Из доказанных лемм непосредственно вытекает следующее предложение:

Теорема 3. *Частично упорядоченное множество $M(X)$ представляет собой полную атомную алгебру Буля изоморфную $D^{\Delta(X)}$. Её неразложимыми идемпотентами (атомами) служат всевозможные классы, содержащие действительные полуметрики, не обращающиеся тождественно в нуль. В алгебре Буля $M(X)$ для любого множества $A \subset M(X)$ элемент ∇A определяется как класс, содержащий насыщенную полуметрику, порождающую прямое произведение всех полуметрик, взятых по одной из каждого класса, составляющих множество A. Единицей в $M(X)$ является класс, содержащий произведение всех действительных полуметрик на X. Дополнение $C\alpha$ элемента $\alpha \in M(X)$ определяется, как класс, содержащий минимальную насыщенную полуметрику, дающую в произведении с полуметрикой $\varrho \in \alpha$ единицу алгебры Буля $M(X)$.*

Цитированная литература

[1] М. Я. Антоновский, В. Г. Болтянский, Т. А. Сарымсаков, *Очерк теории топологический полуполей*, УМН, т. XXI, выпуск 4 (130), 1966.

[2] — *Обобщенные метрики и структуры связанные с ними*, УМН выпуск 1, 1969.

ЦЭМИ АН СССР

Reçu par la Rédaction le 18. 12. 1968

Remarks on generalized analytic sets and the axiom of determinateness

by

B. V. Rao (Calcutta)

§ 1. Summary. The purpose of this note is to treat systematically the generalized analytic sets introduced by Ulam and discuss the problems posed by him. Some results of this note will be strengthened versions of previous results of the author. Others will be conclusions from the axiom of determinateness introduced by Jan Mycielski and H. Steinhaus. After giving preliminaries in Section 2, we discuss the main results in Sections 3 and 4. The results in this paper could equivalently be stated in terms of arbitrary separable metric spaces, but the author has not done so to keep the flavour of the discussion of Ulam.

§ 2. Preliminaries. Let I denote the unit interval $[0,1]$ and $\{A_n, n \geqslant 1\}$ be any sequence of subsets of I. Unless otherwise stated this sequence is fixed throughout the paper. $\boldsymbol{B}$ is the σ-algebra on I generated by this sequence. Following (Ulam [11], p. 9) the projections to I of sets in the σ-algebra on $I \times I$ generated by rectangles with sides in $\boldsymbol{B}$ are called generalized analytic sets (clearly these will depend on the sequence $\{A_n\}$). The σ-algebra on I generated by the generalized analytic sets is denoted by $\boldsymbol{A}$. The generalized analytic sets of $I^2 = I \times I$ are defined as the projections to I^2 of sets in the σ-algebra on I^3 generated by cubes with sides in $\boldsymbol{B}$. One can also define higher projective classes which are, however, not necessary for our purposes.

When $\{A_n\}$ happens to be the sequence of intervals with rational end points then $\boldsymbol{B}$ coincides with the usual Borel sets and the generalized analytic sets coincide with the usual analytic sets. We shall refer to these as standard Borel and standard analytic respectively.

Let f be the Marczewski function [2] on I to I defined as

$$f(x) = \sum \frac{2\chi_{A_n}(x)}{3^n}.$$

The range of f is denoted by R. The σ-algebra $\boldsymbol{R}$ on R is always the standard Borel σ-algebra of I restricted to R. If the atoms of $\boldsymbol{B}$ are singletons, then obviously f is an isomorphism between $(I, \boldsymbol{B})$ and $(R, \boldsymbol{R})$.

(H) denotes the statement: 'R contains a perfect set'. When the hypothesis (H) is satisfied, then without any further explanation, we denote by P any typical compact, perfect subset of R and $Q = f^{-1}(P)$. Observe that f restricted to Q is an isomorphism between Q and P where, of course, $Q \subset I$ and $P \subset R$ have their relative σ-algebras.

Following (Ulam [11], p. 7) we shall denote by class 0 the sets $\{A_n\}$ and their complements. Having defined classes smaller than $\alpha < \Omega$, we define class α as consisting of countable unions of sets in the previous classes and the complements of these sets. We say that $\boldsymbol{B}$ has sets of high Borel class number if given any $\alpha < \Omega$ there is a set $B \in \boldsymbol{B}$ which does not belong to any class smaller than α.

Jan Mycielski and H. Steinhaus [5] have introduced a set theoretic axiom—the axiom of determinateness (A). This axiom is formulated in game-theoretic terminology. It says, that certain games are always determined. We shall not attempt to give a precise statement of the axiom, for it is not needed in its original form for our purposes. Interesting consequences of this axiom have been studied by Jan Mycielski ([3], see also [4]) and by Jan Mycielski and S. Świerczkowski [6]. Main consequences are the following [3]:

$\mathrm{I_A}$. 'Every subset of the real line is Lebesgue measurable' or equivalently 'for every finite denumerably additive measure μ over the field of Borel sets of a separable metric space X and every $Y \subseteq X$ there are Borel sets $B_1 \subseteq Y \subseteq B_2$ such that $\mu(B_1) = \mu(B_2)$'.

$\mathrm{II_A}$. 'Every subset of the Cantor discontinuum has the property of Baire, in the sense that, it is of the form $G \cup K_1 - K_2$ where G is open and K_1, K_2 are of first category' or equivalently 'Every subset of a separable metric space has the property of Baire'.

$\mathrm{III_A}$. 'Every nondenumerable subset of the Cantor discontinuum has a perfect subset' or equivalently 'every nondenumerable separable metric space contains a compact perfect subset'.

$\mathrm{IV_A}$. For every family of pairwise disjoint sets $\boldsymbol{F}$ s. t. $\Phi \notin \boldsymbol{F}$, $|\boldsymbol{F}| \leqslant \aleph_0$ and $|\bigcup_{X \in F} X| \leqslant 2^{\aleph_0}$; there exists a choice set.

Clearly the axiom (A) is inconsistent with the axiom of choice. We remind the reader that the consistency of the axiom (A) with the usual axioms of set theory (without the axiom of choice) is not known. In [4] an alternative of (A) is proposed which has the same main consequences as (A). In fact this latter axiom gives a more refined version of $\mathrm{IV_A}$.

§ 3. Consequences of (H). Throughout this section we assume that (H) is satisfied. Since $(R, \boldsymbol{R})$ is structure isomorphic to $(I, \boldsymbol{B})$, we can and shall suppose that the atoms of $\boldsymbol{B}$ are singletons. The following two theorems are easy to prove:

THEOREM 1. *A set $X \subset Q$ is generalized analytic iff $f(X) \subset P$ is standard analytic.*

THEOREM 2. *$\boldsymbol{B}$ supports a nonatomic perfect* [9] *probability measure. In fact there is one such concentrated on Q.*

Let Z be any perfect subset of I and $\boldsymbol{B}_Z, \boldsymbol{A}_Z, \boldsymbol{L}_Z$ be the σ-algebras on Z generated by standard Borel, standard analytic and by sets measurable w.r.t. a fixed nonatomic probability measure on $\boldsymbol{B}_Z$. Let $\boldsymbol{E}_Z$ be any σ-algebra on Z satisfying $\boldsymbol{A}_Z \subset \boldsymbol{E}_Z \subset \boldsymbol{L}_Z$. Then the following theorem which is a refinement of Theorem 1 of [7] is in fact equivalent to it and the proof is also along the same lines.

THEOREM 3. *$\boldsymbol{E}_Z$ is not countably generated.*

Let us fix any nonatomic probability measure μ on $\boldsymbol{B}$ giving strictly positive measure to Q. Denote the completion of $\boldsymbol{B}$ w.r.t. this probability measure by $\boldsymbol{L}$. Then as a consequence of Theorem 3, in view of Theorem 1, we have:

THEOREM 4. *If $\boldsymbol{E}$ is any σ-algebra on I s.t. $\boldsymbol{A} \subset \boldsymbol{E} \subset \boldsymbol{L}$ then $\boldsymbol{E}$ is not countably generated.*

This is clearly an analogue of Theorem 1 of [7]. We can define a map $\tilde{f}$ from $I \times I$ onto $R \times R$ by $\tilde{f}(x, y) = (f(x), f(y))$. Observe that if $U \subset P \times P$ is a standard analytic set universal w.r.t. the standard analytic subsets of P (for the existence see [10], p. 252) then $V = \tilde{f}^{-1}(U)$ is a generalized analytic subset of $I \times I$ contained in $Q \times Q$ which is universal w.r.t. the generalized analytic subsets of I contained in Q. Just as Theorem 2 of [7] is a consequence of Theorem 1 of [7] the following theorem is a consequence of Theorem 4.

THEOREM 5. *$V \notin \boldsymbol{C} \times \boldsymbol{L}$ where $\boldsymbol{C}$ denotes the class of all subsets of I.*

We shall now conclude this section with two theorems which are analogues of classical theorems.

THEOREM 6. *There are generalized analytic sets which do not belong to $\boldsymbol{B}$.*

Proof. Clear from Theorem 1 and [10], p. 254.

THEOREM 7. *$\boldsymbol{B}$ has sets of high Borel class number.*

Proof. Observe that I can be metrized so as to be homeomorphic to R by defining $d(x, y) = |f(x) - f(y)|$. It is not difficult to verify that each A_n is a clopen subset of I and that Q is a perfect subset when I is given this metric. Now the result is a consequence of [1], p. 278.

§ 4. Consequences of (A). Throughout the section we assume the axiom (A).

THEOREM 8. *There does not exist a sequence of sets $\{A_n\}$ in I with the following properties:*

(a) *The σ-algebra generated by the sets $\{A_n\}$ contains noncountable set of atoms.*

(b) *All the analytic sets over $\{A_n\}$ coincide with Borel sets over $\{A_n\}$.*

Proof. (a) implies that the range of the Marczewski function for $\{A_n\}$ is uncountable and hence by III_A it contains a perfect set. Consequently we can apply Theorem 6, if we have nowhere used the axiom of choice in the proof of that theorem. But since an effective way of obtaining non Borel analytic sets is known ([10], p. 254) we are through.

Compare this theorem with [11], p. 10, lines 1–3.

THEOREM 9. *There does not exist a sequence of sets $\{A_n\}$ in I with the following properties:*

(a) *The σ-algebra over $\{A_n\}$ contains sets of arbitrarily high Borel class number.*

(b) *All analytic sets over $\{A_n\}$ are Borel sets over $\{A_n\}$.*

Proof. Take any sequence of sets $\{A_n\}$ in I. Clearly, if (a) is to be satisfied then the range of the Marczewski function should be uncountable. Then (b) can not be satisfied in view of Theorem 8.

Compare this with [11], p. 10, lines 4–11. Though the above two theorems are apparently different, that they are equivalent is shown by the following theorem.

THEOREM 10. *For any sequence of sets $\{A_n\}$ in I the following two conditions are equivalent.*

a_8. *The σ-algebra generated by $\{A_n\}$ contains non-countable set of atoms.*

a_9. *The σ-algebra over $\{A_n\}$ contains sets of arbitrarily high Borel class number.*

Proof. Clearly a_9 implies a_8. We show the other way. Observe that in view of III_A the range of the Marczewski function contains a perfect set. Consequently, we can apply Theorem 7, if we have nowhere used the axiom of choice in the proof of that theorem. But since an effective way of exhibiting Borel sets of any class ([1], p. 279) is known we are through.

The author could not avoid the use of the axiom of choice in proving Theorem 1 of [7] (To be precise, we have used the Borel isomorphism theorem). Observe that if this is done then Theorems 4 and 5 will follow without the aid of the axiom of choice and hence can be used in conjunction with (A) to yield the following: 'There is no separable σ-algebra on I containing all its standard analytic sets' and 'the product of discrete σ-algebras on I is not the discrete σ-algebra on $I \times I$' which are contradictory to the conclusions (Theorems 1, 4 of [8]) arrived earlier.

The author acknowledges with pleasure, the useful discussions he had with Dr. A. Maitra.

References

[1] C. Kuratowski, *Topologie, I*, Warszawa 1952.
[2] E. Marczewski (Szpilrajn), *The characteristic function of a sequence of sets and some of its applications*, Fund. Math. 31 (1938), p. 222.
[3] Jan Mycielski, *On the axiom of determinateness*, Fund. Math. 53 (1964), p. 205
[4] — *On the axiom of determinateness II*, Fund. Math. 59 (1966), p. 203.
[5] — and H. Steinhaus, *A mathematical axiom contradicting the axiom of choice*, Bull. Acad. Polon. Sci., Series Math. Astr. et. Phys. 10 (1962), p. 1.
[6] — and S. Świerczkowski, *On the Lebesgue measurability and the axiom of determinateness*, Fund. Math. 54 (1964), p. 67.
[7] B. V. Rao, *Remarks on analytic sets*, Fund. Math. 66 (1970), p. 237.
[8] — *On discrete Borel spaces and projective sets*, Bull. Amer. Math. Soc. 75 (1969), p. 614.
[9] C. Ryll-Nardzewski, *On quasi compact measures*, Fund. Math. 40 (1953), p. 125.
[10] W. Sierpiński, *General topology*, Toronto 1952.
[11] S. M. Ulam, *A collection of mathematical problems*, New York 1960.

INDIAN STATISTICAL INSTITUTE

Reçu par la Rédaction le 24. 1. 1969