A characterization of a continuous curve.

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In this paper I will establish two theorems relating to continuous curves.

Lemma 1. *If K is a closed subset of a continuous curve M and H is a maximal connected subset of $M-K$ then either H is identical with $M-K$ or H and $(M-K)-H$ are mutually separated.*[1]

Proof. Since H is a *maximal* connected subset of $M-K$ it is clear that no point of $(M-K)-H$ is a limit point of H. That no point of H is a limit point of $(M-K)-H$ is a consequence of a result established on Page 256 of my paper *Concerning continuous curves in the plane*[2]. It is also a consequence of a theorem of Kuratowski's[3] together with Theorem I of that paper.

Lemma 2. *If N is a closed subset of a continuum M and M_1 is a maximal connected subset of N and L is a bounded subcontinuum of M which contains at least one point of M_1 and at least one point of $M-M_1$ then L contains at least one point of M_1 which is a limit point of $M-N$.*

Proof. Since the connected point set L contains a point of M_1 and M_1 is a *maximal* connected subset of N and L is a subset of M but not of M_1, therefore L must contain a point of $M-N$. Let T denote the derived set of $M-N$ and let $\overline{L}$ denote the set

[1] Two point sets are said to be mutually separated if they have no point in common and neither of them contains a limit point of the other one

[2] Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260.

[3] *Une définition topologique de la ligne de Jordan*, Fund. Math., vol. 1 (1920), pp. 40–43.

of points common to L and M_1. Suppose that L contains no point of M_1 which is a limit point of $M-N$. Then the closed point sets $\overline{L}$ and T have no point in common. But the connected point set L contains at least one point of $\overline{L}$ and at least one point of T. Hence [1]) L contains a connected subset H such that neither $\overline{L}$ nor T contains a point of H but each of them contains at least one limit point of H. Let H_1 denote the point set composed of H together with all those limit points of H which belong to $\overline{L}$. Since the connected point set H_1 contains a point of M_1 and is a subset of N and M_1 is a maximal connected subset of N therefore H_1 is a subset of M_1. But M_1 is closed. Hence H' (the derived set of H) is a subset of M_1. But H' contains at least one point which belongs to T and which is therefore a limit point of $M-N$. Furthermore H' is a subset of L. The truth of Lemma 2 is therefore established.

Definition. The subset K of the connected point set M is said to *separate A from B in M* if $M-K$ is the sum of two mutually separated point sets which contain A and B respectively.

Theorem I. *In order that a continuum M should be a continuous curve it is necessary and sufficient that for every two distinct points A and B of M there should exist a subset of M which consists of a finite number of continua and which separates A from B in M.*

P r o o f. I will first show that this condition is sufficient. Suppose, on the contrary, that there exists a continuum M which is not a continuous curve but which has the property that every two of its points can be separated from each other in M by a subset of M which consists of a finite number of continua. Since M is not a continuous curve there exist [2]) two concentric circles k_1 and k_2 (k_2 being within k_1) and a countable infinity of continua $\overline{M}, M_1, M_2, M_3, \ldots$ such that (1) each of these continua is a subset of M and contains at least one point of k_1 and at least one point of k_2 and is a subset of the point set H which is composed of the two circles k_1

[1]) See Theorem 1 of the thesis of Miss Anna M. Mullikin, Transactions of the American Mathematical Society, vol. 24 (1922), pp. 144–162.

[2]) Cf. my *Report on continuous curves from the viewpoint of analysis situs*, Bull. Amer. Math. Soc, vol. 29 (1923), p. 296. See also my papers *A characterization of Jordan regions by properties having no reference to their boundaries*, Proc. Nat Acad. Sc., vol. 4 (1918), pp. 364–370, and *Continuous sets that have no continuous sets of condensation*, Bull. Amer. Math. Soc., vol. 25 (1918), pp. 174–176.

and k_2 together with all those points of the plane which lie between these circles, (2) no two of these continua have a point in common and, indeed, no one of them is a proper subset of any connected point set which is common to M and H, (3) the set $\overline{M}$ is the sequential limiting set [1]) of the sequence of sets $M_1, M_2, M_3, \ldots$ For each n, let a_n denote the closed set of points common to M_n and k_1 and let b_n denote the set common to M_n and k_2. For each n, let A_n and B_n denote definite points belonging to a_n and b_n respectively. There clearly exist two points A and B and a sequence of distinct positive integers $n_1, n_2, n_3, \ldots$ such that A and B are sequential limit points of the sequences $A_{n_1}, A_{n_2}, \ldots$ and $B_{n_1}, B_{n_2}, \ldots$ respectively. By hypothesis the continuum M contains a subcontinuum L such that (a) $M-L$ is the sum of two mutually separated point sets U and V which contain A and B respectively, (b) L is the sum of a finite number of continua $L_1, L_2, L_3, \ldots L_n$. Since neither A nor B belongs to the closed point set L and A is not a limit point of V and B is not a limit point of U, therefore there exist circles C_A and C_B, with centers at A and B respectively, such that C_A encloses no point of $L+V$ and C_B encloses no point of $L+U$. There exists an integer δ such that, for every j greater than δ, the point set a_{n_j} is wholly within C_A and the point set b_{n_j} is wholly within C_B. Thus, for every j greater than δ, M_{n_j} contains a point A_{n_j} which belongs to U and a point B_{n_j} which belongs to V. But M_{n_j} is a subcontinuum of M and every subcontinuum of M which contains a point of U and a point of V must contain a point of L. Hence, for every j greater than δ, M_{n_j} contains a point of L and therefore of some one of the sets $L_1, L_2, \ldots L_m$. It follows that there exists an integer g and an infinite sequence of distinct integers $t_1, t_2, t_3, \ldots$ such that, for every j, L_g contains at least one point in common with M_{t_j}. Since, for

[1]) The point set M is said to be the *limiting set* of the sequence of point sets $M_1, M_2, M_3, \ldots$ provided that (a) each point of M is the sequential limit of an infinite subsequence of some sequence of points $P_1, P_2, P_3, \ldots$ such that, for every n, P_n belongs to M_n, (b) if $P_1, P_2, P_3, \ldots$ is a sequence of points such that, for every n, P_n belongs to M_n then M contains the sequential limit point of every infinite subsequence of $P_1, P_2, P_3, \ldots$ that has a sequential limit point. If the further condition is satisfied that every infinite subsequence of the sequence $M_1, M_2, M_3, \ldots$ has the same limiting set M then M is said to be the *sequential* limiting set of the sequence $M_1, M_2, M_3, \ldots$

every j, the subcontinuum L_g of the continuum M contains a point of M_{t_j} and a point of $M_{t_{j+1}}$ and M_{t_j} and $M_{t_{j+1}}$ are maximal subcontinua of the point set which is common to H and M, therefore, by Lemma 2, L_g must contain a point either of a_{t_j} or of b_{t_j}. Thus there exists an infinite sequence of distinct integers $j_1, j_2, j_3, \ldots$ such that either L_g has at least one point in common with each point set of the sequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ or it has at least one point in common with each point set of the sequence $b_{j_1}, b_{j_2}, b_{j_3}, \ldots$ In the first case A is a limit point of L_g while in the second case B is a limit point of L_g. But L_g is closed. Hence it contains either A or B. But this is not the case. Thus the supposition that the condition of Theorem 1 is not sufficient has led to a contradiction.

That the condition of Theorem I is necessary may be proved as follows. Suppose that A and B are two points belonging to the continuous curve M. Let C_1 denote a circle with center at A and radius equal to one half the distance between A and B and let C_2 denote a circle concentric with C_1 and lying within it. Let H denote the set of points consisting of C_1 and C_2 together with all those points which lie between C_1 and C_2. The curve M contains [1]) at least one subcontinuum N which is a subset of H and which contains at least one point of C_1 and at least one point of C_2. The greatest connected point set which contains N and is common to H and M is a continuum. Let G denote the set of all such continua, that is to say the set of all those maximal connected subsets of M which lie wholly in H and contain one or more points of C_1 and one or more points of C_2. Since M is a continuous curve there are not infinitely many of these continua. But is has been proved that there is at least one of them. Hence G is a finite set of mutually exclusive continua $L_1, L_2, L_3, \ldots L_n$. Let L denote the closed point set obtained by adding together the points of these continua. Let M_A denote the greatest connected subset of $M - L$ which contains A. The point B does not belong to M_A. For if it did then [2]) it could be joined to A by a simple continuous arc which lies wholly in M_A and this arc would contain as a subset an arc t which is a subset of H and which has its endpoints on C_1 and

[1]) Cf. Miss Anna M. Mullikin, loc. cit.

[2]) See Theorem I of my paper *Concerning continuous curves in the plane*, loc. cit.

C_2 respectively and the arc t would necessarily be a subset of some continuum of the set G, contrary to the fact that M_A and L have no point common. It follows that B belongs to the set $(M—L)—M_A$. But, since L is closed and M is a continuous curve it follows by Lemma I that the point sets M_A and $(M-L)-M_A$ are mutually separated. Hence L separates A from B in M

Theorem 2. *In order that a bounded continuum M should be a continuous curve which contains no domain and does not separate the plane it is necessary and sufficient that for every two distinct points A and B which belong to M there should exist a point which separates A from B in M.*

Proof. This condition is sufficient. For suppose that M is a bounded continuum which fulfills this condition. By Theorem 1, M is a continuous curve. If M contained a domain no two points of that domain would be separated from each other in M by any point of M, contrary to hypothesis. Suppose that M separates the plane. Then [1]) M contains a simple closed curve J. If A and B are any two points of J and P is any point of J distinct from A and from B then, of the two arcs of J which have A and B as endpoints, at least one fails to contain P. Hence P does not separate A from B in M. The sufficiency of the condition of Theorem 2 is therefore established.

This condition is also necessary. For suppose that M is a bounded continuous curve which neither contains a domain nor separates the plane. Let A and B denote two distinct points of M. The curve M contains [2]) a simple continous arc t with extremities at A and B. Let P denote any point, other than A and B, which belongs to the arc t. If $M—P$ contained a connected subset containing A and B then it would [3]) contain a simple continuous arc $\bar{t}$, with extremities at A and B, and the point set composed of the arcs t and $\bar{t}$ would contain a simple closed curve and, therefore, since M contains no domain, M would separate the plane, contrary to hypothesis. Hence,

[1]) Loc. cit. Theorem 5.

[2]) Cf. R. L. Moore, *A theorem concerning continuous curves*, Bull. Amer. Math. Soc, vol. 23 (1917), pp. 233—236; S. Mazurkiewicz, *Sur les lignes de Jordan*, Fund. Math., vol. 1. pp. 166—209; H. Tietze, *Ueber stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven*, Math. Zeitschr. vol. 5 (1919), pp. 284—291.

[3]) R. L. Moore, Mathematische Zeitschrift, loc. cit.

if M_A denotes the greatest connected subset of $M-P$ which contains A, the set M_A does not contain B. It follows, by Lemma 1, that P separates A from B in M.

If A and B are two distinct points of a continuum M and K is a closed subset of M which contains neither A nor B but which has at least one point in common with every subcontinuum of M which contains both A and B, then, according to a definition given by Mazurkiewicz [1]), it is said that „K découpe M entre A et B“ or that K constitutes „une coupure de M entre A et B“. In this case I will say that K separates A from B in M *in the weak sense.*

Neither Theorem 1 nor Theorem 2 remains true if the phrase „separates A from B in M“ is replaced by the phrase „separates A from B in M in the weak sense“. The truth of this statement may be seen with the help of the following example.

Example. Let O denote the point (0, 0), let C denote the point (1, 0) and, for every positive integer n, let C_n denote the point (1, $1/n$). Let M denote the point set composed of the straight line intervals OC, OC_1, OC_2, OC_3,... If A and B are any two distinct points of M there exists a point X which separates A from B in M in the weak sense. But M is not a continuous curve. It is easy to verify the fact that if, in this example, A and B are any two distinct points which lie between A and B then there exists no subcontinuum of M which consists of a finite number of subcontinua of M and which separates A from B in M (in the strong sense).

[1]) S. Mazurkiewicz, *Sur un ensemble G_δ, punctiforme, qui n'est homéomorphe avec aucun ensemble linéaire*, Fund. Math. vol. I, p. 62.