Some remarks concerning the theory of deduction.

by

E. Żyliński (Lwów)

§ 1. Mathematical preliminaries.

Let us consider one-valued functions $\varphi(x, y)$ of two variables x and y „on" the set (a, b) „to" the same set (a, b), where a and b are two distinct elements. Evidently we have as many such „φ-functions" as there are different „coverings" of the set of ordered pairs $((a, a), (a, b), (b, a), (b, b))$ by elements of the set (a, b) itself, it is $2^4 = 16$.

Let us introduce the notation: $\varphi_i(x, y)$ is this of our 16 φ-functions which values $\varphi_i(a, a)$, $\varphi_i(a, b)$, $\varphi_i(b, a)$ and $\varphi_i(b, b)$ we found on the right of φ_i in the graph of the following table:

	$\varphi(a, a)$	$\varphi(a, b)$	$\varphi(b, a)$	$\varphi(b, b)$
φ_1	a	a	a	a
φ_2	a	a	a	b
φ_3	a	a	b	a
φ_4	a	a	b	b
φ_5	a	b	a	a
φ_6	a	b	a	b
φ_7	a	b	b	a
φ_8	a	b	b	b

	$\varphi(a,a)$	$\varphi(a,b)$	$\varphi(b,a)$	$\varphi(b,b)$
φ_9	b	a	a	a
φ_{10}	b	a	a	b
φ_{11}	b	a	b	a
φ_{12}	l	a	b	b
φ_{13}	b	b	a	a
φ_{14}	b	b	a	b
φ_{15}	b	b	b	a
φ_{16}	b	b	b	b

The way in which this numeration of φ-functions is made is quite unessential for our purposes.

Let us denote:

$$\varphi_4(x,y)=\lambda, \quad \varphi_6(x,y)=\mu.$$

We see immediately that for all possible values of x and y we have:

$$\varphi_i(x,y)=\varphi_i(\varphi_4(x,y),\ \varphi_6(x,y))=\varphi_i(\lambda,\mu) \qquad (i=1,2,\dots 16)$$

and φ_4 and φ_6 (not φ_6 and φ_4) is the unique ordered pair of φ-functions which posseses this property.

Let us now consider expressions obtained of an arbitrary φ-function $\varphi=\varphi(\lambda,\mu)$ by consecutive replacing of λ and μ by arbitrary φ-functions expressed in terms of λ and μ, e. g.

$$\varphi_{12}(\varphi_3(\mu,\lambda),\mu);$$

we shall call them „φ-expressions". Each φ-expression represents evidently a definite φ-function, e. g. the expression above represents the φ-function φ_{15}.

By equalising φ-expressions which represent the same φ-functions we obtain an enumerable infinity of functional equations; we get, for example, the equation

$$\varphi_9(\varphi_9(\mu,\lambda),\ \varphi_{13}(\lambda,\mu))=\varphi_5(\varphi_5(\lambda,\lambda),\lambda), \tag{1}$$

both sides of which represent the same function φ_4.

Let us transpose the letters a and b *everywhere* in the table of φ-functions. Each of these φ-functions is transformed by this in an definite φ-function, e. g. the function φ_6 with values a, b, a, a for (a, a), (a, b), (b, a), (b, b) respectively becomes the function φ_{14} with values b, b, a, b for (a, a), (a, b), (b, a) and (b, b). In particular the φ-functions φ_4, φ_6, φ_{11} and φ_{13} are invariant (selfcorresponding) under this transformation. The transformation of the φ-functions being of order 2 (involutory), this transformation is a permutation

$$(1, 16)\ (2, 8)\ (3, 12)\ (5, 14)\ (7, 10)\ (9, 15),$$

i being written here instead of φ_i.

This permutation (as a permutation of indices) can now be aplied to every equation between φ-expressions, e. g. we get from (1) the „conjugated" equation

$$\varphi_{15}(\varphi_{15}(\mu, \lambda), \ \varphi_{13}(\lambda, \mu)) = \varphi_{14}(\varphi_{14}(\lambda, \lambda), \lambda).$$

The truth of this can be easily seen by observing, that both conjugated φ-functional equations can be regarded as instances of the same functional equation relative to φ-functions constructed on the set (m, n) (instead of the set (a, b), obtained by two specialisations: $m = a$, $n = b$ and $m = b$, $n = a$. Thus we have here a *„principle of duality"* ordering in pairs [1]) φ-functional equations and permitting to pass automaticaly from one conjugated functional equation to the other.

Let us consider now the set D of all the φ-expressions representing the function φ_1. We have a following theorem important for applications:

I If ε_1 and ε_2 are φ-expressions and ε_1 and $\varphi_5(\varepsilon_1, \varepsilon_2)$ are contained in D, then ε_2 is also contained in D.

In fact, $\varphi_5(\varepsilon_1, \varepsilon_2)$ being contained in D, it is having always the value a, we see immediately from the φ-functions table that this occur only for following values of ε_1 and ε_2 resp.

$$(a, a), \quad (b, a), \quad (b, b).$$

But ε_1 is always a by hypothesis Thus we have in all the cases for ε_1 and ε_2 the values

$$(a, a)$$

[1]) The φ-expressions involving uniquely the functions $\varphi_4 = \lambda$, $\varphi_6 = \mu$, φ_{11} and φ_{13} and only these are self-corresponding.

resp., thus always the value a for ε_2, q. e. d. Beside the theorem I we note the following fundamental *„principle of substitution“*:

II. When the φ-expression $\Theta(\lambda, \mu)$ is contained in D and $\psi(\lambda, \mu)$, $\chi(\lambda, \mu)$ are two quite arbitrary φ-expressions, then the φ-expression $\Theta(\psi(\lambda, \mu), \chi(\lambda, \mu))$ is also contained in D.

In fact, the value of $\Theta(\lambda, \mu)$ is equal to a for all possible combinations of values of their arguments and therefore the same occur for the φ-expression $\Theta(\psi(\lambda, \mu), \chi(\lambda, \mu))$ for quite arbitrary $\psi(\lambda, \mu)$, and $\chi(\lambda, \mu)$.

It is of interest to know which φ-functions can be represented as φ-expressions involving *only one* given φ-function φ_i, λ and μ being admitted exclusively as final arguments.

For this purpose let us consider the set $\Phi_i^{(1)}$ of following expressions:

$$\varphi_i(\lambda, \mu),\ \varphi_i(\mu, \lambda),\ \varphi_i(\lambda, \lambda),\ \varphi_i(\mu, \mu).$$

The set $\Phi_i^{(n)}$ will be by definition the set of all expressions obtained by putting in $\varphi_i(x, y)$ in place of x and y expressions from $\sum_{k=1}^{n-1} \Phi_i^{(k)}$, λ or μ, but necessarily one expression, at least, from $\Phi_i^{(n-1)}$ in the place of x or y.

Let us now denote by $F_i^{(n)}$ the aggregate of all φ-functions which are represented through expressions of $\Phi_i^{(n)}$ Evidently the set to be computed is equal to $\sum_k F_i^{(k)} = F_i$, where Σ denotes the ordinary summation of aggregates without repetition of equal elements. It is easy to see that in the case when $F_i^{(n)}$ does not contain other φ-functions but those contained already in $\sum_{k=1}^{n-1} F_i^{(k)}$, we have $F_i = \sum_{k=1}^{n-1} F_i^{(k)}$; that follows immediately from the fact, that the φ-function represented through the expression $\varphi_i(\varepsilon_1, \varepsilon_2)$ depends uniquely of φ-functions represented through ε_1 and ε_2, the form of those expressions being of no influence.

This must occur not later than for $k = 16$, the number of φ functions being finite and equal to 16. Thus each F_i can be computed in a finite number of operational steps.

These computations were executed by the indicated method in the Mathematical Seminary of the University of Lwów. The chief results are as follows.

For 2 φ-functions (φ_1 and φ_{16}) F_i (F_1 and F_{16}) consist of one only φ-function (φ_1 and φ_{16} respectively); for 2 (φ_4 and φ_6) — of 2; for 2 (φ_2 and φ_8) — of 3; for 4 (φ_7, φ_{10}, φ_{11} and φ_{13}) — of 4; for 4 (φ_3, φ_5, φ_{12} and φ_{14}) — of 6; for 2 (φ_9 and φ_{15}) — of all the 16 φ-functions.

By the computation of all the F_i except F_9 and F_{15} aggregates $\Phi^{(1)}$ and $\Phi^{(2)}$ are quite sufficient. In the case of F_9 and F_{15} the needed expressions for φ_7, φ_{10}, φ_{12}, φ_{14}, φ_{15}, φ_{16} and φ_1, φ_3, φ_5, φ_7, φ_9, φ_{10} respectively are found only in $\Phi_9^{(3)}$ and $\Phi_{15}^{(3)}$.

A φ-expression belonging to a $\Phi_i^{(n)}$, will be called „of order n". There can be many expressions of the *lowest* order k (in a $\Phi_i^{(k)}$) representing the same φ-function contained in F_i, e. g. φ_{12} and φ_3 have in $\Phi_9^{(3)}$ and $\Phi_{15}^{(3)}$ resp. 84 different expressions of lowest order $k=3$ foı each of them.

The calculated table of all these lowest order representations can not bo inserted here because of its considerable extension.

Remark. The whole theory of φ-functions of two variables x and y „on" (a, b) „to" (a, b) can be obviously generalised in an adequate .heory of φ-functions of n variables [1]). We have then, generally, 2^{2^n} different φ-functions; n fundamental φ-functions $\lambda, \mu, \ldots \varrho, \sigma$ instead of two λ and μ; theorems I and II are valid after obvious alterations, theorem II being true for every φ-function the value of which depends uniquely of values of two of their arguments and which has respectively to them the character of φ_5. A similar method of computation is applicable to analogous representation questions. All results as in the case of two variables have here also (because of the thorough finiteness) the highest grade of mathematical evidence.

§ 2. Applicatio to mathematical logic.

Let us consider propositional „truth-functions", that is propositional functions the truth-value of which depends uniquely of the truth-values of their arguments and which have the only meaning defined by this dependence.

Denoting by a the „truth", by b the „falsehood" of two propo-

[1]) Considering φ-functions of lower number of variables as instances of φ-functions of higher number of variables independent of the values of new variables, we can proceed with our developments dealing exclusively with φ-functions of an arbitrary great (say enumerable) number of variables

sitions λ and μ, each of the 16 φ-functions of § 1 corresponds with one of the 16 possible truth-functions of λ and μ, namely φ_{13} and φ_{11} to *negations* of λ and μ resp., φ_2 — to their *disjunction* „λ or μ", φ_8 — to *conjunction* „λ and μ", φ_5 — to *implication* „λ implies μ", φ_{15} — to joint-falsehood [1]) „not-λ and not-μ" and so on. Russell and Whitehead in their *Principia Mathematica* reduce truth functions (primitive ideas) they there employ to two of them: negation and disjunction. It means in the language of φ-functions that the φ-expressions occuring in *Principia* involve uniquely the φ-functions φ_2, φ_{13} and φ_{11}. H. M. Sheffer in the paper mentioned above has shown that all primitive ideas of *Principia* can be expressed (in our ordinary meaning) through joint-falsehood φ_{15} alone; the same was noted then as true for the incompatibility (φ_9).

The theory of § 1 and relative calculatory results supply us with a further intelligence in this matter.

1° Through joint-falsehood and incompatibility can be expressed (in our ordinary meaning) *each* of the 16 possible truth-functions of two propositions.

2°. There are *no other* truth-functions (of two propositions) possesing this property.

Russell and Whitehead in *Principia Mathematica* deduce the theorems of their theory of deduction from 5 fundamental φ-expressions of φ_1 involving only φ-functions of two propositions and containing at most 3 different fundamental φ-functions (formal principles of deduction) by applying to them essentially the theorems I and II of § 1. M. Nicod [2]) proceeds in a similar way with but one formal principle (involving 5 different fundamental φ-functions) and employing instead of theorem I an other one. which can be expressed as follows: „the φ-expression ε_3 represents φ_1, if ε_1 and $\varphi_9(\varepsilon_1, \varphi_9(\varepsilon_2, \varepsilon_3))$, represent φ_1".

It is very probable although not proved as far as I know that there are φ-representations of φ_1 not to be obtained in Russell's or Nicod's systems. It seems ablso to be of interest to investigate

[1]) *rejection* in the terminology of H. M. Sheffer: A set of five independent postulates for Boolean Algebras with application to logical constants, Trans. Am. Math. Soc. Vol. 14 (1913) p. 487.

[2]) Proc. Camb. Phil. Soc. vol. XIX. 1, January 1917.

whether all φ-functions of finite number of variables can be reduced to our 16 φ-functions of two variables, and in consequence to incompatibility or joint-falsehood. The last question is obviously equivalent with the problem of sufficiency of our contemporary language to „express“ (not to „describe“) every true relation between arbitrary propositions.

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