A large cardinal in the constructible universe*

by

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In the first section of this paper, I propose to show that one of the large cardinal properties introduced by Erdös and his school, namely $\varkappa\to(\omega)^{<\aleph_0}$, is consistent with the axiom of constructibility (called '$V = L$' for short). More precisely, so our argument runs, if $\varkappa\to(\omega)^{<\aleph_0}$ in the real world, then $\varkappa\to(\omega)^{<\aleph_0}$ in the constructible universe, giving the desired relative consistency result. For the proof, we adapt to our purpose the familiar technique from descriptive set theory of expressing a Π_1^1 statement in number theory as a statement about well-foundedness [6].

From Reinhardt–Silver [3], it is known that a cardinal $\varkappa$ for which $\varkappa\to(\omega)^{<\aleph_0}$ must be quite large, exceeding as it does the first inaccessible cardinal, the first weakly compact cardinal, and a host of other still larger cardinals previously introduced. (On the other hand, as Rowbottom and others have observed ([4], [7]), the first cardinal for which $\varkappa\to(\omega)^{<\aleph_0}$ is vastly smaller than the first measurable cardinal.) Moreover, Rowbottom [4], refining Scott's [5] celebrated result on measurability, showed that $\varkappa\to(\omega_1)^{<\aleph_0}$ contradicts $V = L$ and, indeed, he conjectured that the same would prove true of $\varkappa\to(\omega)^{<\aleph_0}$, the conjecture which is refuted here. It does not seem extravagant, then, to assert that, for all practical purposes, $\varkappa\to(\omega)^{<\aleph_0}$ is the strongest strong axiom of infinity known to be consistent with $V = L$, and therein lies its chief interest (cf. also the corollary of section 1 and the subsequent paragraph).

In the second section, we obtain a pseudo-algebraic characterization of $\varkappa\to(\omega)^{<\aleph_0}$ to the following effect: every algebra of cardinality $\varkappa$ has a subalgebra which admits a non-trivial monomorphism into itself. This will answer a question posed in my thesis [7]. The proof utilizes techniques from model theory.

1. Let us begin by defining the Erdös notation [1] in rather distressing generality. $[X]^{<\aleph_0}$ is always the set of finite subsets of X.

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DEFINITION. (a) If $f\colon [\varkappa]^{<\aleph_0} \to S$ where S is any set and $X \subseteq \varkappa$, then X is said to be a set of *indiscernibles* for f if any two members of $[X]^{<\aleph_0}$ having the same cardinality also have the same image under f.

(b) We say that $\varkappa \to (\alpha)_\lambda^{<\aleph_0}$ if any function $f\colon [\varkappa]^{<\aleph_0} \to \lambda$ has a set of indiscernibles of order type α.

When the subscript is omitted, it is to be understood that $\lambda = 2$ is intended.

L will denote the constructible universe. If θ is a formula of set theory, $\theta^{(L)}$ is the result of relativizing all the quantifiers of θ to L in the customary way. The same will apply to terms τ built up using set-theoretic formulas and $\{\,:\,\}$.

I propose to prove the following theorem, which will, in particular, prove all the relevant claims made in the introduction.

THEOREM. *If $\varkappa \to (\alpha)^{<\aleph_0}$ and $\alpha < \omega_1^L$, then $[\varkappa \to (\alpha)^{<\aleph_0}]^{(L)}$.*

The significance of this more general form over and above the case $\alpha = \omega$ is best pointed up by an immediate corollary.

COROLLARY. *If $\varkappa \to (\alpha)^{<\aleph_0}$ for all countable α, then the same is true of $\varkappa$ in L.*

Comparing this with the theorem of Rowbottom that $\varkappa \to (\omega_1)^{<\aleph_0}$ contradicts $V = L$, we seem to be skating on the edge of a contradiction when we have both $V = L$ and $\varkappa \to (\alpha)^{<\aleph_0}$ for all countable α. ω_1 is the exact break-off point. (It should be noted that, by Rowbottom ([4], [7]), $\varkappa \to (\alpha+\omega)^{<\aleph_0}$ is properly stronger than $\varkappa \to (\alpha)^{<\aleph_0}$. However, it seems fair to regard $\varkappa \to (\omega)^{<\aleph_0}$ as entirely representative of the whole class of statements $\varkappa \to (\alpha)^{<\aleph_0}$ where α is countably infinite, since none of them is known to have any particularly distinctive consequences, as compared with the others. In view of Martin [2], I would hesitate to say the same of 'for all countable α, $\varkappa \to (\alpha)^{<\aleph_0}$'.)

Pursuant to the proof of the theorem, this lemma is needed.(¹)

LEMMA. *Suppose $f\colon [\varkappa]^{<\aleph_0} \to 2$, g is a 1-1 map of ω onto α, and further suppose that*

$$S(f, g) = \{h\colon \text{for some } n \in \omega,\ h \text{ maps } n \text{ onto a set of indiscernibles for } f, \text{ and, for any } i, j \in n,\ h(i) \in h(j) \text{ iff } g(i) \in g(j)\}.$$

Then f has a set of indiscernibles of order type α if and only if $S(f, g)$ has an infinite ascending $\subseteq$ chain.

Proof of the Lemma. If X is a set of indiscernibles or order type α for f, say k is an order preserving map of α onto X, then $\{k \circ (g|n)\colon n \in \omega\}$ is an infinite ascending chain. On the other hand, if C is an infinite ascending chain in $S(f, g)$, then plainly $\bigcup C$ is a function, say H, from ω onto some subset Y of $\varkappa$. By the definition of $S(f, g)$, $H(i) \in H(j)$ iff $g(i) \in g(j)$, whence Y also has order type α, $H \circ g^{-1}$ being an order preserving map of α onto Y. Moreover, Y, an ascending union of sets of indiscernibles for f, must itself be a set of indiscernibles for f.

(¹) R. L. Vaught suggested that I reformulate my original proof in terms of well-foundedness.

Notice that the condition stated in the lemma can be recast in terms of well-foundedness by reversing $\subseteq$. We can conclude, then, that "f has a set of indiscernibles of order type α iff $S(f, g)$ is not well-founded under $\supseteq$". The *prima facie* dependence on the axiom of choice involved in passing from the absence of infinite descending chains to well-foundedness can easily be eliminated by noticing that $S(f, g)$ can be effectively well-ordered.

Regarding S as a formal term of ZF defined as in the lemma, we are now in a position to prove the main theorem. Formally speaking, we shall be making use of the fact that

(#) *If $f\colon [\varkappa]^{<\aleph_0} \to 2$ and $g\colon \omega \xrightarrow[\text{onto}]{1-1} \alpha$, then f has a set of indiscernibles of order type α iff $S(f, g)$ is not well-founded under $\supseteq$.*

is a theorem of ZF.

Suppose $\varkappa \to (\alpha)^{<\aleph_0}$ where $\alpha < \omega_1^L$. Since $\alpha < \omega_1^L$, there is a function $g \in L$ such that $g\colon \omega \xrightarrow{1-1} \alpha$. By some very trivial absoluteness considerations, $[\varkappa \to (\alpha)^{<\aleph_0}]^{(L)}$ will be established once we show: for any $f \in L$ such that $f\colon [\varkappa]^{<\aleph_0} \to 2$, [$f$ has a set of indiscernibles of order type $\alpha]^{(L)}$.

We now argue this claim. Suppose $f\colon [\varkappa]^{<\aleph_0} \to 2$ and $f \in L$. Since $\varkappa \to (\alpha)^{<\aleph_0}$, f has a set of indiscernibles of order type α in the real world, so, in the real world, $S(f, g)$ is not well-founded under $\supseteq$. But, manifestly, $S(f, g)^{(L)}$ is equal to $S(f, g)$. This is so because the definition of $S(f, g)$ involves only finite functions and thoroughly absolute notions.

Furthermore, it is well-known that well-foundedness is absolute, since it can be defined on the one hand in terms of the absence of sets with no minimal elements (universally) and on the other hand in terms of the existence of an order-preserving map into the ordinals. The precise statement needed here is: for any relation R, R is well-founded iff [R is well-founded$]^{(L)}$. So $S(f, g) = S(f, g)^{(L)}$ is still not well-founded under $\supseteq$, even in the sense of L. Thus

$$[S(f, g) \text{ is not well-founded under } \supseteq]^{(L)}.$$

Since (#) is true in L, we conclude

$$[f \text{ has a set of indiscernibles of order type } \alpha]^{(L)}.$$

This completes the proof of the theorem.

Several glosses should now be made on this section. The arguments go over with negligible alteration to any transitive model $\mathcal{M}$ of ZF in place of L, whether $\mathcal{M}$ is an "inner model" containing all ordinals or just a common, garden-variety set. For example, we have:

THEOREM. *If $\mathcal{M}$ is a transitive model of* ZF, $\alpha, \varkappa \in \mathcal{M}$, $\alpha < \omega_1^{\mathcal{M}}$, *and* $\varkappa \to (\alpha)^{<\aleph_0}$, *then* $\mathcal{M} \models \varkappa \to (\alpha)^{<\aleph_0}$. (²)

Nor would there have been any harm in retaining the subscript λ throughout our discussion. Sticking to 2 was purely a matter of expository felicity.

Sticklers for generality will reproach me for not stating a general theorem, corresponding to the Π_1^1 result, from which our particular absoluteness result can be trivially derived. The following statement falls somewhat short of this, but it is nonetheless of some small interest. By way of preliminary definitions, let L_{ω_1,ω_1} be the infinitary first-order language having countably infinite conjunctions and countably infinite homogeneous (i.e., non-alternating) blocks of quantifiers but only *finitary* predicate and function symbols. A sentence of L_{ω_1,ω_1} is called *existential conjunctive* if it consists of an initial string of existential quantifiers followed by a conjunction of atomic formulas and negations of atomic formulas.

THEOREM. *Suppose $\mathcal{M}$ is a transitive model of* ZF+AC, $\mathfrak{A}$ *is a structure which is an element of $\mathcal{M}$, and σ is a sentence such that $\mathcal{M} \models \sigma$ is an existential conjunctive sentence of L_{ω_1,ω_1}.* (³) *Then σ is true in $\mathfrak{A}$ if and only if $\mathcal{M} \models \sigma$ is true in $\mathfrak{A}$.*

Finally, a word is in order concerning the precise statements of our consistency results. Of course they are all relative consistency results. Thus, from the main theorem of this section, we conclude at once: if $\mathrm{ZF} + \exists\varkappa\big(\varkappa \to (\omega)^{<\aleph_0}\big)$ is consistent, then $\mathrm{ZF} + \mathrm{V} = \mathrm{L} + \exists\varkappa\big(\varkappa \to (\omega)^{<\aleph_0}\big)$ is consistent. The same is true as regards the statement: there exists a $\varkappa$ such that, for all countable α, $\varkappa \to (\alpha)^{<\aleph_0}$.

2. In this section we shall investigate a property closely akin to $\varkappa \to (\omega)^{<\aleph_0}$ and shall in fact finally establish the equivalence of the two properties (in ZF+AC).

DEFINITION. We say $\varkappa \underset{W}{\to} (\omega)_\lambda^{<\aleph_0}$ if the following condition holds: for all $f\colon [\varkappa]^{<\aleph_0} \to \lambda$ there is an increasing sequence $\langle \alpha_i\colon i \in \omega\rangle$ of ordinals such that, for each n, $f(\{\alpha_0, \ldots, \alpha_{n-1}\}) = f(\{\alpha_1, \ldots, \alpha_u\})$.

Transparently, $\varkappa \to (\omega)_\lambda^{<\aleph_0}$ implies $\varkappa \underset{W}{\to} (\omega)_\lambda^{<\aleph_0}$. As before, $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$ will mean $\varkappa \underset{W}{\to} (\omega)_2^{<\aleph_0}$ and will be the focus of our discussion. Ultimately, its equivalence with $\varkappa \to (\omega)^{<\aleph_0}$ will be demonstrated, after first restating it in the pseudo-algebraic terms alluded to before. That the equivalence

(²) As usual, $\mathcal{M} \models \theta$ means that θ is true in $\mathcal{M}$.

(³) ω_1 is to be understood here as a term of ZF.

holds for general λ is largely a curiosity whose proof does not warrant the added pains.

It is of some, though hardly compelling, interest that

LEMMA. *For all $\varkappa$, $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$ iff $\varkappa \underset{W}{\to} (\omega)_c^{<\aleph_0}$ (where $c = 2^{\aleph_0}$).*

Proof. This proof is closely modeled on Rowbottom's proof of the analogous statement for $\varkappa \to (\omega)^{<\aleph_0}$ ([7]). Suppose $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$. To prove $\varkappa \underset{W}{\to} (\omega)_c^{<\aleph_0}$, let f be an arbitrary function from $[\varkappa]^{<\aleph_0}$ into ${}^\omega 2$, the set of functions from ω into 2 (there is no harm in replacing c by ${}^\omega 2$, a set of cardinality c). We wish to find a sequence $\langle \alpha_i\colon i \in \omega\rangle$ with the property mentioned above in the definition. Define a new function g as follows: if $m = 2^t(2s+1)$ and $\beta_1 < \ldots < \beta_m$ are elements of $\varkappa$, let $g(\{\beta_1, \ldots, \beta_m\}) = f(\{\beta_1, \ldots, \beta_s\})(t)$. (Recall that $f(\{\beta_1, \ldots, \beta_s\})$ maps ω into 2.) Since g maps $[\varkappa]^{<\aleph_0}$ into 2, $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$ gives us a sequence $\langle \alpha_i\colon i \in \omega\rangle$ with $g(\{\alpha_0, \ldots, \alpha_{m-1}\}) = g(\{\alpha_1, \ldots, \alpha_m\})$ for all m. To see that $f(\{\alpha_0, \ldots, \alpha_{s-1}\}) = f(\{\alpha_1, \ldots, \alpha_m\})$, we show that their ith coordinates are the same for each i. But, if $m = 2^i(2s+1)$ then $g(\{\alpha_0, \ldots, \alpha_{m-1}\})$, the ith coordinate of $f(\{\alpha_0, \ldots, \alpha_{s-1}\})$, and $g(\{\alpha_1, \ldots, \alpha_m\})$, the ith coordinate of $f(\{\alpha_1, \ldots, \alpha_s\})$, are indeed equal. Thus $\langle \alpha_i\colon i \in \omega\rangle$ has the requisite property.

Much of the mathematical interest of the property $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$ derives from the following characterization (in which an algebra is understood to be a structure with countably many operations).

PROPOSITION. $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$ *if and only if, for every algebra $\mathfrak{A}$ of cardinality $\varkappa$, there is some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that there is a non-trivial monomorphism of $\mathfrak{B}$ into itself.*

Necessity. Suppose $\mathfrak{A} = \langle A, g_i\rangle_{i\in\omega}$. We may assume $A = \varkappa$ without loss of generality. If $\beta_1 < \ldots < \beta_n$ are elements of $\varkappa$, let $f(\{\beta_1, \ldots, \beta_n\}) =$ the set of open formulas in first-order logic satisfied by $\langle \beta_1, \ldots, \beta_n\rangle$ in the structure $\mathfrak{A}$. (It will be understood throughout that we have a fixed enumeration $v_0, v_1, \ldots$, of variables. In the present context, β_1 corresponds to v_0, β_2 to v_1, etc.). The number of possible sets of formulas being c, we may apply $\varkappa \underset{W}{\to} (\omega)_c^{<\aleph_0}$ (invoking the lemma) to get an increasing sequence $\langle \alpha_i\colon i \in \omega\rangle$ so that, for any n, $f(\{\alpha_0, \ldots, \alpha_{n-1}\}) = f(\{\alpha_1, \ldots, \alpha_n\})$, i.e. $\langle \alpha_0, \ldots, \alpha_{n-1}\rangle$ and $\langle \alpha_1, \ldots, \alpha_n\rangle$ satisfy the same open formulas in $\mathfrak{A}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\{\alpha_i\colon i \in \omega\}$. We now claim that there is a monomorphism F of $\mathfrak{B}$ into itself defined by $F\big(\tau(\alpha_0, \ldots, \alpha_{n-1})\big) = \tau(\alpha_1, \ldots, \alpha_n)$, where τ is any term of the relevant first-order language and where $\tau(\alpha_0, \ldots, \alpha_{n-1})$ is the result of evaluating, in the structure $\mathfrak{B}$, the term τ with α_i assigned to the ith variable v_i for each i. For example to see that the above equation defines a 1-1 function, notice that $\tau(\alpha_0, \ldots, \alpha_{n-1}) = \tau'(\alpha_0, \ldots, \alpha_{n-1})$ iff $\tau(\alpha_1, \ldots, \alpha_n) = \tau'(\alpha_1, \ldots, \alpha_n)$, since

$\langle a_0, \ldots, a_{n-1}\rangle$ satisfies the formula $\tau(v_0, \ldots, v_{n-1}) = \tau'(v_0, \ldots, v_{n-1})$ iff $\langle a_1, \ldots, a_n\rangle$ does. Of course, it is evident that every member of $\mathfrak{B}$ can be written in the form $\tau(a_0, \ldots, a_{n-1})$. That F is a monomorphism is immediate from the defining equation of F. F is non-trivial because $F(a_0) = a_1$.

Sufficiency. Suppose $f\colon [\varkappa]^{<\aleph_0} \to 2$. Split this into infinitely many functions by setting

$$g_i(\beta_1, \ldots, \beta_i) = f(\{\beta_1, \ldots, \beta_i\})$$

for each $i \in \omega$ and $\beta_1, \ldots, \beta_i \in \varkappa$. Let J be the characteristic function of the $\in$ relation on $\varkappa$. Consider the algebra $\mathfrak{A} = \langle \varkappa, J, g_i, 0, 1\rangle_{i\in\omega}$. Our condition guarantees the existence of a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ having a non-trivial monomorphism F into itself. Let a_0 be an element moved by F. Define inductively $a_{n+1} = F(a_n)$. If $a_1 < a_0$, then $J(a_1, a_0) = 1$, so applying F successively, we would have $J(a_{n+1}, a_n) = 1$, giving us a descending chain. Hence $a_0 < a_1 < a_2 < \ldots$ This sequence has the desired property because (since F preserves 0 and 1)

$$f(\{a_0, \ldots, a_{n-1}\}) = g_n(a_0, \ldots, a_{n-1}) = F\big(g_n(a_0, \ldots, a_{n-1})\big)$$
$$= g_n(a_1, \ldots, a_n) = f(\{a_1, \ldots, a_n\}) .$$

THEOREM. *If* $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$, *then* $\varkappa \to (\omega)^{<\aleph_0}$.

Proof. Assume $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$, and let $f\colon [\varkappa]^{<\aleph_0} \to 2$ be arbitrary. We seek an infinite set of indiscernibles for f. Consider the structure $\mathfrak{D} = \langle \varkappa, <, f_i\rangle_{i\in\omega}$, $<$ the usual ordering of $\varkappa$, where f_i is i-ary, defined by:

$$f_i(\beta_1, \ldots, \beta_i) = f(\{\beta_1, \ldots, \beta_i\}) \quad \text{for} \quad \beta_1, \ldots, \beta_i \in \varkappa .$$

In a standard way, we can endow this structure with Skolem functions g_i, so that any subalgebra of $\langle \varkappa, g_i\rangle_{i\in\omega}$ is an elementary substructure of $\mathfrak{D}$ (for brevity, I am confounding structures with their universes). By the proposition, $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$ guarantees the existence of a subalgebra $\mathfrak{B}'$ of $\langle \varkappa, g_i\rangle$ with a non-trivial monomorphism F of $\mathfrak{B}'$ into itself. Let $\mathfrak{B} = \langle B, <, f_i\rangle$ be the substructure of $\mathfrak{D}$ with the same universe as $\mathfrak{B}'$. $\mathfrak{B}$ is an elementary substructure of $\mathfrak{D}$, and F is an elementary monomorphism of $\mathfrak{B}$ into itself.

Let a_0 be the smallest element of B moved by F. Define inductively: $a_{n+1} = F(a_n)$. If $a_0 > a_1$, then successively applying F to both sides, $a_1 > a_2$, $a_2 > a_3$, etc. giving us a descending chain. Hence $a_0 < a_1$ and, successively applying F, $a_0 < a_1 < a_2 < \ldots$ We intend to show that $\{a_i\colon i \in \omega\}$ is a set of indiscernibles for f.

This is done most aptly by proving, using induction on m, the following statement: if $1 \leqslant m \leqslant n$ and $i_1 < \ldots < i_n$, then

$$f(\{a_0, \ldots, a_{n-1}\}) = f(\{a_{i_1}, \ldots, a_{i_m}, a_{i_m+1}, \ldots, a_{i_m+n-m}\}) ,$$

i.e.

$$f_n(a_0, \ldots, a_{n-1}) = f_n(a_{i_1}, \ldots, a_{i_m}, a_{i_m+1}, \ldots, a_{i_m+n-m}) .$$

The case $m = n$ directly implies the desired indiscernibility of the a_i. Turning to the proof, notice that F^{i_1}, the result of iterating F i_1 times, is still an elementary monomorphism of $\mathfrak{B}$, from which we infer that $f_n(a_0, \ldots, a_{n-1}) = 0$ iff $f_n(a_{i_1}, a_{i_1+1}, \ldots, a_{i_1+n-1}) = 0$, 0 being definable and hence fixed by elementary monomorphisms. Since f_n assumes only the values 0 and 1, the desired equality follows in case $m = 1$.

For the induction step, suppose the above statement is true for m and suppose that $m+1 \leqslant n$. Let $k = m+1$. It will suffice, by induction hypothesis, to see that

$$(*) \qquad f_n(a_{i_1}, \ldots, a_{i_m}, a_{i_m+1}, \ldots) = f_n(a_1, \ldots, a_{i_m}, a_{i_k}, a_{i_k+1}, \ldots) .$$

For the moment, we examine a situation which is ostensibly quite different. Let $p = i_k - (i_m+1)$. If $\beta_1, \ldots, \beta_n$ are elements of B less than a_0, then

$$f_n(\beta_1, \ldots, \beta_m, a_0, \ldots, a_{n-m-1}) = 0 \quad \text{iff} \quad f_n(\beta_1, \ldots, \beta_m, a_p, \ldots, a_{p+n-m-1}) = 0 ,$$

since F^p is an elementary monomorphism of $\mathfrak{B}$ which fixes $\beta_1, \ldots, \beta_m$. Thus,

$$(\forall x_1, \ldots, x_m < a_0)\big(f_n(x_1, \ldots, x_m, a_0, \ldots, a_{n-m-1})$$
$$= f_n(x_1, \ldots, x_m, a_p, \ldots, a_{p+n-m-1})\big)$$

is true in $\mathfrak{B}$. Applying F^{i_m+1} to this sentence, we conclude that

$$(\forall x_1, \ldots, x_m < a_{i_m+1})\big(f_n(x_1, \ldots, x_m, a_{i_m+1}, \ldots, a_{i_m+n-m})$$
$$= f_n(x_1, \ldots, x_m, a_{i_k}, a_{i_k+1}, \ldots)\big)$$

is also true in $\mathfrak{B}$. Setting $x_1 = a_{i_1}, \ldots, x_m = a_{i_m}$ we obtain just the thing we set out to prove, the equality $(*)$.

The hard direction completed, the desired equivalence of $\varkappa \to (\omega)^{<\aleph_0}$ with $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}$ is now established. We conclude with a word on generalizing to arbitrary λ (i.e., $\varkappa \to (\omega)^{<\aleph_0}_{\lambda}$ iff $\varkappa \underset{W}{\to} (\omega)^{<\aleph_0}_{\lambda}$). The proof of the last theorem must be modified slightly to insure that F preserves every element less than λ. This can be done by first improving the lemma to assert the equivalence of $\varkappa \to (\omega)^{<\aleph_0}_{\lambda}$ with $\varkappa \to (\omega)^{<\aleph_0}_{\lambda^{\aleph_0}}$ and by then refining the definition of f in the proof of the proposition to deal with elements $< \lambda$ (so that, under our revised definition of f, two sets of cardinality n will have the same image if and only if, when viewed as increasing n-triples, they satisfy the same open formulas in $\mathfrak{A}$ and, for each term τ, give the same values in $\mathfrak{A}$ upon substitution in τ, provided either value is less than λ). It should be stressed that we have replaced the existing proposition not by its full analogue for general λ but rather by a modified form in which the algebras still have countably many operations but the monomorphism is required to fix every element of λ present in $\mathfrak{B}$. In the full analogue, we would replace 'algebra' by 'algebra having λ operations'. This full analogue does in fact hold, but its proof depends upon the equivalence of $\varkappa\to(\omega)_\lambda^{<\aleph_0}$ with $\varkappa\underset{W}{\to}(\omega)_\lambda^{<\aleph_0}$ and uses results of [3], in particular the equivalence of $\varkappa\to(\omega)_\lambda^{<\aleph_0}$ with $\varkappa\to(\omega)_{2^\lambda}^{<\aleph_0}$.

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