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On locally compact abelian groups which are topologically pure in their Bohr compactifications

by

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1. Introduction. We adopt the notations and terminology of Hewitt and Ross [3] which will be referred to as H. & R. in the sequel.

It is known (Alfsen and Holm [2]) that every topological group G admits a unique (up to topological isomorphism) *maximal compact representation* $(\varrho, \hat{G})$ where $\hat{G}$ is a compact topological group and ϱ a continuous homomorphism of G to a dense subgroup of $\hat{G}$. Here the word *maximal* is used to mean that $(\varrho, \hat{G})$ has the *universal factorization property*; that is to say, if ξ is a continuous homomorphism of G to a dense subgroup of a compact group H, then $\xi = \xi' \circ \varrho$ where ξ' is a continuous homomorphism of $\hat{G}$ onto H.

Let now G be a locally compact abelian group. The maximal compact representation of G is called the *Bohr compactification* of G and is denoted by $(\Phi, \mathrm{b}G)$. The topological space underlying $\mathrm{b}G$ is the maximal ideal space of the Banach algebra of all almost periodic functions on G. Furthermore, if X is the character group of G, $\mathrm{b}G$ is the (compact) character group of the abstract group underlying X taken with the discrete topology. Thus, while G has as elements *all continuous characters* of X, $\mathrm{b}G$ has as elements *all characters* of X and Φ is the map which identifies a continuous character of X as a character of X.

A subgroup P of an abelian topological group G is said to be *topologically pure*, if $g \in G$, $g^n \in P$, n an integer implies that for every open set $V \ni g$, there exists $g_0 \in V \cap P$ such that $g_0^n = g^n$.

It is clear that topological purity implies purity and the converse is not necessarily true. Recalling the well-known definition that a subgroup P of an abelian group G is *pure closed* if $g \in G$, $g^n \in P$, n an integer implies $g \in P$, we observe that any pure closed subgroup of an abelian topological group is topologically pure. In particular, any pure subgroup of a torsion free abelian topological group is topologically pure.

The object of this paper is twofold:

(i) to characterize the class of all locally compact abelian groups G with the property: $\Phi(G)$ is topologically pure in bG. (By abuse of language we shall say that G is topologically pure in its Bohr compactification.)

(ii) to show that the class of locally compact abelian groups which are pure in their Bohr compactifications is invariant under Pontryagin duality, i.e., if G is a locally compact abelian group and X its character group, G is pure in bG if and only if X is pure in bX. (c.f. Corollary to Theorem 1.)

2. Preliminaries. A topological group G is said to be *power open*, if the map $f_n\colon x \to x^n$ of G onto $G^{(n)}$ where $G^{(n)} = \{y \in G \mid \exists x \in G,\ x^n = y\}$ is open. So if V is open in G, $f_n(V)$ is open in the relative topology of $G^{(n)}$. A topological group is said to be *absolutely power open*, if for every open set V in G, $f_n(V)$ is open in G.

Every group with discrete topology is absolutely power open. Obviously, if a topological group is absolutely power open, it is power open.

(*) We observe that every division closed σ-compact locally compact abelian group G is power open. (An abelian topological group G is said to be *division closed*, if $G^{(n)}$ is closed for every integer n.) For, $G^{(n)}$ is locally compact and so by Theorem 5.29, p. 42 of H. & R., f_n is open.

Let G be a locally compact abelian group, bG its Bohr compactification and Φ the canonical continuous isomorphism of G into bG. An open set U of G such that $U = \Phi^{-1}(V)$, V open in bG is said to be *strongly open*.

A locally compact abelian group G is said to be *weakly power open*, if for every strongly open set U of G, $f_n(U)$ is open in $G^{(n)}$. Clearly, a power open or absolutely power open locally compact abelian group is weakly power open.

Our main theorem is:

THEOREM 1. *A locally compact abelian group G is pure in bG if and only if it is division closed and weakly power open.*

Łoś [4] proved that every abelian group can be imbedded as a pure subgroup of a compact abelian group. Adimoolam [1] proved a stronger form of this result using Pontryagin duality and showed that every abelian group G is pure in the compact group bG. Since a discrete abelian group is clearly division closed and weakly power open, we can deduce from Theorem 1, a stronger form of Adimoolam's result: Every abelian group G is a topologically pure subgroup of bG. Theorem 1 further implies that every division closed σ-compact locally compact abelian group G is topologically pure in bG.

Proposition 5 below proves the necessity part of Theorem 1. For the sufficiency part, we prove the crucial Proposition 6 which relates the properties of division closure and weakly power openness of the character group of G to the topological purity of G in bG. This relationship immediately yields the proofs of the Theorem 1 and its Corollary.

3. Proof of Theorem 1. We first prove a series of propositions which will be used in the proof of Theorem 1.

PROPOSITION 1. *Let G be a locally compact abelian group, χ a continuous character of G and W an open set of the compact abelian group T of the unimodular complex numbers under multiplication. Then $\chi^{-1}(W)$ is strongly open in G.*

Proof. Let X be the character group of G. Then bG is the character group of X_d, viz., the group X taken with the discrete topology. Any $\chi \in X$, a continuous character of G, can also be interpreted as a continuous character χ_b of bG. Let (,) and [,] denote the pairings of G and X, bG and X_d respectively with reference to duality. By definition of Φ, for any $x \in G$, we have $(x, \chi) = [\Phi(x), \chi_b]$. As Φ is 1-1 and continuous, in order to prove that $\chi^{-1}(W)$ is strongly open, it suffices to prove that $\Phi\chi^{-1}(W)$ is relatively open in $\Phi(G)$. In fact, we prove that $\Phi\chi^{-1}(W) = \chi_b^{-1}(W) \cap \Phi(G)$. For,

$$\begin{aligned} y \in \Phi\chi^{-1}(W) &\Leftrightarrow y \in bG,\ y = \Phi(z),\ (z, \chi) \in W \text{ for some } z \in G \\ &\Leftrightarrow y \in bG,\ y = \Phi(z),\ [\Phi(z), \chi_b] \in W \text{ for some } z \in G \\ &\Leftrightarrow y \in \Phi(G) \text{ and } y \in \chi_b^{-1}(W)\,. \end{aligned}$$

Our proposition is thus proved.

PROPOSITION 2. *Let G be a locally compact abelian group which is weakly power open. Let χ be a character of G such that χ^n which is also a character of G is continuous. Then the restriction $\chi|G^{(n)}$ of χ to $G^{(n)}$ is a continuous character of $G^{(n)}$.*

Proof. That $\chi|G^{(n)}$ is a character of $G^{(n)}$ is obvious. Let W be an open set of T. Then $\chi^{-1}(W) \cap G^{(n)} = f_n\big((\chi^n)^{-1}(W)\big)$. Since χ^n is a continuous character of G, $(\chi^n)^{-1}(W)$ is strongly open in G by Proposition 1 above. As G is weakly power open, $f_n\big((\chi^n)^{-1}(W)\big)$ is open in $G^{(n)}$. This implies that $\chi|G^{(n)}$ is continuous and thus the proposition is proved.

PROPOSITION 3. (Adimoolam [1]). *Let G be a locally compact abelian group such that $\Phi(G)$ is pure in bG. Then G is division closed.*

Proof. As bG is compact, bG is division closed. So, for every integer n, $\big(\Phi(G)\big)^{(n)} = (bG)^{(n)} \cap \Phi(G)$ is relatively closed in $\Phi(G)$. As Φ is a continuous isomorphism, it follows that $G^{(n)}$ is closed in G for every integer n. This completes the proof.

PROPOSITION 4. *Let G be a locally compact abelian group such that $\Phi(G)$ is topologically pure in bG. Then G is weakly power open.*

Proof. As bG is a compact abelian group, bG is power open. So, if V is open in bG, $f_n(V)$ is open in $(\mathrm{b}G)^{(n)}$. As $\Phi(G)$ is topologically pure in bG, it is pure in bG and hence $(\Phi(G))^{(n)} = (\mathrm{b}G)^{(n)} \cap \Phi(G)$. Hence $f_n(V) \cap (\Phi(G))^{(n)}$ is open relative to $(\Phi(G))^{(n)}$. We claim that $f_n(V \cap (\Phi(G)) = f_n(V) \cap (\Phi(G))^{(n)}$. That the L.H.S. is contained in the R.H.S. is obvious. Let $y \in$ R.H.S. Then there exists an $x \in V$ such that $f_n(x) = x^n = y \in \Phi(G)$. As $\Phi(G)$ is topologically pure in bG, there exists $z \in V \cap \Phi(G)$ such that $z^n = x^n = y$. So $y \in$ L.H.S. Thus our claim is proved. This implies that $\Phi(G)$ is power open in the relative topology and consequently, gives that G is weakly power open. The proof of Proposition 4 is completed.

PROPOSITION 5. *Let G be a locally compact abelian group such that $\Phi(G)$ is topologically pure in* bG. *Then G is division closed and weakly power open.*

Proof. Immediate from Propositions 3 and 4 above.

PROPOSITION 6. *Let G be a locally compact abelian group such that its character group X is division closed and weakly power open. Then $\Phi(G)$ is topologically pure in* bG.

Proof. Let $y \in \mathrm{b}G$ and n an integer such that $y^n \in \Phi(G)$. Let V be an open set of bG such that $y \in V$. Without loss of generality we may assume that $V = yP(S, \varepsilon)$ where S is a finite subset of X, ε a real number >0 and $P(S, \varepsilon) = \{x \in \mathrm{b}G \mid |x(s)-1| < \varepsilon \text{ for all } s \in S\}$. As y is a character of bG and $y^n \in \Phi(G)$, y^n can be considered as a continuous character of X. As X is weakly power open, by Proposition 2 above this implies that $y|X^{(n)}$ is continuous. As X is division closed, $X^{(n)}$ is closed in X. Then in the relative topology of the subgroup Y of X generated by $X^{(n)} \cup S$, $X^{(n)}$ is clearly closed. Since for every $s \in S$, $s^n \in X^{(n)}$ and S is a finite subset of the abelian group X, $X^{(n)}$ is of finite index in Y. So we can deduce that $X^{(n)}$ is an open subgroup of Y. As $y|X^{(n)}$ is continuous on the open subgroup $X^{(n)}$ of Y, $y|Y$ is a continuous character of Y. By Remark 26.30 p. 369 of H. & R., $y|Y$ can be extended to a continuous character y_1 of $\overline{Y}$. Now $\overline{Y}$ is a closed subgroup of the locally compact abelian group X, y_1 can be extended to a continuous character z of X. As z extends $y|Y$, $z(s) = y(s)$ for every $s \in S$ and hence $z \in V$. Also as $Y \supset X^{(n)}$ $z^n = y^n$. This proves that $\Phi(G)$ is topologically pure in bG and completes the proof of our proposition.

Proof of Theorem 1. Proposition 5 is the necessity part of Theorem 1. We prove the sufficiency part. Let G be a division closed locally compact abelian group which is weakly power open. By Proposition 6, its character group X is such that $\Phi(X)$ is topologically pure in bX. So by Proposition 5, X is division closed and weakly power open. Now applying Proposition 6 to G, we deduce that $\Phi(G)$ is topologically pure in bG. Our proof is complete.

The following Corollary is immediate.

COROLLARY. *If G is a locally compact abelian group, G is division closed and weakly power open (equivalently $\Phi(G)$ is topologically pure in* bG) *if and only if its character group is so.*

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References

[1] C. Adimoolam, *Master of Science Thesis* (unpublished), University of Madras, 1962.
[2] E. M. Alfsen and Holm Per, *A note on compact representations and almost periodicity in topological groups*, Math. Scand. 10 (1962), pp. 127–136.
[3] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, Grundlehren der Mathematischen Wissenschaften, Bd. 115, Berlin–Göttingen–Heidelberg 1963.
[4] J. Łoś, *Abelian groups that are direct summands of every abelian group which contains them as pure subgroups*, Fund. Math. 44 (1957), pp. 84–90.

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