ $(i = 1, 2)$ is a continu A must lie in H_1 and B in H_2. Then $(S-N)' + H_1 + (S-M)'$ is a continu of S containing C_1 and C_2 and not containing B. Thus in this case we are led to a contradiction.

(b) $[N-(S-M)']'$ is a continu. Then it follows from the fact that $(S-N)'$ and $(S-M)'$ have no points in common and that N is irreducible between A and B, that B is in $(S-M)'$ and A is in $[N-(S-M)']'$. Now as $[N-(S-M)']'$ and $[M-(S-N)']'$ are continua common to M and N, they can have no point in common, otherwise we would have a continu common to M and N, containing both A and B.

Now $(S-N)' + [M-(S-N)']' + (S-M)'$ is a continu containing both C_1 and C_2 and therefore is identical with S. Hence as $[N-(S-M)']'$ has no point in common with $[M-(S-N)']'$, $[N-(S-M)']'$ is a subset of $(S-N)' + (S-M)'$. But $(S-N)' + [N-(S-M)']' + (S-M)'$ is a continu containing both C_1 and C_2 and is hence identical with S. But as $[N-(S-M)']'$ is a subset of $(S-N)' + (S-M)'$, it follows that $S = (S-N)' + (S-M)'$.

Case a II. $[M-(S-N)']' = H_1 + H_2$, two mutually separated continua. It follows as before that A is in H_1 and B in H_2. Suppose $S \neq (S-N)' + (S-M)'$. Then $(S-N)'$ and $(S-M)'$ have no points in common. It also follows that as $H_i + (S-N)'$ is connected, then H_i $(i = 1, 2)$ has at least one point in $(S-N)'$.

Two cases may arise:

(i) $[N-(S-M)']'$ is a continu. Then A is in $(S-M)'$ and B

1) Two point sets are said to be *mutually separated* if they have no point or limit point in common.

is in $[N-(S-M)']'$. Then $(S-N)'+H_1+(S-M)'$ is a continu from C_1 to C_2 not containing B. Thus in this case we are led to a contradiction.

(β) The set $[N-(S-M)']'=K_1+K_2$, two mutually separated continua, A contained in K_1 and B contained in K_2. Then as $(S-N)'+H_1+K_1+(S-M)'$ is a continu containing C_1 and C_2, it must be identical with S. As $(S-N)'+(S-M)'\neq S$, $(S-N)'$ and $(S-M)'$ are mutually separated. As H_1 and H_2 are mutually separated, H_2 is a subset of $(S-N)'+K_1+(S-M)'$. The set K_1 can have no point in common with H_2, otherwise H_2+K_1 is a continu common to M and N containing A and B. Hence H_2 is a subset of $(S-N)'+(S-M)'$. But we know H_2 has at least one point in common with $(S-N)'$ and that H_2 is connected. Hence H_2 has no point in common with $(S-M)'$, a set mutually separated from $(S-N)'$. Thus H_2 is a subset of $(S-N)'$. As B is in H_2, then $[M-(S-N)']'$ is a continu [1]). Hence in this case we are led to a contradiction if $S\neq(S-N)'+(S-M)'$.

We shall now show that M_i $(i=1,2)$ is irreducible between C_i and every point common to M_1 and M_2. For suppose M_1 were not irreducible between C_1 and a point P, common to M_1 and M_2. Then there is a proper subcontinu H of M_1 which contains C_1 and P. As $H+M_2=H+(S-M)'$ is a continu containing C_1 and C_2, it follows that $H+(S-M)'$ is identical with S. Hence all points of $S-N$ are in $H+(S-M)'$. Hence all points of $(S-N)$ are in H. But H is closed. Hence all points of $(S-N)'$ are in H, contrary to the assumption that H is a proper subset of $(S-N)'$. In like manner we may show that M_2 is irreducible between C_2 and every point common to M_1 and M_2.

Proof that condition (*a*) *is satisfied*. Nine concevable cases may arise. —

(Case 1) $(S-N)'\equiv M$ and $(S-M)'\equiv N$. The points $\overline{A}$ and $\overline{B}$ are then the points A and B, respectively.

(Cese 2) $(S-N)'\equiv M$ and $[N-(S-M)']'$ is a continu containing B. Consider a point B^*, common to $[N-(S-M)']'$ and $(S-M)'$. Now B^* is in $(S-N)'$ and $(S-M)'$. Suppose $(S-N)'$ is not irreducible between A and B^*. Then there is a proper subcontinu K of $(S-N)'$ containing A and B^*. Consider two possibilities

[1]) Cf. C. Kuratowski, loc. cit. Theorem II, p. 202.

(a) The set K contains C_1. As K is a proper subcontinu of $(S-N)'$ and contains A, it follows that K cannot contain B, for $M\equiv(S-N)'$ is irreducible between A and B. Then $K+(S-M)'$ is a continu from C_1 to C_2 not containing B contrary to the hypothesis of our theorem.

(b) The set K does not contain C_1. Now $[N-(S-M)']'$ is common to M and N and hence does not contain C_1. But $[N-(S-M)']'$ is a continu of M containing B^* and B. Hence $K+[N-(S-M)']'$ is a continu of M from A to B, not containing C_1. This is contrary to our hypothesis concerning M.

Suppose $(S-M)'$ is not irreducible from A to B^*. Then there is a proper subcontinu R of $(S-M)'$ containing A and B^* Consider the two possibilities:

(i) R contains C_2. Then $R+M$ is a continu as R and M have point A in common. But $R+M$ contains C_1 and C_2. Hence $R+M=S$. Hence all points of $S-M$ are in R. But R is closed. Hence $(S-M)'$ is a subset of R. But R was supposed a proper subset of $(S-M)'$. Hence in this case we are led to a contradiction.

(ii). R does not contain C_2. Then $R+[N-(S-M)']'$ is a continu belonging to N and containing A and B but not containing C_2, contrary to our hypothesis concerning N.

In case 2, let A be the point $\overline{A}$ and B^* the point $\overline{B}$.

Case 3. $(S-N)'=M$ and $[N-(S-M)']'=H_1+H_2$, two mutually separated continua, H_1 containing A and H_2 containing B. Let A^* be a point common to H_1 and $(S-M)'$ and B^* a point common to H_2 and $(S-M)'$. We shall show that $M\equiv(S-N)'$ is irreducible between A^* and B^*. For suppose it were not. Then there exists a proper subcontinu K of M containing A^* and B^*.

Suppose K contains C_1. Then $K+(S-M)'$ is a continu containing C_1 and C_2 and hence is identical with S. Hence all points of $S-N$ are in the closed set K. Hence $(S-N)'$ is in K, contrary to the supposition that the set K is a proper subset of $(S-N)'$.

Suppose K does not contain C_1. Then H_1+K+H_2 is a continu of M containing A and B and not containing C_1, contrary to our assumption concerning M.

In like manner we may prove that $(S-M)'$ is irreducible between A^* and B^*.

In case 3 $\overline{A}$ and $\overline{B}$ will be, respectively, the points A^* and B^*

Case 4. $(S-M)' \equiv N$ and $[M-(S-N)']'$ is a continu. In this case we proceed exactly as in case 2 above.

Case 5. $[M-(S-N)']'$ is a continu and $[N-(S-M)']'$ is a continu. Suppose $(S-N)'$ contains A. Then B is in $[M-(S-N)']'$ and also in $(S-M)'$. Let B^* be a point common to $[M-(S-N)']'$ and $(S-N)'$ and A^* a point common to $[N-(S-M)']'$ and $(S-M)'$. It follows as before that A^* and B^* satisfy respectively the requirements for $\overline{A}$ and $\overline{B}$.

Case 6. $[M-(S-N)']'$ is a continu containing B and $[N-(S-M)']' = H_1 + H_2$, two mutually separated continua where H_1 contains A and H_2 contains B. Let A^* denote a point common to $(S-M)'$, $(S-N)'$ and H_1 while B^* is a point common to $(S-M)'$ and H_2. It may be proved easily that A^* and B^* satisfy the conditions for $\overline{A}$ and $\overline{B}$.

It may easily be proved by arguments similar to those above that we may find the points $\overline{A}$ and $\overline{B}$ in the remaining three cases which follow: —

Case 7. $[M-(S-N)']' = H_1 + H_2$, two mutually separated continua and $(S-M)' \equiv N$.

Case 8. $[M-(S-N)']' = H_1 + H_2$, two mutually separated continua and $[N-(S-M)']'$ is a continu.

Case 9. $[M-(S-N)']' = H_1 + H_2$ two mutually separated continua and $[N-(S-M)']' = K_1 + K_2$, two mutually separated continua.

It is thus true that each of the continua M_i $(i = 1, 2)$ is irreducible between any two of the three points $\overline{A}$, $\overline{B}$ and C_i. Hence by a theorem due to Janiszewski and Kuratowski [1]), M_i is indecomposable.

We shall now show that $(S-M_1)' \equiv M_2$. Now as $S \equiv (S-N)' + (S-M)'$, it follows that $S-(S-N)'$ is a subset of $(S-M)'$. Hence $[S-(S-N)']'$ is a subset of $(S-M)'$. Suppose $[S-(S-N)']'$ is not identical with $(S-M)'$. As $(S-M)'$ is an irreducible continu between the point C_2 and every point common to $(S-M)'$ and $(S-N)'$ and the continu $[S-(S-N)']'$ contains C_2, then it follows that $[(S-M)' - \{S-(S-N)'\}']'$ is a continu. Now $[S-(S-N)']'$ consists of points in $(S-M)'$ which are not also

[1]) Cf. Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1. p. 285.

in $(S-N)'$ plus all points $[X]$ such that X is a limit point of those points of $(S-M)'$ which are not in $(S-N)'$. Hence $\{(S-M)'-[S-(S-N)']'\}$ is made up of points $[Y]$ such that (1) Y is common to $(S-M)'$ and $(S-N)'$ and (2) Y is not a limit point of the set of points which belong to $(S-M)'$ and not to $(S-N)'$. Hence all points of $\{(S-M)'-[S-(S-N)']'\}$ are common to $(S-N)'$ and $(S-M)'$ and thus $\{(S-M)'-[S-(S-N)']'\}'$ is a continu common to $(S-M)'$ and $(S-N)'$ and hence not containing C_2. Hence $(S-M)'$ is the sum of two continua $[S-(S-N)']'$ and $[(S-M)'-\{S-(S-N)'\}]'$ each different from $(S-M)'$. This contradicts the fact that $(S-M)'$ is indecomposable. In like manner it follows that $[S-(S-M)']'\equiv(S-N)'$.

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