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Homotopy dependence of fundamental sequences, relative fundamental equivalence of sets and a generalization of cohomotopy groups

by

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In order to extend some standard notions of the homotopy theory onto arbitrary compacta K. Borsuk introduced in [4] the notion of the *fundamental sequence*. Replacing maps by fundamental sequences one can obtain generalizations of many standard notions. In such a manner we obtain the notions of homotopy dependence of fundamental sequences (§ 4), relative fundamental domination and relative fundamental equivalence of sets (§ 6) and fundamental skeletons (§ 7). All these notions are generalizations of the notions introduced by K. Borsuk in [1] and [2]. Using the notion of the fundamental skeleton, we define groups $\underline{\pi}_k^n(X)$ which are generalizations of the generalized cohomotopy groups $\pi_k^n(X)$ introduced by K. Borsuk in [3].

§ 1. Basic notions. In [4], [5], and [6] K. Borsuk introduced the basic notions of theory of shape. We recall some of the basic definitions. All spaces considered in this paper are compact and metric, and thus we can assume that they lie in the Hilbert cube Q.

By a *fundamental sequence from X to Y* (notation $\underline{f} = \{f_k, X, Y\}$ or $\underline{f}\colon X \to Y$) we understand an ordered triple consisting of the compacta $X, Y \subset Q$ and of a sequence of maps $f_k\colon Q \to Q$, $k = 1, 2, \ldots$, such that for every neighborhood V of Y there exists a neighborhood U of X such that $f_k|U \simeq f_{k+1}|U$ in V for almost all k.

We say that the fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, X, Y\}$ are *homotopic* (notation $\underline{f} \simeq \underline{g}$) if for every neighborhood V of Y there exists a neighborhood U of X such that $f_k|U \simeq g_k|U$ in V for almost all k. This relation is reflexive, symmetric and transitive and it decomposes all fundamental sequences into *fundamental classes*. The fundamental class with representative $\underline{f}$ is denoted by $[\underline{f}]$ or, precisely, by $[\underline{f}]\colon X \to Y$.

If $f\colon X \to Y$ is a map, then there exists a map $\hat{f}\colon Q \to Q$ such that $\hat{f}(x) = f(x)$ for $x \in X$. Setting $f_k = \hat{f}$ for $k = 1, 2, \ldots$ we get a fundamental sequence $\underline{f} = \{f_k, X, Y\}$ called the *fundamental sequence generated by the map* f. Then we say that the fundamental class $[\underline{f}]$ is *generated* by the map f.

If $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, Y, Z\}$ are two fundamental sequences, then the triple $\{g_k f_k, X, Z\}$ is also a fundamental sequence. It is called the *composition of the fundamental sequences* $\underline{f}$ and $\underline{g}$ and denoted by $\underline{gf}$. The fundamental class $[\underline{gf}]$ is called the *composition of the fundamental classes* $[\underline{f}]$ and $[\underline{g}]$ and it is denoted by $[\underline{g}][\underline{f}]$.

If f_k is the identity map $i\colon Q \to Q$ for every $k = 1, 2, \ldots$, then the fundamental sequence $\{f_k, X, X\}$ is said to be the *fundamental identity sequence for* X and it is denoted by $\underline{i}_X$. The fundamental class $[\underline{i}_X]$ is called the *fundamental identity class for* X.

By the *shape* of a compactum X we understand the collection $\mathrm{Sh}(X)$ of all compacta Y such that there exist fundamental sequences $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to X$ such that $\underline{gf} \simeq \underline{i}_X$, and $\underline{fg} \simeq \underline{i}_Y$. Then the fundamental sequence $\underline{f}$ (and also the fundamental sequence $\underline{g}$) is called the *fundamental equivalence*. If for compacta X, Y there exist fundamental sequences $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to X$ such that $\underline{fg} \simeq \underline{i}_Y$, then we say that the shape of X *dominates* the shape of Y and write $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$.

§ 2. Extendability of fundamental sequences. If $X \subset X'$ and $\underline{f} = \{f_k, X, Y\}$ and $\underline{f}' = \{f'_k, X', Y\}$ are fundamental sequences such that $f'_k|X = f_k|X$, then we say that the fundamental sequence $\underline{f}'$ is an *extension of* the fundamental sequence $\underline{f}$ (see [5], p. 56). If for a fundamental sequence $\underline{f}\colon X \to Y$ there exists an extension $\underline{f}'\colon X' \to Y$, then we say that the fundamental sequence $\underline{f}$ is *extendable over* X'. The collection of all fundamental sequences from X to Y will be denoted by $\{X, Y\}$ and the collection of all fundamental classes from X to Y by $[X, Y]$. If $X \subset X'$, then by $\{X \subset X', Y\}$ we denote the collection of all fundamental sequences from X to Y extendable over X'.

In [9] H. Patkowska has proved the following

(2.1) THEOREM. *If fundamental sequences* $\underline{f}\colon X \to Y$ *and* $\underline{g}\colon X \to Y$ *are homotopic and* $\underline{f}$ *has an extension* $\underline{f}'\colon X' \to Y$, *then* $\underline{g}$ *has an extension* $\underline{g}'\colon X' \to Y$ *homotopic to* $\underline{f}'$.

By Theorem (2.1) we can introduce the notion of extendability for fundamental classes. We say that a fundamental class $[\underline{f}]\colon X \to Y$ is *extendable* over X' if its representative $\underline{f}\colon X \to Y$ is extendable over X'. If $\underline{f}'$ is an extension of $\underline{f}$, then we say that the fundamental class $[\underline{f}']$ is an *extension* of the fundamental class $[\underline{f}]$. The collection of all fundamental classes from X to Y extendable over X' is denoted by $[X \subset X', Y]$.

The collection of all maps of X into Y is denoted by Y^X and the collection of all homotopy classes of maps belonging to Y^X is denoted by $[Y^X]$. If $X \subset X'$, then the subset of Y^X consisting of all maps extendable over X' is denoted by $Y^{X \subset X'}$ (compare [3], p. 616).

It is well known (see [1], p. 94) that

(2.2) *If* $Y \in \mathrm{ANR}$, *then every map* $g\colon X \to Y$ *homotopic to a map* $f\colon X \to Y$ *extendable to a map* $f'\colon X' \to Y$ *is extendable to a map* $g'\colon X' \to Y$ *homotopic to* f'.

It follows that if $Y \in \mathrm{ANR}$, then the set $Y^{X \subset X'}$ is the union of some homotopy classes belonging to $[Y^X]$. In the case of $Y \in \mathrm{ANR}$ the set of all homotopy classes of maps belonging to $Y^{X \subset X'}$ is denoted by $[Y^{X \subset X'}]$ (compare [3], p. 616).

(2.3) *If* $\underline{f} = \{f_k, X, Y\}$ *is a fundamental sequence generated by a map* $f\colon X \to Y$ *and* $f'\colon X' \to Y$ *is an extension of* f, *then* $\underline{f}$ *is extendable over* X' *to a fundamental sequence homotopic to a sequence generated by* f'.

Proof. Let $\underline{f}' = \{f'_k, X', Y\}$ be a fundamental sequence generated by f'. It is easy to see that $\underline{f}'' = \{f'_k, X, Y\}$ is a fundamental sequence (because $X \subset X'$). Since $f'_k|X = f_k|X$, then by (1.1) of [5] $\underline{f} \simeq \underline{f}''$. The fundamental sequence $\underline{f}'$ is an extension of $\underline{f}''$. Hence by Patkowska's Theorem (2.1) the fundamental sequence $\underline{f}$ is extendable over X' to a fundamental sequence homotopic to $\underline{f}'$.

From (2.3) we infer that

(2.4) *A fundamental class generated by a map* $f\colon X \to Y$ *extendable over* X' *is extendable over* X'.

Let us prove that

(2.5) *If* $Y \in \mathrm{ANR}$ *and a fundamental sequence* $\underline{f} = \{f_k, X, Y\}$ *generated by a map* $f\colon X \to Y$ *is extendable to a fundamental sequence* $\underline{f}' = \{f'_k, X', Y\}$, *then the map* f *is extendable to a map* $f'\colon X' \to Y$ *such that the fundamental sequence generated by* f' *is homotopic to* $\underline{f}'$.

Proof. By (5.1) of [4] there exists a fundamental sequence $\underline{f}'' = \{f''_k, X', Y\}$ generated by a map $f''\colon X' \to Y$ and homotopic to $\underline{f}'$. It is easy to see that $\underline{g}' = \{f'_k, X, Y\}$ and $\underline{g}'' = \{f''_k, X, Y\}$ are fundamental sequences and $\underline{g}' \simeq \underline{g}''$ (because $X \subset X'$). Since $f'_k|X = f_k|X$, then $\underline{f} \simeq \underline{g}'$ (see [5], (1.1), p. 57). Hence $\underline{f} \simeq \underline{g}''$. Moreover, $\underline{g}''$ is generated by $f''|X$. Hence by (4.3) and (2.1) of [4] $f \simeq f''|X$. Therefore, since $Y \in \mathrm{ANR}$, f is extendable to a map $f'\colon X \to Y$ and $f' \simeq f''$. Therefore the fundamental sequence generated by f' is homotopic to $\underline{f}''$, and thus it is homotopic to $\underline{f}'$.

(2.6) Remark. Without the hypothesis $Y \in \mathrm{ANR}$ (2.5) is not true. For instance, let X' be a circle given in the plane E^2 by the equation $x^2 + y^2 = 1$ and let X be a subset of X' consisting of all points $(x, y) \in X'$ satisfying the inequality $x \geqslant 0$. Let Y_1 by a subset of the plane E^2 consisting of all points $(x, y) \in E^2$ satisfying condition $y = \sin(1/x)$ and $0 < x \leqslant 1$. Let Y_2 be the segment with end-points $(0,1)$ and $(0,-2)$, Y_3 the segment with end-points $(0,-2)$ and $(1,-2)$ and Y_4 the segment with end-points $(1,-2)$ and $(1, \sin 1)$. Let $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. Let $f\colon X \to Y$ be a map such that $f(X) = Y_4$ and the map $h\colon X \to Y_4$ defined by the formula $h(x) = f(x)$ is a homeomorphism. It is easy to see that there exists a fundamental sequence $\underline{f}\colon X \to Y$ satisfying hypothesis of (2.5) and such that its extension $\underline{f}'$ is a fundamental equivalence. Then no fundamental sequence homotopic to $\underline{f}'$ is generated by a map $f'\colon X' \to Y$, because every map of X' into Y is homotopic to constant map.

From (2.5) we obtain

(2.7) *If $Y \in$ ANR and a fundamental class $[\underline{f}]\colon X \to Y$ generated by a map $f\colon X \to Y$ is extendable to a fundamental class $[\underline{f}']\colon X' \to Y$, then the map f is extendable to a map $f'\colon X' \to Y$ generating the fundamental class $[\underline{f}']$.*

§ 3. Cartesian products and diagonals of fundamental sequences. The notions introduced in this section for fundamental sequences are analogous to the well-known notions for maps. They will be needed in the next section. First we recall the well-known notions of the Cartesian product of maps and the diagonal of maps. The Cartesian product of sets $X_1, X_2, \ldots, X_n$ is denoted by $\mathop{P}\limits_{i=1}^{n} X_i$. If $X_i = X$ for $i = 1, 2, \ldots, n$, then the Cartesian product $\mathop{P}\limits_{i=1}^{n} X_i$ is denoted by X^n.

Consider maps $f_i\colon X_i \to Y_i$ for $i = 1, 2, \ldots, n$. The map $f\colon \mathop{P}\limits_{i=1}^{n} X_i \to \mathop{P}\limits_{i=1}^{n} Y_i$ defined by the formula

$$f(x_1, x_2, \ldots, x_n) = \big(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\big) \quad \text{for} \quad x_i \in X_i$$

is called the *Cartesian product of the maps* $f_1, f_2, \ldots, f_n$ and denoted by $f_1 \times f_2 \times \ldots \times f_n$ or, shortly, by $\mathop{P}\limits_{i=1}^{n} f_i$.

If $f_i\colon X \to Y_i$ are maps for $i = 1, 2, \ldots, n$, then the map $f\colon X \to \mathop{P}\limits_{i=1}^{n} Y_i$ defined by the formula

$$f(x) = \big(f_1(x), f_2(x), \ldots, f_n(x)\big) \quad \text{for} \quad x \in X$$

is called the *diagonal of the maps* $f_1, f_2, \ldots, f_n$ and denoted by $f_1 \triangle f_2 \triangle \ldots \triangle f_n$ or, shortly, by $\mathop{\triangle}\limits_{i=1}^{n} f_i$.

Let us prove the following

(3.1) LEMMA. *Suppose $X_i, Y_i \subset Q$ for $i = 1, 2, \ldots, n$ are compacta and let $X = \mathop{P}\limits_{i=1}^{n} X_i \subset Q^n$ and $Y = \mathop{P}\limits_{i=1}^{n} Y_i \subset Q^n$. If $\underline{f_i} = \{f_{ik}, X_i, Y_i\}$ (for $i = 1, 2, \ldots, n$) are fundamental sequences and $f_k = \mathop{P}\limits_{i=1}^{n} f_{ik}\colon Q^n \to Q^n$ (for $k = 1, 2, \ldots$), then $\underline{f} = \{f_k, X, Y\}$ is a fundamental sequence.*

Proof. Take an arbitrary neighborhood V of Y in Q^n. Then there exist neighborhoods V_i of Y_i in Q (for $i = 1, 2, \ldots, n$) such that $\mathop{P}\limits_{i=1}^{n} V_i \subset V$. Since $\underline{f_i}$ (for $i = 1, 2, \ldots, n$) is a fundamental sequence then there exists a neighborhood U_i of X_i in Q such that $f_{ik}|U_i \simeq f_{i,k+1}|U_i$ in V_i for almost all k. Therefore there exists a map

$$h_{ik}\colon U_i \times \langle 0, 1\rangle \to V_i \quad (\text{for } i = 1, 2, \ldots, n \text{ and almost all } k)$$

such that

(3.2) $$h_{ik}(x_i, 0) = f_{ik}(x_i), \quad h_{ik}(x_i, 1) = f_{i,k+1}(x_i) \quad \text{for} \quad x_i \in U_i.$$

The set $U = \mathop{P}\limits_{i=1}^{n} U_i$ is a neighborhood of X in Q^n. Let us define the map $h_k\colon U \times \langle 0, 1\rangle \to V$ by the formula

(3.3) $$h_k\big((x_1, x_2, \ldots, x_n), t\big) = \big(h_{1k}(x_1, t), h_{2k}(x_2, t), \ldots, h_{nk}(x_n, t)\big).$$

Since the maps h_{ik} $(i = 1, 2, \ldots, n)$ are defined for almost all k, then the maps h_k are also defined for almost all k. It follows from (3.2) and (3.3) that

$$h_k(x, 0) = f_k(x), \quad h_k(x, 1) = f_{k+1}(x) \quad \text{for} \quad x \in U.$$

Hence $f_k|U \simeq f_{k+1}|U$ in V for almost all k. Thus $\underline{f} = \{f_k, X, Y\}$ is a fundamental sequence.

The fundamental sequence $\underline{f}$ associated with fundamental sequences $\underline{f_1}, \underline{f_2}, \ldots, \underline{f_n}$ by Lemma (3.1) will be called the *Cartesian product of the fundamental sequences* $\underline{f_1}, \underline{f_2}, \ldots, \underline{f_n}$ and denoted by $\underline{f_1} \times \underline{f_2} \times \ldots \times \underline{f_n}$ or, shortly, by $\mathop{P}\limits_{i=1}^{n} \underline{f_i}$.

It follows at once from the definition of the Cartesian product of fundamental sequences that

(3.4) *If $\underline{f_i}\colon X_i \to Y_i$ and $\underline{g_i}\colon Y_i \to Z_i$ for $i = 1, 2, \ldots, n$, then*

$$\mathop{P}\limits_{i=1}^{n} (\underline{g_i}\underline{f_i}) = \mathop{P}\limits_{i=1}^{n} \underline{g_i} \mathop{P}\limits_{i=1}^{n} \underline{f_i}.$$

(3.5) LEMMA. *If $\underline{f_i} \simeq \underline{g_i}$ for $i = 1, 2, \ldots, n$,* then

$$\underset{i=1}{\overset{n}{P}} \underline{f_i} \simeq \underset{i=1}{\overset{n}{P}} \underline{g_i} .$$

A simple proof of Lemma (3.5), analogous to the proof of Lemma (3.1), is left to the reader.

By Lemma (3.5) the fundamental class $[\underset{i=1}{\overset{n}{P}} \underline{f_i}]$ depends only on the fundamental classes $[\underline{f_1}], [\underline{f_2}], \ldots, [\underline{f_n}]$. It will be called the *Cartesian product of the fundamental classes* $[\underline{f_1}], [\underline{f_2}], \ldots, [\underline{f_n}]$ and denoted by $[\underline{f_1}] \times [\underline{f_2}] \times \ldots \times [\underline{f_n}]$ or, shortly, by $\underset{i=1}{\overset{n}{P}} [\underline{f_i}]$.

From (3.4) we obtain

$$(3.6) \qquad \underset{i=1}{\overset{n}{P}} ([\underline{g_i}][\underline{f_i}]) = \underset{i=1}{\overset{n}{P}} [\underline{g_i}] \underset{i=1}{\overset{n}{P}} [\underline{f_i}] .$$

The following two properties of Cartesian products of fundamental sequences are obvious.

(3.7) *If a fundamental sequence $\underline{f_i'}\colon X_i' \to Y_i$ is an extension of a fundamental sequence $\underline{f_i}\colon X_i \to Y_i$ for $i = 1, 2, \ldots, n$, then the Cartesian product $\underset{i=1}{\overset{n}{P}} \underline{f_i'}\colon \underset{i=1}{\overset{n}{P}} X_i' \to \underset{i=1}{\overset{n}{P}} Y_i$ is an extension of the Cartesian product $\underset{i=1}{\overset{n}{P}} \underline{f_i}\colon \underset{i=1}{\overset{n}{P}} X_i \to \underset{i=1}{\overset{n}{P}} Y_i$.*

(3.8) *If fundamental sequences $\underline{f_i}\colon X_i \to Y_i$ are generated by maps $f_i\colon X_i \to Y_i$ for $i = 1, 2, \ldots, n$, respectively, then the Cartesian product $\underset{i=1}{\overset{n}{P}} \underline{f_i}$ of the fundamental sequences $\underline{f_1}, \underline{f_2}, \ldots, \underline{f_n}$ is generated by the Cartesian product $\underset{i=1}{\overset{n}{P}} f_i$ of the maps $f_1, f_2, \ldots, f_n$.*

Let us prove the following

(3.9) LEMMA. *Suppose $X, Y_1, Y_2, \ldots, Y_n \subset Q$ are compacta and let $Y = \underset{i=1}{\overset{n}{P}} Y_i \subset Q^n$. If $\underline{f_i} = \{f_{ik}, X, Y_i\}$ (for $i = 1, 2, \ldots, n$) are fundamental sequences and $f_k = \underset{i=1}{\overset{n}{\triangle}} f_{ik}\colon Q \to Q^n$ (for $k = 1, 2, \ldots$), then $\underline{f} = \{f_k, X, Y\}$ is a fundamental sequence.*

Proof. Take an arbitrary neighborhood V of Y in Q^n. Then there exist neighborhoods V_i of sets Y_i in Q (for $i = 1, 2, \ldots, n$) such that $\underset{i=1}{\overset{n}{P}} V_i \subset V$. Since $\underline{f_i}$ (for $i = 1, 2, \ldots, n$) is a fundamental sequence, there exists a neighborhood U_i of X_i in Q such that $f_{ik}|U_i \simeq f_{i,k+1}|U_i$ in V_i for almost all k. Therefore there exists a map $h_{ik}\colon U_i \times \langle 0, 1\rangle \to V_i$ (for $i = 1, 2, \ldots, n$ and almost all k) such that

$$(3.10) \qquad h_{ik}(x, 0) = f_{ik}(x), \quad h_{ik}(x, 1) = f_{i,k+1}(x) \quad \text{for} \quad x \in U_i .$$

The set $U = \bigcap_{i=1}^{n} U_i$ is a neighborhood of X in Q. Let us define the map $h_k\colon U \times \langle 0, 1\rangle \to V$ by the formula

$$(3.11) \qquad h_k(x, t) = \big(h_{1k}(x, t), h_{2k}(x, t), \ldots, h_{nk}(x, t)\big) .$$

Since the maps h_{ik} ($i = 1, 2, \ldots, n$) are defined for almost all k, the maps h_k are also defined for almost all k. It follows from (3.10) and (3.11) that

$$h_k(x, 0) = f_k(x), \quad h_k(x, 1) = f_{k+1}(x) \quad \text{for} \quad x \in U .$$

Hence $f_k|U \simeq f_{k+1}|U$ in V for almost all k. Thus $\underline{f} = \{f_k, X, y\}$ is a fundamental sequence.

The fundamental sequence $\underline{f}$ associated with fundamental sequences $\underline{f_1}, \underline{f_2}, \ldots, \underline{f_n}$ by Lemma (3.9) will be called the *diagonal of the fundamental sequences* $\underline{f_1}, \underline{f_2}, \ldots, \underline{f_n}$ and denoted by $\underline{f_1} \triangle \underline{f_1} \triangle \ldots \triangle \underline{f_n}$ or shortly, by $\underset{i=1}{\overset{n}{\triangle}} \underline{f_i}$.

It follows at once from the definitions of the Cartesian product of fundamental sequences and of the diagonal of fundamental sequences that

(3.12) *If $\underline{f_i}\colon X \to Y_i$ and $\underline{g_i}\colon Y_i \to Z_i$ for $i = 1, 2, \ldots, n$, then*

$$\underset{i=1}{\overset{n}{\triangle}} (\underline{g_i}\underline{f_i}) = (\underset{i=1}{\overset{n}{P}} \underline{g_i})(\underset{i=1}{\overset{n}{\triangle}} \underline{f_i})\colon X \to \underset{i=1}{\overset{n}{P}} Z_i .$$

(3.13) LEMMA. *If $\underline{f_i} \simeq \underline{g_i}\colon X \to Y_i$ for $i = 1, 2, \ldots, n$, then*

$$\underset{i=1}{\overset{n}{\triangle}} \underline{f_i} \simeq \underset{i=1}{\overset{n}{\triangle}} \underline{g_i}\colon X \to \underset{i=1}{\overset{n}{P}} Y_i .$$

A simple proof of Lemma (3.13), analogous to the proof of Lemma (3.9), is left to the reader.

By Lemma (3.13) the fundamental class $[\underset{i=1}{\overset{n}{\triangle}} \underline{f_i}]$ depends only on the fundamental classes $[\underline{f_1}], [\underline{f_2}], \ldots, [\underline{f_n}]$. It will be called the *diagonal of the fundamental classes* $[\underline{f_1}], [\underline{f_2}], \ldots, [\underline{f_n}]$ and denoted by $[\underline{f_1}] \triangle [\underline{f_2}] \triangle \ldots \triangle [\underline{f_n}]$ or, shortly, by $\underset{i=1}{\overset{n}{\triangle}} [\underline{f_i}]$.

From (3.12) we obtain

(3.14) *If $[\underline{f_i}]\colon X \to Y_i$* and *$[\underline{g_i}]\colon Y_i \to Z_i$ for $i = 1, 2, \ldots, n$, then*

$$\underset{i=1}{\overset{n}{\triangle}} ([\underline{g_i}][\underline{f_i}]) = (\underset{i=1}{\overset{n}{P}} [\underline{g_i}])(\underset{i=1}{\overset{n}{\triangle}} [\underline{f_i}])\colon X \to \underset{i=1}{\overset{n}{P}} Z_i .$$

The following two properties of diagonals of fundamental sequences are obvious.

(3.15) *If a fundamental sequence* $\underline{f}'_i\colon X' \to Y_i$ *is an extension of a fundamental sequence* $\underline{f}_i\colon X \to Y_i$ *for* $i = 1, 2, \ldots, n$, *then the diagonal* $\mathop{\triangle}\limits_{i=1}^{n} \underline{f}'_i\colon X' \to \mathop{P}\limits_{i=1}^{n} Y_i$ *is an extension of the diagonal* $\mathop{\triangle}\limits_{i=1}^{n} \underline{f}_i\colon X \to \mathop{P}\limits_{i=1}^{n} Y_i$.

(3.16) *If fundamental sequences* $\underline{f}_i\colon X \to Y_i$ *are generated by maps* $f_i\colon X \to Y_i$ *for* $i = 1, 2, \ldots, n$, *respectively, then the diagonal* $\mathop{\triangle}\limits_{i=1}^{n} \underline{f}_i$ *of the fundamental sequences* $\underline{f}_1, \underline{f}_2, \ldots, \underline{f}_n$ *is generated by the diagonal* $\mathop{\triangle}\limits_{i=1}^{n} f_i$ *of the maps* $f_1, f_2, \ldots, f_n$.

§ 4. Homotopy dependence of fundamental sequences. In [1] K. Borsuk has introduced the notion of the homotopy dependence of maps. In an analogous manner we introduce the notion of the homotopy dependence of fundamental sequences.

Suppose $X, Y \subset Q$ are two compacta and a subset A of the set $\{X, Y\}$ of all fundamental sequences from X to Y. We shall say that a fundamental sequence $\underline{f}\colon X \to Y$ is *homotopically dependent on* A if there exist $\underline{f}_1, \underline{f}_2, \ldots, \underline{f}_n \in A$ and a fundamental sequence $\underline{\vartheta}\colon Y^n \to Y$, called the *fundamental multisequence on* Y, such that $\underline{f} \simeq \underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f}_i$ (compare the definitions of the *multimap* and the *homotopy dependence of maps* in [1], p. 64). The set of all fundamental sequences homotopically dependent on A will be denoted by $\omega(A)$.

It follows at once from the definition that

(4.1) *If* $\underline{f} \in \omega(A)$ *and* $\underline{f} \simeq \underline{g}$ *then* $\underline{g} \in \omega(A)$.

Let us denote by 2^N the family of all subsets of a set N. A function $\lambda\colon 2^N \to 2^N$ satisfying the conditions:

$$A \subset \lambda(A) \text{ for every set } A \subset N,$$

$$\text{if } A \subset B \subset N \text{ then } \lambda(A) \subset \lambda(B),$$

$$\lambda(\lambda(A)) = \lambda(A) \text{ for every set } A \subset N$$

is said to be the *dependence operation* in the set N and then the set N is said to be the *dependence domain* (see [1], p. 66).

Let us prove the following

(4.2) Theorem. *The operation* ω *in the set* $\{X, Y\}$ *is a dependence operation, i.e.,*

(i) $A \subset \omega(A)$ *for every set* $A \subset \{X, Y\}$,

(ii) *if* $A \subset B \subset \{X, Y\}$ *then* $\omega(A) \subset \omega(B)$,

(iii) $\omega(\omega(A)) = \omega(A)$ *for every set* $A \subset \{X, Y\}$.

Proof. (i). Take an arbitrary fundamental sequence $\underline{f} \in A$. Let $\underline{f}_1 = \underline{f}$ and let $\underline{\vartheta}\colon Y \to Y$ be the fundamental identity sequence. Then $\underline{f} = \underline{\vartheta}\underline{f}_1$, whence $\underline{f} \in \omega(A)$.

(ii). Let $A \subset B \subset \{X, Y\}$ and $\underline{f} \in \omega(A)$. Then $\underline{f} \simeq \underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f}_i$ where $\underline{\vartheta}\colon Y^n \to Y$ and $\underline{f}_i \in A$ for $i = 1, 2, \ldots, n$. Since $A \subset B$, we have $\underline{f}_i \in B$ for $i = 1, 2, \ldots, n$. Hence $\underline{f} \in \omega(B)$.

(iii). It follows from (i) and (ii) that $\omega(A) \subset \omega(\omega(A))$. It remains to prove that $\omega(\omega(A)) \subset \omega(A)$. Take an arbitrary fundamental sequence $\underline{f} \in \omega(\omega(A))$. Then there exist fundamental sequences $\underline{f}_i \in \omega(A)$ for $i = 1, 2, \ldots, n$ and a fundamental multisequence $\underline{\vartheta}\colon Y^n \to Y$ such that $\underline{f} \simeq \underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f}_i$. Since $\underline{f}_i \in \omega(A)$ for $i = 1, 2, \ldots, n$, there exist fundamental sequences $\underline{f}_{ij} \in A$ for $j = 1, 2, \ldots, m_i$ and fundamental multisequences $\underline{\vartheta}_i\colon Y^{m_i} \to Y$ such that $\underline{f}_i \simeq \underline{\vartheta}_i \mathop{\triangle}\limits_{j=1}^{m_i} \underline{f}_{ij}$. Hence by (3.13) and (3.12)

$$\mathop{\triangle}\limits_{i=1}^{n} \underline{f}_i \simeq \mathop{\triangle}\limits_{i=1}^{n} \big(\underline{\vartheta}_i \mathop{\triangle}\limits_{j=1}^{m_i} \underline{f}_{ij}\big) = \big(\mathop{P}\limits_{i=1}^{n} \underline{\vartheta}_i\big) \mathop{\triangle}\limits_{i=1}^{n} \big(\mathop{\triangle}\limits_{j=1}^{m_i} \underline{f}_{ij}\big).$$

Hence we obtain

$$\underline{f} \simeq \underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f}_i \simeq \underline{\vartheta}\big(\mathop{P}\limits_{i=1}^{n} \underline{\vartheta}_i\big) \mathop{\triangle}\limits_{i=1}^{n} \big(\mathop{\triangle}\limits_{j=1}^{m_i} \underline{f}_{ij}\big).$$

Therefore the fundamental sequence $\underline{f}$ is homotopic to the composition of the diagonal

$$\mathop{\triangle}\limits_{i=1}^{n} \big(\mathop{\triangle}\limits_{j=1}^{m_i} \underline{f}_{ij}\big)\colon X \to Y^{m_1+m_2+\ldots+m_n}$$

of the fundamental sequences $\underline{f}_{ij} \in A$ and of the fundamental multisequence

$$\underline{\vartheta} \mathop{P}\limits_{i=1}^{n} \underline{\vartheta}_i\colon\ Y^{m_1+m_2+\ldots+m_n} \to Y.$$

Thus $\underline{f} \in \omega(A)$ and the proof is completed.

Consider an arbitrary subset T of the set $[X, Y]$ of all fundamental classes from X to Y. Let us assign to the set T the set $\lambda(T) \subset [X, Y]$ defined as follows. Let A_T be the subset of the set $\{X, Y\}$ of all fundamental sequences from X to Y such that

(4.3) $\underline{f} \in A_T$ if and only if $[\underline{f}] \in T$

and let

(4.4) $\underline{[f]} \in \lambda(T)$ if an only if $\underline{f} \in \omega(A_T)$.

The operation λ is well defined by (4.1).

Let us prove the following

(4.5) THEOREM. *The operation λ in the set $[X, Y]$ is a dependence operation, i.e.,*

(i) *$T \subset \lambda(T)$ for every set $T \subset [X, Y]$,*

(ii) *if $T \subset S \subset [X, Y]$ then $\lambda(T) \subset \lambda(S)$,*

(iii) *$\lambda(\lambda(T)) = \lambda(T)$ for every set $T \subset [X, Y]$.*

Proof. (i). Take an arbitrary fundamental class $\underline{[f]} \in T$. Then by (4.3) $\underline{f} \in A_T$. Hence by Theorem (4.2) $\underline{f} \in \omega(A_T)$ and by (4.4) $\underline{[f]} \in \lambda(T)$.

(ii). Let $T \subset S$ and $\underline{[f]} \in \lambda(T)$. Then $\underline{f} \in \omega(A_T)$. Obviously $A_T \subset A_S$. Hence by Theorem (4.2) $\omega(A_T) \subset \omega(A_S)$. Therefore $\underline{f} \in \omega(A_S)$ and by (4.4) $\underline{[f]} \in \lambda(S)$.

(iii). It follows from (i) and (ii) that $\lambda(T) \subset \lambda(\lambda(T))$. It remains to show that $\lambda(\lambda(T)) \subset \lambda(T)$. Take an arbitrary fundamental class $\underline{[f]} \in \lambda(\lambda(T))$. Then $\underline{f} \in \omega(A_{\lambda(T)})$. Let us observe that $A_{\lambda(T)} = \omega(A_T)$. Indeed, by (4.3) $\underline{g} \in A_{\lambda(T)}$ if and only if $\underline{[g]} \in \lambda(T)$ and by (4.4) this relation holds if and only if $\underline{g} \in \omega(A_T)$. Hence $\underline{f} \in \omega(\omega(A_T))$. Therefore by Theorem (4.2) $\underline{f} \in \omega(A_T)$. Thus by (4.4) $\underline{[f]} \in \lambda(T)$ and the proof is finished.

It follows from Lemma (3.13) and the definitions of the operations ω and λ that

(4.6) *$\underline{[f]} \in \lambda(T)$ if and only if there exist fundamental classes $\underline{[f_1]}, \underline{[f_2]}, \ldots, \underline{[f_n]} \in T$ and a fundamental class $\underline{[\vartheta]}$: $Y^n \to Y$ such that $\underline{[f]} = \underline{[\vartheta]} \mathop{\triangle}_{i=1}^{n} \underline{[f_i]}$.*

The fundamental class $\underline{[\vartheta]}$: $Y^n \to Y$ will be called the *fundamental multiclass*.

Suppose $X_1, X_2, Y \subset Q$ are three compacta and $\underline{[f]}$: $X_1 \to X_2$ is a fundamental class. Let us define the function

$$\underline{[f]}^*\colon [X_2, Y] \to [X_1, Y]$$

by the formula

(4.7) $$\underline{[f]}^*(\underline{[\varphi]}) = \underline{[\varphi]}\,\underline{[f]} \quad \text{for} \quad \underline{[\varphi]}\colon X_2 \to Y.$$

It is obvious that

(4.8) *If $\underline{[f]}$ is a fundamental identity class, then $\underline{[f]}^*$ is a identity function.*

(4.9) *If a composition $\underline{[g]}\,\underline{[f]}$ is defined, then*

$$(\underline{[g]}\,\underline{[f]})^* = \underline{[f]}^*\underline{[g]}^*.$$

Let N_1 and N_2 be two dependence domains with dependence operations λ_1 and λ_2, respectively. A function f: $N_1 \to N_2$ satisfying the condition

$$f(\lambda_1(A)) \subset \lambda_2(f(A)) \quad \text{for every set } A \subset N_1$$

will be called a *λ-morphism*. A one-to-one λ-morphism for which the inverse function is a λ-morphism is said to be a *λ-isomorphism* (see [1], p. 66).

Let λ_1 and λ_2 be the dependence operations in the sets $[X_1, Y]$ and $[X_2, Y]$ respectively, defined as above. Then we obtain the following

(4.10) THEOREM. *The function $\underline{[f]}^*$: $[X_2, Y] \to [X_1, Y]$ is a λ-morphism. Moreover, if $\underline{f}$ is a fundamental equivalence, then $\underline{[f]}^*$ is a λ-isomorphism.*

Proof. We must show that $\underline{[f]}^*(\lambda_2(T)) \subset \lambda_1(\underline{[f]}^*(T))$ for an arbitrary set $T \subset [X_2, Y]$. Take an arbitrary fundamental class $\underline{[\psi]} \in \underline{[f]}^*(\lambda_2(T))$. Then there exists a fundamental class $\underline{[\varphi]} \in \lambda_2(T)$ such that

$$\underline{[f]}^*(\underline{[\varphi]}) = \underline{[\psi]}, \quad \text{i.e.,} \quad \underline{[\varphi]}\,\underline{[f]} = \underline{[\psi]}.$$

Since $\underline{[\varphi]} \in \lambda_2(T)$, there exist fundamental classes $\underline{[\varphi_1]}, \underline{[\varphi_2]}, \ldots, \underline{[\varphi_n]} \in T$ and a fundamental multiclass $\underline{[\vartheta]}$: $Y^n \to Y$ such that

$$\underline{[\varphi]} = \underline{[\vartheta]} \mathop{\triangle}_{i=1}^{n} \underline{[\varphi_i]}.$$

Hence we obtain

$$\underline{[\psi]} = \underline{[\vartheta]} \mathop{\triangle}_{i=1}^{n} \underline{[\varphi_i]}\,\underline{[f]}.$$

Let $\underline{[\psi_i]} = \underline{[\varphi_i]}\,\underline{[f]}$ for $i = 1, 2, \ldots, n$. Then $\underline{[\psi]} = \underline{[\vartheta]} \mathop{\triangle}_{i=1}^{n} \underline{[\psi_i]}$.

It follows from (4.7) that $\underline{[\psi_i]} \in \underline{[f]}^*(T)$ and hence by (4.6) $\underline{[\psi]} \in \lambda_1(\underline{[f]}^*(T))$; thus the first part of the Theorem is proved.

Now, suppose that $\underline{f}$ is a fundamental equivalence. Let $\underline{[g]}$: $X_2 \to X_1$ be a fundamental class such that the compositions $\underline{[f]}\,\underline{[g]}$ and $\underline{[g]}\,\underline{[f]}$ are both fundamental identity classes. Then by (4.8) and (4.9) the functions

$$(\underline{[g]}\,\underline{[f]})^* = \underline{[f]}^*\underline{[g]}^* \quad \text{and} \quad (\underline{[f]}\,\underline{[g]})^* = \underline{[g]}^*\underline{[f]}^*$$

are identities. Therefore $\underline{[f]}^*$ and $\underline{[g]}^*$ are one-to-one functions and the λ-morphism $\underline{[g]}^*$ is the inverse to the λ-morphism $\underline{[f]}^*$. Thus $\underline{[f]}^*$ is a λ-isomorphism and the proof is completed.

Suppose we are given two compacta $X, Y \subset Q$. Consider an arbitrary subset A of the collection Y^X of all maps of X into Y. We shall say that a map f: $X \to Y$ is *homotopically dependent on the set A* provided that there exist maps $f_1, f_2, \ldots, f_n \in A$ and ϑ: $Y^n \to Y$ such that $f \simeq \vartheta \mathop{\triangle}\limits_{i=1}^{n} f_i$. (It is a modification of the notion given in [1], p. 64). Let us denote by $\{A\}$ the set of all fundamental sequences generated by maps belonging to A. We shall prove that

(4.11) *If a fundamental sequence $\underline{f}$ is homotopic to a fundamental sequence generated by a map homotopically dependent on a set A, then $\underline{f}$ is homotopically dependent on the set $\{A\}$.*

Proof. Let $\underline{g}$: $X \to Y$ be a fundamental sequence homotopic to $\underline{f}$ and generated by a map g: $X \to Y$ homotopically dependent on A. Then $g \simeq \vartheta \mathop{\triangle}\limits_{i=1}^{n} g_i$, where $g_i \in A$ for $i = 1, 2, \ldots, n$ and ϑ: $Y^n \to Y$. Let $\underline{g_i}$ (for $i = 1, 2, \ldots, n$) be a fundamental sequence generated by g_i and $\underline{\vartheta}$: $Y^n \to Y$ fundamental multisequence generated by ϑ. Then by (3.16) the fundamental sequence $\mathop{\triangle}\limits_{i=1}^{n} \underline{g_i}$ is generated by the map $\mathop{\triangle}\limits_{i=1}^{n} g_i$ and hence the fundamental sequence $\underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{g_i}$ is generated by the map $\vartheta \mathop{\triangle}\limits_{i=1}^{n} g_i$. By (4.1) of [4] $\underline{g} \simeq \underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{g_i}$. Hence $\underline{f} \simeq \underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{g_i}$ and $\underline{g_i} \in \{A\}$ by the definition of the set $\{A\}$. Therefore $\underline{f}$ is homotopically dependent on the set $\{A\}$.

(4.12) *If $Y \in$ ANR, $A \subset Y^X$ and a fundamental sequence $\underline{f}$ is homotopically dependent on the set of all fundamental sequences generated by maps belonging to A, then $\underline{f}$ is homotopic to a fundamental sequence generated by a map homotopically dependent on A.*

Proof. By the hypothesis $\underline{f} \simeq \underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f_i}$, where fundamental sequences $\underline{f_i}$ are generated by maps $f_i \in A$ for $i = 1, 2, \ldots, n$ and $\underline{\vartheta}$: $Y^n \to Y$. Since $Y \in$ ANR, then by Theorem (5.1) of [4] (p. 228) the fundamental multisequence $\underline{\vartheta}$ is homotopic to a fundamental multisequence $\underline{\eta}$: $Y^n \to Y$ generated by a map η: $Y^n \to Y$. Hence $\underline{f} \simeq \underline{\eta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f_i}$ and by (3.16) the fundamental sequence $\underline{\eta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f_i}$ is generated by the map $\eta \mathop{\triangle}\limits_{i=1}^{n} f_i$. Since $f_i \in A$ for $i = 1, 2, \ldots, n$, the map $\eta \mathop{\triangle}\limits_{i=1}^{n} f_i$ is homotopically dependent on A.

Let us prove the following

(4.13) THEOREM. *If a fundamental sequence $\underline{f}$ is homotopically dependent on a set $A \subset \{X, Y\}$ and every fundamental sequence belonging to A is extendable over X', then $\underline{f}$ is extendable over X'.*

Proof. By the hypothesis $\underline{f} \simeq \underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f_i}$ where $\underline{f_i} \in A$ for $i = 1, 2, \ldots, n$ and $\underline{\vartheta}$: $Y^n \to Y$. Let $\underline{f_i'}$: $X' \to Y$ be an extension of $\underline{f_i}$ for $i = 1, 2, \ldots, n$. Then by (3.15) $\mathop{\triangle}\limits_{i=1}^{n} \underline{f_i'}$ is an extension of $\mathop{\triangle}\limits_{i=1}^{n} \underline{f_i}$ and hence $\underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f_i'}$ is an extension of $\underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f_i}$. Therefore the fundamental sequence $\underline{\vartheta} \mathop{\triangle}\limits_{i=1}^{n} \underline{f_i}$ is extendable over X'. Hence by Patkowska's Theorem (2.1) $\underline{f}$ is extendable over X'.

(4.14) Remark. Theorem (4.13) is a generalization of Patkowska's Theorem (2.1). Indeed, if a fundamental sequence $\underline{g}$: $X \to Y$ is homotopic to a fundamental sequence $\underline{f}$: $X \to Y$ having an extension $\underline{f}'$: $X' \to Y$, then $\underline{g}$ is homotopically dependent on the set $\{\underline{f}\}$ consisting only of the fundamental sequence $\underline{f}$ (as the required fundamental multisequence we can take a fundamental identity sequence on Y). Hence by Theorem (4.13) $\underline{g}$ is extendable over X'.

§ 5. Fundamental dimension and cohomotopy groups. By a *fundamental dimension* of a compactum X we understand the number $\mathrm{Fd}(X) = \mathrm{Min}\{\dim Y\colon \mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)\}$. This notion is due to K. Borsuk.

In order to define the n-th cohomotopy group (see [10]) of a space X one requires (in the case $n \geqslant 2$) that the dimension $\dim X$ of the space X be less than $2n-1$. In this section we show that for an arbitrary compactum X the inequality $\dim X < 2n-1$ may be replaced by the inequality $\mathrm{Fd}(X) < 2n-1$.

First we prove a few lemmas concerning maps into an ANR.

Suppose we are given three compacta $X, Y, Z \subset Q$, where $Z \in$ ANR. Consider a fundamental sequence $\underline{f}$: $X \to Y$. Take an arbitrary map φ: $Y \to Z$. Let $\underline{\varphi}$: $Y \to Z$ be the fundamental sequence generated by the map φ. Then the fundamental class $[\underline{\varphi f}]$ is generated by a map $\bar{\varphi}$: $X \to Z$ (see [4], Theorem (5.1), p. 228). Therefore to each map φ: $Y \to Z$ we assign a certain map $\bar{\varphi}$: $X \to Z$. This assignment is not unique, but the homotopy class $[\bar{\varphi}]$ of the map $\bar{\varphi}$ depends only on the homotopy class $[\varphi]$ of the map φ, i.e.,

(5.1) *If $\varphi \simeq \psi$: $Y \to Z$, then $\bar{\varphi} \simeq \bar{\psi}$: $X \to Z$.*

Proof. Since $\varphi \simeq \psi$, then by (4.1) of [4] (see p. 226) $\underline{\varphi} \simeq \underline{\psi}$. Hence $\underline{\varphi f} \simeq \underline{\psi f}$ (see (6.4) of [4], p. 232). Therefore maps $\bar{\varphi}$ and $\bar{\psi}$ are weakly homotopic (see [4], p. 224 and Theorem (4.3), p. 228), and since $Z \in$ ANR, we have $\bar{\varphi} \simeq \bar{\psi}$ (see [4] Theorem (2.1), p. 224).

By (5.1) we can assign to an arbitrary fundamental sequence $\underline{f}$: $X \to Y$ a function

$$\underline{f}^{\#}\colon [Z^Y] \to [Z^X]$$

defined by the formula $\underline{f}^{\#}([\varphi]) = [\varphi]$ for $\varphi \in Z^Y$. If follows at once from the definition that

(5.2) *If $\underline{f} \simeq \underline{g}$ then $\underline{f}^{\#} = \underline{g}^{\#}$.*

(5.3) *If a composition $\underline{gf}$ is defined, then $(\underline{gf})^{\#} = \underline{f}^{\#}\underline{g}^{\#}$.*

(5.4) *If $\underline{i}_X$: $X \to X$ is the fundamental identity sequence, then $\underline{i}_X^{\#}$ is the identity function.*

Now, let $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. Then there exist fundamental sequences $\underline{f}$: $X \to Y$ and $\underline{g}$: $Y \to X$ such that $\underline{gf} \simeq \underline{i}_X$. Hence by (5.2), (5.3) and (5.4) $\underline{f}^{\#}\underline{g}^{\#}$ is the identity function. It follows that

(5.5) *If $\underline{f}$: $X \to Y$ and $\underline{g}$: $Y \to X$ are fundamental sequences such that $\underline{gf} \simeq i_X$ and $Z \in \mathrm{ANR}$, then the function $\underline{f}^{\#}$: $[Z^Y] \to [Z^X]$ is onto and $\underline{g}^{\#}$: $[Z^X] \to [Z^Y]$ is a single-valued function.*

Now we recall the definition of the nth cohomotopy group of a space X given in [10].

Let $S = S^n$ be the n-dimensional sphere. Let us choose a point $s_0 \in S$. Consider the subset

$$S \vee S = \big(S \times (s_0)\big) \cup \big((s_0) \times S\big)$$

of the Cartesian product $S \times S$. Let us define the map Ω: $S \vee S \to S$ by the formula

$$\Omega(s, s_0) = \Omega(s_0, s) = s \quad \text{for} \quad s \in S.$$

If $\dim X < 2n-1$, then for arbitrary two maps φ, ψ: $X \to S$ there exists a map

$$\Phi\colon X \times \langle 0, 1\rangle \to S \times S$$

such that

$$\Phi(x, 0) = \big(\varphi(x), \psi(x)\big), \quad \Phi(x, 1) \in S \vee S \quad \text{for} \quad x \in X.$$

The map Φ is called a *normalizing homotopy* for the maps φ and ψ. Then the map χ: $X \to S \vee S$ defined by the formula $\chi(x) = \Phi(x, 1)$ is said to be a *normalization* of the maps φ and ψ (see [10], p. 210). Then in the set $[S^X]$ of all homotopy classes of maps of X into S the group operation may be defined by the formula

(5.6) $$[\varphi] + [\psi] = [\Omega\chi].$$

This group is called the *n-th cohomotopy group* of the space X and denoted by $\pi^n(X)$ (see [10], p. 213).

We show that if the fundamental dimension $\mathrm{Fd}(X)$ of a compactum X is less than $2n-1$, then we can define the nth cohomotopy group $\pi^n(X)$ of the compactum X in the same manner. Since $\mathrm{Fd}(X) < 2n-1$, then there exists a compactum $Y \subset Q$ such that $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ and $\dim Y < 2n-1$. Therefore there exist fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, Y, X\}$ such that $\underline{gf} \simeq \underline{i}_X = \{i, X, X\}$, where i: $Q \to Q$ is the identity map. Take two arbitrary maps φ, ψ: $X \to S$, where $S = S^n$ is the n-dimensional sphere. Since $S \in \mathrm{ANR}$, there exists a neighborhood U of X (in Q) such that there exist extensions $\tilde{\varphi}, \tilde{\psi}$: $U \to S$ of maps φ and ψ, respectively. Since $\underline{gf} \simeq \underline{i}_X$, we have

$$g_k f_k | X \simeq i | X \quad \text{in } U \text{ for almost all } k.$$

Therefore for almost all k there exist maps

$$F_k\colon X \times \langle 0, 1\rangle \to U$$

such that

$$F_k(x, 0) = x \quad \text{and} \quad F_k(x, 1) = g_k f_k(x) \quad \text{for} \quad x \in X.$$

Let us define the map

$$G_k\colon X \times \langle 0, 1\rangle \to S \times S$$

by the formula

$$G_k(x, t) = \big(\tilde{\varphi} F_k(x, t), \tilde{\psi} F_k(x, t)\big) \quad \text{for} \quad x \in X \text{ and } 0 \leqslant t \leqslant 1.$$

Then for $x \in X$

(5.7) $$\begin{cases} G_k(x, 0) = \big(\tilde{\varphi} F_k(x, 0), \tilde{\psi} F_k(x, 0)\big) = \big(\tilde{\varphi}(x), \tilde{\psi}(x)\big) = \big(\varphi(x), \psi(x)\big), \\ G_k(x, 1) = \big(\tilde{\varphi} F_k(x, 1), \tilde{\psi} F_k(x, 1)\big) = \big(\tilde{\varphi} g_k f_k(x), \tilde{\psi} g_k f_k(x)\big). \end{cases}$$

Since $\underline{g}$ is a fundamental sequence, there exists a neighborhood V' of Y such that $g_k(V') \subset U$ for almost all k. Then for almost all k we define the maps φ_k, ψ_k: $Y \to S$ by the formulae

$$\varphi_k(y) = \tilde{\varphi} g_k(y), \quad \psi_k(y) = \tilde{\psi} g_k(y) \quad \text{for} \quad y \in Y.$$

Since $\dim Y < 2n-1$, for the maps φ_k and ψ_k there exists a normalizing homotopy

$$H_k\colon Y \times \langle 0, 1\rangle \to S \times S.$$

Then

$$H_k(y, 0) = \big(\varphi_k(y), \psi_k(y)\big) = \big(\tilde{\varphi} g_k(y), \tilde{\psi} g_k(y)\big) \quad \text{and} \quad H_k(y, 1) \in S \vee S \quad \text{for} \quad y \in Y.$$

Since $S \times S$, $S \vee S \in \mathrm{ANR}$, there exists an extension

$$\tilde{H}_k\colon V \times \langle 0, 1\rangle \to S \times S$$

of H_k, where $V \subset V'$ is a neighborhood of Y, such that

(5.8) $$\begin{cases} \tilde{H}_k(y, 0) = \big(\tilde{\varphi} g_k(y), \tilde{\psi} g_k(y)\big), \\ \tilde{H}_k(y, 1) \in S \vee S \end{cases} \quad \text{for} \quad y \in V.$$

Let k_0 be a natural number so great that the maps G_{k_0} and H_{k_0} exist and $f_{k_0}(X) \subset V$. Let us define the map

$$\Phi\colon X \times \langle 0, 1\rangle \to S \times S$$

by the formula

$$\Phi(x, t) = \begin{cases} G_{k_0}(x, 2t) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ \widetilde{H}_{k_0}(f_{k_0}(x), 2t-1) & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

By (5.7) and (5.8) Φ is well defined and continuous and it is a normalizing homotopy of the maps φ and ψ. Thus we have proved that

(5.9) *If* $\mathrm{Fd}(X) < 2n-1$, *then for an arbitrary two maps* $\varphi, \psi\colon X \to S$ *there exists a normalizing homotopy.*

Now we show that the addition in the set $[S^X]$, given by the formula (5.6), is well defined, i.e.,

(5.10) *If* $\mathrm{Fd}(X) < 2n-1$ *and* $\chi\colon X \to S \vee S$ *is a normalization of maps* $\varphi, \psi\colon X \to S$, *then the homotopy class* $[\Omega\chi]$ *depends only on the homotopy classes* $[\varphi]$ *and* $[\psi]$.

Proof. Since $\mathrm{Fd}(X) < 2n-1$, then there exists a compactum $Y \subset Q$ such that $\dim Y < 2n-1$ and $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. Therefore there exist fundamental sequences $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to X$ such that $\underline{g}\underline{f} \simeq \underline{i}_X$. Hence by (5.5)

(5.11) $\underline{g}^{\#}\colon [S^X] \to \pi^n(Y)$ *is a single-valued function.*

Let $\overline{\varphi}, \overline{\psi}\colon Y \to S$ be maps such that

$$(5.12) \qquad \underline{g}^{\#}([\varphi]) = [\overline{\varphi}] \quad \text{and} \quad \underline{g}^{\#}([\psi]) = [\overline{\psi}].$$

Let $\overline{\chi}\colon Y \to S \vee S$ be a normalization of the maps $\overline{\varphi}$ and $\overline{\psi}$. Then the homotopy class $[\Omega\overline{\chi}]$ depends only on the homotopy classes $[\overline{\varphi}]$ and $[\overline{\psi}]$ (see [10], Lemma 6.2, p. 211). In order to prove (5.10) it suffices by (5.11) to show that

$$(5.13) \qquad \underline{g}^{\#}([\Omega\chi]) = [\Omega\overline{\chi}].$$

Since $\chi\colon X \to S \vee S$ is a normalization of the maps φ and ψ, then there exists a normalizing homotopy

$$\Phi\colon X \times \langle 0, 1\rangle \to S \times S$$

such that

$$\Phi(x, 0) = (\varphi(x), \psi(x)) \quad \text{and} \quad \Phi(x, 1) = \chi(x) \quad \text{for} \quad x \in X.$$

Since $S, S \vee S, S \times S \in \mathrm{ANR}$, then there exists a closed neighborhood U of X (in Q) such that the two following conditions are satisfied:

(5.14) *There exist extensions* $\widetilde{\varphi}\colon U \to S$, $\widetilde{\psi}\colon U \to S$, $\widetilde{\chi}\colon U \to S \vee S$ *of the maps* φ, ψ, χ *respectively.*

(5.15) *There exists an extension* $\widetilde{\Phi}\colon U \times \langle 0, 1\rangle \to S \times S$ *of the homotopy* Φ *such that*

$$\widetilde{\Phi}(x, 0) = (\widetilde{\varphi}(x), \widetilde{\psi}(x)) \quad \text{and} \quad \widetilde{\Phi}(x, 1) = \widetilde{\chi}(x) \quad \text{for} \quad x \in U.$$

Since $\underline{g} = \{g_k, Y, X\}$ is a fundamental sequence, there exists a closed neighborhood V of Y and a natural number k_0 such that

$$(5.16) \qquad g_k|V \simeq g_{k_0}|V \quad \text{in } U \text{ for } k \geqslant k_0.$$

It follows from (5.16) that $g_k(V) \subset U$ for $k \geqslant k_0$. Then for $k \geqslant k_0$ we can define maps

$$\widetilde{\varphi}_k, \widetilde{\psi}_k\colon V \to S \quad \text{and} \quad \widetilde{\chi}_k\colon V \to S \vee S$$

by the formulae

$$(5.17) \qquad \widetilde{\varphi}_k(y) = \widetilde{\varphi}g_k(y), \quad \widetilde{\psi}_k(y) = \widetilde{\psi}g_k(y), \quad \widetilde{\chi}_k(y) = \widetilde{\chi}g_k(y) \quad \text{for} \quad y \in V.$$

It follows by (5.16) and (5.17) that

$$(5.18) \qquad \widetilde{\varphi}_k \simeq \widetilde{\varphi}_{k_0}, \quad \widetilde{\psi}_k \simeq \widetilde{\psi}_{k_0}, \quad \widetilde{\chi}_k \simeq \widetilde{\chi}_{k_0} \quad \text{for} \quad k \geqslant k_0.$$

Let

$$(5.19) \qquad \overline{\varphi}_k = \widetilde{\varphi}_k|Y, \quad \overline{\psi}_k = \widetilde{\psi}_k|Y, \quad \overline{\chi}_k = \widetilde{\chi}_k|Y.$$

We shall prove that

$$(5.20) \qquad \underline{g}^{\#}([\varphi]) = [\overline{\varphi}_{k_0}], \quad \underline{g}^{\#}([\psi]) = [\overline{\psi}_{k_0}], \quad \underline{g}^{\#}([\Omega\chi]) = [\Omega\overline{\chi}_{k_0}].$$

Let $\hat{\varphi}\colon Q \to Q$ be a map such that $\hat{\varphi}(x) = \widetilde{\varphi}(x)$ for $x \in U$ and $\hat{\varphi}_k\colon Q \to Q$ be a map such that $\hat{\varphi}_k(y) = \widetilde{\varphi}_k(y)$ for $y \in V$ and $k \geqslant k_0$. By the definition of maps $\hat{\varphi}$ and $\hat{\varphi}_k$ and by (5.17) and (5.18) it follows that the fundamental sequence $\{\hat{\varphi}g_k, Y, S\}$ is homotopic to the constant fundamental sequence $\{\hat{\varphi}_{k_0}, Y, S\}$. The fundamental sequence $\{\hat{\varphi}_{k_0}, Y, S\}$ is generated by the map $\overline{\varphi}_{k_0}\colon Y \to S$. Therefore the fundamental sequence $\{\hat{\varphi}g_k, Y, S\}$ belongs to the fundamental class generated by $\overline{\varphi}_{k_0}$, and thus by the definition of the function $\underline{g}^{\#}$ we obtain $\underline{g}^{\#}([\varphi]) = [\overline{\varphi}_{k_0}]$. Analogously one can prove the remaining two conditions (5.20).

Now, let us define the map

$$\Psi\colon Y \times \langle 0, 1\rangle \to S \times S$$

by the formula

$$(5.21) \qquad \Psi(y, t) = \widetilde{\Phi}(g_{k_0}(y), t) \quad \text{for} \quad y \in Y \text{ and } 0 \leqslant t \leqslant 1.$$

Then from (5.21), (5.15), (5.17) and (5.19) we obtain

$$\Psi(y,0)=\tilde{\Phi}(g_{k_0}(y),0)=(\tilde{\varphi}g_{k_0}(y),\tilde{\psi}g_{k_0}(y))$$
$$=(\tilde{\varphi}_{k_0}(y),\tilde{\psi}_{k_0}(y))=(\overline{\varphi}_{k_0}(y),\overline{\psi}_{k_0}(y))\quad\text{for}\quad y\in Y,$$
$$\Psi(y,1)=\tilde{\Phi}(g_{k_0}(y),1)=\tilde{\chi}g_{k_0}(y)=\tilde{\chi}_{k_0}(y)=\overline{\chi}_{k_0}(y)\quad\text{for}\quad y\in Y.$$

It follows that $\overline{\chi}_{k_0}$ is a normalization of the maps $\overline{\varphi}_{k_0}$ and $\overline{\psi}_{k_0}$. By (5.12) and (5.20) we have

(5.22) $$[\varphi]=[\overline{\varphi}_{k_0}],\qquad[\psi]=[\overline{\psi}_{k_0}].$$

Since $\dim Y<2n-1$, the homotopy class $[\Omega\overline{\chi}]$ depends only on the homotopy classes $[\varphi]$ and $[\psi]$ and the homotopy class $[\Omega\overline{\chi}_{k_0}]$ depends only on the homotopy classes $[\overline{\varphi}_{k_0}]$ and $[\overline{\psi}_{k_0}]$ (see [10], Lemma 6.2, p. 211). Therefore by (5.22) $[\Omega\overline{\chi}]=[\Omega\overline{\chi}_{k_0}]$ and hence by (5.20) we obtain (5.13) and the proof of (5.10) is completed.

If $\mathrm{Fd}(X)<2n-1$, then by (5.9) and (5.10) we can define the addition in the set $[S^X]$ by the formula (5.6). We show that the set $[S^X]$ with this addition is an Abelian group.

Since $\mathrm{Fd}(X)<2n-1$, then there exists a compactum $Y\subset Q$ such that $\dim Y<2n-1$ and $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$. Therefore there exist fundamental sequences $\underline{f}\colon X\to Y$ and $\underline{g}\colon Y\to Y$ such that $\underline{gf}\simeq\underline{i}_X$. Hence by (5.5) the function $\underline{f}^{\#}\colon\pi^n(Y)\to[S^X]$ is onto and $\underline{g}^{\#}\colon[S^X]\to\pi^n(Y)$ is a single-valued function. Moreover, by (5.2), (5.3) and (5.4) $\underline{f}^{\#}\underline{g}^{\#}$ is the identity function on $[S^X]$. It follows by (5.6), (5.12) and (5.13) that for arbitrary two maps $\varphi,\psi\colon X\to S$

(5.23) $$\underline{g}^{\#}([\varphi]+[\psi])=\underline{g}^{\#}([\varphi])+\underline{g}^{\#}([\psi]).$$

Analogously one can prove that for arbitrary two maps $\overline{\varphi},\overline{\psi}\colon Y\to S$

(5.24) $$\underline{f}^{\#}([\overline{\varphi}]+[\overline{\psi}])=\underline{f}^{\#}([\overline{\varphi}])+\underline{f}^{\#}([\overline{\psi}]).$$

(Compare also the proof of Theorem (2.3) of [7]). Let $[\overline{\varphi}_0]$ be the zero of the group $\pi^n(Y)$ and let $[\varphi_0]=\underline{f}^{\#}([\overline{\varphi}_0])$. Then

(5.25) $[\varphi_0]+[\psi]=[\psi]$ for an arbitrary map $\psi\colon X\to S$.

Indeed, let $[\overline{\psi}]=\underline{g}^{\#}([\psi])$. Then $[\overline{\varphi}_0]+[\overline{\psi}]=[\overline{\psi}]$. Hence by (5.24)

$$\underline{f}^{\#}([\overline{\varphi}_0])+\underline{f}^{\#}[(\overline{\psi}])=\underline{f}^{\#}([\overline{\psi}]).$$

But

$$\underline{f}^{\#}([\overline{\varphi}_0])=[\varphi_0]\quad\text{and}\quad\underline{f}^{\#}([\overline{\psi}])=\underline{f}^{\#}\underline{g}^{\#}[\psi])=[\psi].$$

Hence we obtain (5.25).

It is easy to see that

(5.26) *For an arbitrary map* $\varphi\colon X\to S$ *there exists a map* $\psi\colon X\to S$ *such that* $[\varphi]+[\psi]=[\varphi_0]$.

Indeed, if $\overline{\psi}\colon Y\to S$ is a map such that $\underline{g}^{\#}([\varphi])+[\overline{\psi}]=[\overline{\varphi}_0]$, then by (5.24) an arbitrary map $\psi\in\underline{f}^{\#}([\overline{\psi}])$ satisfies the required condition.

Since $\underline{g}^{\#}\colon[S^X]\to\pi^n(Y)$ is a single-valued function and the addition in the group $\pi^n(Y)$ is associative and commutative, then by (5.23) it follows that

(5.27) *The addition in the set* $[S^X]$ *given by the formula* (5.6) *is associative and commutative.*

Therefore by (5.25), (5.26) and (5.27) the set $[S^X]$ with the addition given by the formula (5.6) is an Abelian group.

Thus we obtain the following

(5.28) THEOREM. *If* $\mathrm{Fd}(X)<2n-1$, *then the n-th cohomotopy group* $\pi^n(X)$ *of the compactum* X *exists.*

In [8] we have proved (see (3.2) of [8]) that if T is a triangulation of a polyhedron P and $P^{(k)}$ is the combinatorial k-skeleton of T and $\mathrm{Fd}(X)\leqslant k$, then for every map $f\colon X\to P$ there exists a map $g\colon X\to P$ homotopic to f and such that $g(X)\subset P^{(k)}$. Hence we obtain the following

(5.29) COROLLARY. *If* $\mathrm{Fd}(X)<n$, *then the n-th cohomotopy group* $\pi^n(X$ *of the compactum* X *is trivial.*

(5.30) THEOREM. *If* $\mathrm{Fd}(X)<2n-1$, $\mathrm{Fd}(Y)<2n-1$ *and* $\underline{f}\colon X\to Y$ *is a fundamental sequence, then the function* $\underline{f}^{\#}\colon\pi^n(Y)\to\pi^n(X)$ *is a homomorphism.*

The proof of Theorem (5.30) is precisely the same as the proof of the analogous Theorem (2.3) of [7]. In the proof of Theorem (2.3) of [7] we utilize the hypotheses $\dim X<2n-1$ and $\dim Y<2n-1$. These hypotheses were needed only for the existence of groups $\pi^n(X)$ and $\pi^n(Y)$.

From (5.2), (5.3), (5.4) and (5.30) we obtain the following corollaries.

(5.31) COROLLARY. *If* $\mathrm{Fd}(X)<2n-1$ *and* $\mathrm{Sh}(X)\geqslant\mathrm{Sh}(Y)$, *then* $\pi^n(X)\underset{r}{\geqslant}\pi^n(Y)$ *i.e. the group* $\pi^n(Y)$ *is a divisor of the group* $\pi^n(X)$.

(5.32) COROLLARY. *If* $\mathrm{Fd}(X)<2n-1$, $\mathrm{Sh}(X)\geqslant\mathrm{Sh}(Y)$ *and* $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$ *then* $\pi^n(X)\underset{r}{=}\pi^n(Y)$.

(5.33) COROLLARY. *If* $\mathrm{Fd}(X)<2n-1$ *and* $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$, *then the groups* $\pi^n(X)$ *and* $\pi^n(Y)$ *are isomorphic i.e. Cohomotopy groups are invariances of shape.*

§ 6. Relative fundamental domination and relative fundamental equivalence of sets. In [2] K. Borsuk introduced the notions of homotopic domination and homotopic equivalence of sets in a space. In this section we introduce analogous notions of relative fundamental domination and relative fundamental equivalence, replacing maps by fundamental sequences.

First we recall the notions of homotopic domination and homotopic equivalence of sets.

Let X_1 and X_2 be closed subsets of a space X and let $i_1\colon X_1\to X$ and $i_2\colon X_2\to X$ denote the inclusion maps. One says that the set X_2 *homotopically dominates* the set X_1 in X (notation $X_1 \underset{h}{\leqslant} X_2$ in X) if there exists a map $\alpha\colon X_1\to X_2$ such that $i_1\simeq i_2\alpha$ (see [2], p. 609). If $X_1 \underset{h}{\leqslant} X_2$ in X and $X_2 \underset{h}{\leqslant} X_1$ in X, then we say that sets X_1 and X_2 are *homotopically equivalent in* X and write $X_1 \underset{h}{=} X_2$ in X (see [2], p. 610).

Now, let X_1 and X_2 be closed subsets of a compactum X lying in the Hilbert cube Q. Let $\underline{i_1}\colon X_1\to X$ and $\underline{i_2}\colon X_2\to X$ be fundamental sequences generated the inclusion maps $i_1\colon X_1\to X$ and $i_2\colon X_2\to X$, respectively. We say that the set X_2 *fundamentally dominates the set* X_1 *in* X (notation $X_1 \underset{F}{\leqslant} X_2$ in X) if there exists a fundamental sequence $\underline{\alpha}\colon X_1\to X_2$ such that $\underline{i_1}\simeq \underline{i_2}\underline{\alpha}$. If $X_1 \underset{F}{\leqslant} X_2$ in X and $X_2 \underset{F}{\leqslant} X_1$ in X, then we say that the sets X_1 and X_2 are *fundamentally equivalent in* X and write $X_1 \underset{F}{=} X_2$ in X.

From the definitions it follows at once that

(6.1) *$X_1 \underset{F}{=} X_1$ in X for every closed subset X_1 of a compactum X.*

(6.2) *$X_1 \underset{F}{=} X_2$ in X if and only if $X_2 \underset{F}{=} X_1$ in X.*

(6.3) *If $X_1 \underset{F}{\leqslant} X_2$ in X and $X_2 \underset{F}{\leqslant} X_3$ in X, then $X_1 \underset{F}{\leqslant} X_3$ in X.*

(6.4) *If $X_1 \underset{F}{=} X_2$ in X and $X_2 \underset{F}{=} X_3$ in X, then $X_1 \underset{F}{=} X_3$ in X.*

It is easy to see that

(6.5) *If $X_1 \underset{h}{\leqslant} X_2$ in X, then $X_1 \underset{F}{\leqslant} X_2$ in X.*

Indeed, if $\alpha\colon X_1\to X_2$ is a map such that $i_1\simeq i_2\alpha$, then $\underline{i_1}\simeq \underline{i_2}\underline{\alpha}$, where $\underline{\alpha}$ is the fundamental sequence generated by α.

(6.6) Remark. The converse of (6.5) is not true. For instance, let X_1 be the segment lying in the plane E^2 with endpoints $(0,1)$ and $(0,-1)$. Let X_3 be a subset of the plane E^2 consisting of all points $(x,y)\in E^2$ satisfying conditions $y=\sin(1/x)$ and $0<x\leqslant 1$. Let $X=X_1\cup X_3$ and denote by X_2 the subset of X_3 consisting of all points $(x,y)\in X_3$ satisfying the inequality $\frac{1}{2}\leqslant x\leqslant 1$. It is easy to see that $X_1 \underset{F}{\leqslant} X_2$ in X, but the relation $X_1 \underset{h}{\leqslant} X_2$ in X does not hold.

It follows by (6.5) that

(6.7) *If $X_1 \underset{h}{=} X_2$ in X, then $X_1 \underset{F}{=} X_2$ in X.*

Now we show that

(6.8) *If $X_1 \underset{F}{\leqslant} X_2$ in X and $X, X_2\in \mathrm{ANR}$, then $X_1 \underset{h}{\leqslant} X_2$ in X.*

Proof. By hypothesis there exists a fundamental sequence $\underline{\alpha}\colon X_1\to X_2$ such that $\underline{i_1}\simeq \underline{i_2}\underline{\alpha}$. Since $X_2\in\mathrm{ANR}$, then the fundamental sequence $\underline{\alpha}$ is homotopic to a fundamental sequence generated by a map $\alpha\colon X_1\to X_2$ (see [4], Theorem (5.1), p. 228). Then the fundamental sequence $\underline{i_2}\underline{\alpha}$ is homotopic to a fundamental sequence generated by the map $i_2\alpha$. Thereore, sinfce $X\in\mathrm{ANR}$, by (2.1) and (4.3) of [4] we obtain $i_1\simeq i_2\alpha$. Thus $X_1 \underset{h}{\leqslant} X_2$ in X.

(6.9) Remark. Any of the hypotheses $X\in\mathrm{ANR}$ and $X_2\in\mathrm{ANR}$ of (6.8) cannot be omitted. For instance, if we define compacta X, X_1 and X_2 as in Remark (6.6), then $X_1 \underset{F}{\leqslant} X_2$ in X, but the relation $X_1 \underset{h}{\leqslant} X_2$ in X does not hold. Moreover, let us observe that $X_2\in\mathrm{ANR}$. Now, let X_2 be the set denoted by Y in Remark (2.6). Let X be the subset of the plane E consisting of all points (x,y) satisfying the inequality $\frac{1}{100}\leqslant (x-\frac{1}{2})^2+y^2\leqslant 9$ and denote by X_1 the circle given in the plane E^2 by the equation $(x-\frac{1}{2})^2+y^2=\frac{1}{100}$. It is easy to see that $X_1 \underset{F}{\leqslant} X_2$ in X, but the relation $X_1 \underset{h}{\leqslant} X_2$ in X does not hold. Moreover, let us observe that $X\in\mathrm{ANR}$.

It follows from (6.8) that

(6.10) *If $X_1 \underset{F}{=} X_2$ in X and $X, X_1, X_2\in\mathrm{ANR}$ then $X_1 \underset{h}{=} X_2$ in X.*

It is easy to see that

(6.11) *If X is a closed subset of a compactum X' and $X_1 \underset{F}{\leqslant} X_2$ in X, then $X_1 \underset{F}{\leqslant} X_2$ in X'.*

Indeed, if $\underline{i_1}\colon X_1\to X$, $\underline{i_2}\colon X_2\to X$, $\underline{i}\colon X\to X'$, $\underline{i_1'}\colon X_1\to X'$, $\underline{i_2'}\colon X_2\to X'$ are fundamental sequences generated by the inclusion maps and $\underline{\alpha}\colon X_1\to X_2$ is a fundamental sequence such that $\underline{i_1}\simeq\underline{i_2}\underline{\alpha}$, then $\underline{i}\underline{i_1}\simeq\underline{i}\underline{i_2}\underline{\alpha}$, but $\underline{i_1'}\simeq\underline{i}\underline{i_1}$ and $\underline{i_2'}\simeq\underline{i}\underline{i_2}$, whence $\underline{i_1'}\simeq\underline{i_2'}\underline{\alpha}$.

(6.12) Remark. There exist compacta X_1, X_2, X, X' such that $X_1, X_2\subset X\subset X'$ and $X_1 \underset{F}{\leqslant} X_2$ in X' but the relation $X_1 \underset{F}{\leqslant} X_2$ in X does not hold. For instance, let X' be a segment with end points x and y (where $x \neq y$), X consists of the points x and y, X_1 consists of the point x, and X_2 consists of the point y.

It is evident that $X_0 \underset{h}{\leqslant} X$ in X for every closed subset X_0 of X (see [2] p. 610). Hence by (6.5)

(6.13) *$X_0 \underset{F}{\leqslant} X$ in X for every closed subset X_0 of a compactum X.*

Let us observe that

(6.14) *If X_0 is a closed subset of a compactum X such that $X \underset{F}{\leqslant} X_0$ in X, then* $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(X_0)$.

Indeed, by hypothesis there exists a fundamental sequence $\underline{\alpha}\colon X \to X_0$ such that $\underline{i_X} \simeq \underline{i_0}\underline{\alpha}$, where $\underline{i_0}\colon X_0 \to X$ is a fundamental sequence generated by the inclusion map $i_0\colon X_0 \to X$.

(6.15) Remark. The converse of (6.14) is not true. For instance, suppose we are given the following two subsets of the straight line E^1: $X_0 = \{0\} \cup \bigcup_{n=1}^{\infty} \{1/n\}$ and $X = X_0 \cup \{2\}$, where $\{x\}$ denotes the set consisting of a point x. It is easy to see that $\mathrm{Sh}(X) = \mathrm{Sh}(X_0)$ (because X is homeomorphic to X_0) and the relation $X \underset{F}{\leqslant} X_0$ in X does not hold.

Let $X_1 \underset{F}{\leqslant} X_2$ in X. Then there exists a fundamental sequence $\underline{\alpha}\colon X_1 \to X$ such that $\underline{i_1} \simeq \underline{i_2}\underline{\alpha}$. Consider the function $\underline{\alpha}^*\colon [X_2, Y] \to [X_1, Y]$ defined by the formula

$$\underline{\alpha}^*([\underline{\varphi}]) = [\underline{\varphi}][\underline{\alpha}]\,.$$

(Compare the definition of $[\underline{f}]^*$ given by (4.7). For definitions of sets $[X, Y]$, $[X \subset X', Y]$ see § 2).

Let us prove that

(6.16) $$\underline{\alpha}^*([X_2 \subset X, Y]) \subset [X_1 \subset X, Y]\,.$$

Proof. Take an arbitrary fundamental sequence $\underline{\varphi} = \{\varphi_k, X_2, Y\}$ having an extension $\underline{\varphi}' = \{\varphi_k', X, Y\}$. It suffices to show that the fundamental sequence $\underline{\varphi}\underline{\alpha}\colon X_1 \to Y$ has an extension over X. Since $\underline{\varphi}'$ is an extension of $\underline{\varphi}$, then $\varphi_k'(x) = \varphi_k(x)$ for $x \in X_2$. Hence $\varphi_k' i_2(x) = \varphi_k(x)$ for $x \in X_2$. Therefore by (1.1) of [6] we have $\underline{\varphi}'\underline{i_2} \simeq \underline{\varphi}$. Hence $\underline{\varphi}'\underline{i_2}\underline{\alpha} \simeq \underline{\varphi}\underline{\alpha}$ and since $\underline{i_2}\underline{\alpha} \simeq \underline{i_1}$, then $\underline{\varphi}'\underline{i_1} \simeq \underline{\varphi}\underline{\alpha}$. The fundamental sequence $\underline{\varphi}'\colon X \to Y$ is an extension of $\underline{\varphi}'\underline{i_1}\colon X_1 \to Y$. Therefore by Patkowska's Theorem (2.1) the fundamental sequence $\underline{\varphi}\underline{\alpha}$ has an extension over X and the proof is finished.

Keeping the hypotheses and the notations as above, let us suppose that $Y \in \mathrm{ANR}$ and consider the function

$$\underline{\alpha}^{\#}\colon [Y^{X_2}] \to [Y^{X_1}]\,.$$

(For the definition of $\underline{f}^{\#}$ see § 5). Let us prove that

(6.17) $$\underline{\alpha}^{\#}([Y^{X_2 \subset X}]) \subset [Y^{X_1 \subset X}]\,.$$

Proof. Take an arbitrary map $\varphi\colon X_2 \to Y$. Let $\underline{\varphi}$ be a fundamental sequence generated by φ. Since $Y \in \mathrm{ANR}$, the fundamental class $[\underline{\varphi}\underline{\alpha}]$ is generated by a map $\bar{\varphi}\colon X_1 \to Y$ (see [4], Theorem (5.1), p. 228). By definition $\underline{\alpha}^{\#}([\varphi]) = [\bar{\varphi}]$. It suffices to show that if the map $\varphi\colon X_2 \to Y$ is extendable over X, then the map $\bar{\varphi}\colon X_1 \to Y$ has an extension over X. Since φ is extandable over X, by (2.3) the fundamental sequence $\underline{\varphi}$ has an extension over X, i.e., $[\underline{\varphi}] \in [X_2 \subset X, Y]$. By (6.16)

$$\underline{\alpha}^*([\underline{\varphi}]) \in [X_1 \subset X, Y]\,, \quad \text{i.e.,} \quad [\underline{\varphi}\underline{\alpha}] \in [X_1 \subset X, Y]\,;$$

therefore the fundamental sequence $\underline{\varphi}\underline{\alpha}\colon X_1 \to Y$ has an extension over X. Hence by (2.5) the map $\bar{\varphi}$ has an extension over X and the proof is finished.

Now, let $X_1 \underset{F}{=} X_2$ in X. Then there exist fundamental sequences $\underline{\alpha_1}\colon X_1 \to X_2$ and $\underline{\alpha_2}\colon X_2 \to X_1$ such that $\underline{i_1} \simeq \underline{i_2}\underline{\alpha_1}$ and $\underline{i_2} \simeq \underline{i_1}\underline{\alpha_2}$. Consider the functions

$$\underline{\alpha_1}^*\colon [X_2, Y] \to [X_1, Y] \quad \text{and} \quad \underline{\alpha_2}^*\colon [X_1, Y] \to [X_2, Y]\,.$$

By (6.16)

$$\underline{\alpha_1}^*([X_2 \subset X, Y]) \subset [X_1 \subset X, Y] \quad \text{and} \quad \underline{\alpha_2}^*([X_1 \subset X, Y]) \subset [X_2 \subset X, Y]$$

Therefore we can define the functions

$$\tilde{\underline{\alpha}}_1\colon [X_2 \subset X, Y] \to [X_1 \subset X, Y], \quad \tilde{\underline{\alpha}}_2\colon [X_1 \subset X, Y] \to [X_2 \subset X, Y].$$

by the formulae

$$\tilde{\underline{\alpha}}_1([\underline{\varphi}]) = \underline{\alpha_1}^*([\underline{\varphi}]) \quad \text{for} \quad [\underline{\varphi}] \in [X_2 \subset X, Y]\,,$$

$$\tilde{\underline{\alpha}}_2([\underline{\psi}]) = \underline{\alpha_2}^*([\underline{\psi}]) \quad \text{for} \quad [\underline{\psi}] \in [X_1 \subset X, Y]\,.$$

Let us prove the following

(6.18) Theorem. *The compositions*

$$\tilde{\underline{\alpha}}_2\tilde{\underline{\alpha}}_1\colon [X_2 \subset X, Y] \to [X_2 \subset X, Y]$$

and

$$\tilde{\underline{\alpha}}_1\tilde{\underline{\alpha}}_2\colon [X_1 \subset X, Y] \to [X_1 \subset X, Y]$$

are identities.

Proof. Take an arbitrary fundamental sequence $\underline{\varphi}\colon X_2\to Y$ extendable over X. In order to prove that $\underline{\tilde{\alpha}_2}\underline{\tilde{\alpha}_1}$ is an identity it suffices to show that $\underline{\varphi}\underline{\alpha_1}\underline{\alpha_2}\simeq\underline{\varphi}$. Let $\underline{\varphi}'\colon X\to Y$ be an extension of $\underline{\varphi}$ over X. Then by (1.1) of [6] $\underline{\varphi}'\underline{i_2}\simeq\underline{\varphi}$. Applying the hypotheses $\underline{i_1}\simeq\underline{i_2}\underline{\alpha_1}$ and $\underline{i_2}\simeq\underline{i_1}\underline{\alpha_2}$, we obtain

$$\underline{\varphi}\simeq\underline{\varphi}'\underline{i_2}\simeq\underline{\varphi}'\underline{i_1}\underline{\alpha_2}\simeq\underline{\varphi}'\underline{i_2}\underline{\alpha_1}\underline{\alpha_2}\simeq\underline{\varphi}\underline{\alpha_1}\underline{\alpha_2}\,.$$

Therefore $\underline{\tilde{\alpha}_2}\underline{\tilde{\alpha}_1}$ is an identity. Analogously one can prove that $\underline{\tilde{\alpha}_1}\underline{\tilde{\alpha}_2}$ is an identity.

By (6.18) we obtain the following

(6.19) COROLLARY. *The functions*

$$\underline{\tilde{\alpha}_1}\colon [X_2\subset X,\,Y]\to[X_1\subset X,\,Y] \quad \textit{and} \quad \underline{\tilde{\alpha}_2}\colon [X_1\subset X,\,Y]\to[X_2\subset X,\,Y]$$

are one-to-one and $\underline{\tilde{\alpha}_2}$ *is the inverse of* $\underline{\tilde{\alpha}_1}$.

Keeping the notations as above and the hypothesis $X_1 \underset{F}{=} X_2$ in X let us suppose that $Y\in \mathrm{ANR}$ and consider the functions

$$\underline{\alpha_1}^{\#}\colon [Y^{X_2}]\to[Y^{X_1}] \quad\text{and}\quad \underline{\alpha_2}^{\#}\colon [Y^{X_1}]\to[Y^{X_2}]\,.$$

By (6.17)

$$\underline{\alpha_1}^{\#}([Y^{X_2\subset X}])\subset[Y^{X_1\subset X}] \quad\text{and}\quad \underline{\alpha_2}^{\#}([Y^{X_1\subset X}])\subset[Y^{X_2\subset X}].$$

Therefore we can define the functions

$$\underline{\hat{\alpha}_1}\colon [Y^{X_2\subset X}]\to[Y^{X_1\subset X}]\,,\qquad \underline{\hat{\alpha}_2}\colon [Y^{X_1\subset X}]\to[Y^{X_2\subset X}]$$

by the formulae

$$\underline{\hat{\alpha}_1}([\varphi])=\underline{\alpha_1}^{\#}([\varphi]) \quad\text{for}\quad \varphi\in Y^{X_2\subset X},$$

$$\underline{\hat{\alpha}_2}([\psi])=\underline{\alpha_2}^{\#}([\psi]) \quad\text{for}\quad \psi\in Y^{X_1\subset X}.$$

Let us prove the following

(6.20) THEOREM. *The compositions*

$$\underline{\hat{\alpha}_2}\underline{\hat{\alpha}_1}\colon [Y^{X_2\subset X}]\to[Y^{X_2\subset X}] \quad\textit{and}\quad \underline{\hat{\alpha}_1}\underline{\hat{\alpha}_2}\colon [Y^{X_1\subset X}]\to[Y^{X_1\subset X}]$$

are identities.

Proof. Take an arbitrary map $\varphi\colon X_2\to Y$ extendable over X. Let $\underline{\varphi}\colon X_2\to Y$ be a fundamental sequence generated by φ. Since $Y\in\mathrm{ANR}$, then the fundamental class $[\underline{\varphi}\underline{\alpha_1}\underline{\alpha_2}]$ is generated by a map $\bar{\varphi}$. By definition $\underline{\hat{\alpha}_2}\underline{\hat{\alpha}_1}([\varphi])=[\bar{\varphi}]$. In order to prove that $\underline{\hat{\alpha}_2}\underline{\hat{\alpha}_1}$ is an identity, it suffices to show that $\varphi\simeq\bar{\varphi}$. Since the map $\varphi\colon X_2\to Y$ has an extension over X, by (2.3) the fundamental sequence $\underline{\varphi}\colon X_2\to Y$ is extendable over X. Hence by (6.18)

$$\underline{\tilde{\alpha}_2}\underline{\tilde{\alpha}_1}([\underline{\varphi}])=[\underline{\varphi}]\,,\quad\text{i.e.,}\quad [\underline{\varphi}\underline{\alpha_1}\underline{\alpha_2}]=[\underline{\varphi}]\,.$$

Therefore the maps φ and $\bar{\varphi}$ generate the same fundamental class, and thus by (4.3) and (2.1) of [4] $\varphi\simeq\bar{\varphi}$. Analogously one can prove that $\underline{\hat{\alpha}_1}\underline{\hat{\alpha}_2}$ is an identity function.

By (6.20) we obtain the following

(6.21) COROLLARY. *The functions*

$$\underline{\hat{\alpha}_1}\colon [Y^{X_2\subset X}]\to[Y^{X_1\subset X}] \quad\textit{and}\quad \underline{\hat{\alpha}_2}\colon [Y^{X_1\subset X}]\to[Y^{X_2\subset X}]$$

are one-to-one and $\underline{\hat{\alpha}_2}$ *is the inverse of* $\underline{\hat{\alpha}_1}$.

§ 7. Fundamental skeletons.

In [2] K. Borsuk introduced the notion of a homotopic k-skeleton of a space. In this section we introduce the analogous notion of fundamental k-skeleton of a compactum. First we recall the definition of a homotopic k-skeleton. A closed subset X_1 of a space X is said to be a *homotopic k-skeleton* of X, provided $\dim X_1\leqslant k$ and for each closed subset X_2 of X with $\dim X_2\leqslant k$ we have $X_2\underset{h}{\leqslant}X_1$ in X.

Now, let X be a compactum lying in the Hilbert cube Q and let X_1 be a closed subset of X. By a *relative fundamental dimension of X_1 in X* we understand the number

$$\mathrm{Fd}_{\mathrm{rel}\,X}(X_1)=\mathrm{Min}\{\dim X_2\colon X_1\underset{F}{\leqslant}X_2 \text{ in } X\}\,.$$

It is obvious that

(7.1) $$\mathrm{Fd}_{\mathrm{rel}\,X}(X_1)\leqslant\dim X_1\,.$$

It follows by (6.11) that

(7.2) *If X is a closed subset of a compactum X' then* $\mathrm{Fd}_{\mathrm{rel}\,X}(X_1)\geqslant\mathrm{Fd}_{\mathrm{rel}\,X'}(X_1)$.

(7.3) EXAMPLE. Let $X_1=X$ be a circle given in the plane E^2 by the equation $x^2+y^2=1$ and let X' be a disc in E^2 given by the inequality $x^2+y^2\leqslant 1$. Then $\mathrm{Fd}_{\mathrm{rel}\,X}(X_1)=1$ and $\mathrm{Fd}_{\mathrm{rel}\,X'}(X_1)=0$.

It follows by (6.14) that

(7.4) $$\mathrm{Fd}_{\mathrm{rel}\,X}(X)\geqslant\mathrm{Fd}(X)\,.$$

A closed subset X_1 of a compactum X is said to be a *fundamental k-skeleton* of X provided that $\mathrm{Fd}_{\mathrm{rel}\,X}(X_1)\leqslant k$ and for each closed subset X_2 of X with $\mathrm{Fd}_{\mathrm{rel}\,X}(X_2)\leqslant k$ we have $X_2\underset{F}{\leqslant}X_1$ in X. By a *strong fundamental k-skeleton* of X we understand a fundamental k-skeleton X_1 of X such that $\mathrm{Fd}(X_1)\leqslant k$.

It is evident that

(7.5) *If X_1 and X_2 are two fundamental k-skeletons of X, then $X_1\underset{F}{=}X_2$ in X.*

Let us observe that

(7.6) *Every homotopic k-skeleton of a compactum X is a strong fundamental k-skeleton of X.*

Indeed, if X_1 is a homotopic k-skeleton of X, then $\dim X_1 \leqslant k$ and hence by (7.1) $\mathrm{Fd}_{\mathrm{rel}X}(X_1) \leqslant k$. Take an arbitrary closed subset X_2 of X such that $\mathrm{Fd}_{\mathrm{rel}X}(X_2) \leqslant k$. Then there exists a closed subset X_3 of X such that $X_2 \underset{F}{\leqslant} X_3$ in X and $\dim X_3 \leqslant k$. Since X_1 is a homotopic k-skeleton of X, then $X_3 \underset{h}{\leqslant} X_1$ in X. Hence by (6.5) $X_3 \underset{F}{\leqslant} X_1$ in X. Therefore by (6.3) $X_2 \underset{F}{\leqslant} X_1$ in X. Thus X_1 is a fundamental k-skeleton of X. By the definition of the fundamental dimension $\mathrm{Fd}(X_1) \leqslant k$. Thus X_1 is a strong fundamental k-skeleton of X.

It follows by (6.13) and (7.4) that

(7.7) *If $\mathrm{Fd}_{\mathrm{rel}X}(X) \leqslant k$ then X is a strong fundamental k-skeleton of X.*

A space X is said to have the property (Δ) (see [1], p. 163) provided for every point $x \in X$ and for every neighborhood U of x (in X) there exists a neighborhood V of x such that every compact subset A of V is contractible to a point in a subset B of U with $\dim B \leqslant \dim A + 1$.

In [2] K. Borsuk has proved (see p. 612) that for every ANR-space X with property (Δ) and for every $k = 0, 1, 2, \ldots$ there exists a homotopic k-skeleton of X. Therefore by (7.6) for every ANR-space X with property (Δ) there exists a strong fundamental k-skeleton of X for every $k = 0, 1, 2, \ldots$

Let us prove that

(7.8) *If X_0 is a closed subset of a compactum X with $\mathrm{Fd}_{\mathrm{rel}X}(X_0) \leqslant k$ and Y_0 is a fundamental k-skeleton of an ANR-set Y with property (Δ), then every fundamental sequence $\underline{f}\colon X \to Y$ is homotopic to a fundamental sequence $\underline{f'}\colon X \to Y$ which is an extension of a fundamental sequence $\underline{j_0}\underline{f_0}\colon X_0 \to Y$, where $\underline{f_0}\colon X_0 \to Y_0$ is a fundamental sequence and $\underline{j_0}\colon Y_0 \to Y$ is the fundamental sequence generated by the inclusion map $j_0\colon Y_0 \to Y$.*

Proof. Since $Y \in \mathrm{ANR}$, the fundamental sequence $\underline{f}\colon X \to Y$ is homotopic to a fundamental sequence $\underline{g}\colon X \to Y$ generated by a map $g\colon X \to Y$. Since $\mathrm{Fd}_{\mathrm{rel}X}(X_0) \leqslant k$, there exists a closed subset X_1 of X such that $\dim X_1 \leqslant k$ and $X_0 \underset{F}{\leqslant} X_1$ in X. Therefore there exists a fundamental sequence $\underline{\alpha}\colon X_0 \to X_1$ such that $\underline{i_0} \simeq \underline{i_1}\underline{\alpha}$, where $\underline{i_0}$ and $\underline{i_1}$ are fundamental sequences generated by the inclusion maps $i_0\colon X_0 \to X$ and $i_1\colon X_1 \to X$, respectively. Since ANR-set Y has the property (Δ), by Theorems (8.1) and (2.1) of [1] (pp. 164 and 94) there exists a map $g'\colon X \to Y$ homotopic to g and such that $\dim g'(X_1) \leqslant k$. Let $Y_1 = g'(X_1)$. Then $\dim Y_1 \leqslant k$ and hence by (7.1) $\mathrm{Fd}_{\mathrm{rel}Y}(Y_1) \leqslant k$. Since Y_0 is a fundamental k-skeleton of Y we have $Y_1 \underset{F}{\leqslant} Y_0$ in Y. Therefore there exists a fundamental sequence $\underline{\beta}\colon Y_1 \to Y_0$ such that $\underline{j_1} \simeq \underline{j_0}\underline{\beta}$, where $\underline{j_0}$ and $\underline{j_1}$ are fundamental sequences generated by the inclusion maps $j_0\colon Y_0 \to Y$ and $j_1\colon Y_1 \to Y$, respectively. Let us define the map $g''\colon X_1 \to Y_1$ by the formula $g''(x) = g'(x)$ for $x \in X_1$. Let $\underline{g''}$ be a fundamental sequence generated by the map g''. Let us set $\underline{f_0} = \underline{\beta}\underline{g''}\underline{\alpha}\colon X_0 \to Y_0$. It remains to show that the fundamental sequence $\underline{j_0}\underline{f_0}\colon X_0 \to Y$ has an extension $\underline{f'}\colon X \to Y$ homotopic to $\underline{f}$. Since $\underline{i_0} \simeq \underline{i_1}\underline{\alpha}$, we have $\underline{g}\underline{i_0} \simeq \underline{g}\underline{i_1}\underline{\alpha}\colon X_0 \to Y$. The fundamental sequence $\underline{g}\colon X \to Y$ is an extension of the fundamental sequence $\underline{g}\underline{i_0}\colon X_0 \to Y$. Hence by Patkowska's Theorem (2.1) the fundamental sequence $\underline{g}\underline{i_1}\underline{\alpha}\colon X_0 \to Y$ has an extension $\underline{\tilde{g}}\colon X \to Y$ homotopic to $\underline{g}$. It follows from the definition of g'' that $g' i_1 = j_1 g''\colon X_1 \to Y$. Since $g \simeq g'$, we have $g i_1 \simeq j_1 g''\colon X_1 \to Y$. Hence $\underline{g}\underline{i_1} \simeq \underline{j_1}\underline{g''}$, and therefore $\underline{g}\underline{i_1}\underline{\alpha} \simeq \underline{j_1}\underline{g''}\underline{\alpha}\colon X_0 \to Y$. Since $\underline{j_1} \simeq \underline{j_0}\underline{\beta}$, we have $\underline{j_1}\underline{g''}\underline{\alpha} \simeq \underline{j_0}\underline{\beta}\underline{g''}\underline{\alpha} = \underline{j_0}\underline{f_0}$. Thus we have $\underline{g}\underline{i_1}\underline{\alpha} \simeq \underline{j_0}\underline{f_0}$. Therefore by Patkowska's Theorem (2.1) the fundamental sequence $\underline{j_0}\underline{f_0}$ has an extension $\underline{f'}$ homotopic to $\underline{g}$. Since $\underline{g} \simeq \underline{f}$, we have $\underline{f'} \simeq \underline{f}$ and the proof is finished.

§ 8. Groups $\underline{\pi}^n_k(X)$. In [3] K. Borsuk, using the notion of the homotopic k-skeleton, introduced the notion of the generalized cohomotopy group $\pi^n_k(X)$. In this section, replacing homotopic k-skeletons by strong fundamental k-skeletons, we define analogously groups $\underline{\pi}^n_k(X)$ as generalizations of groups $\pi^n_k(X)$. First we recall the definition of generalized cohomotopy groups $\pi^n_k(X)$ of a space X.

Let $S = S^n$ be the n-dimensional sphere and let X_0 be a closed subset of a space X with $\dim X_0 < 2n-1$. Then $[S^{X_0 \subset X}] \subset [S^{X_0}]$ (see § 2). Introducing in the set $[S^{X_0}]$ the group operation by the formula (5.6), we obtain the nth cohomotopy group $\pi^n(X_0)$ of X_0. The set $[S^{X_0 \subset X}] \subset \pi^n(X_0)$ generates the subgroup $\pi^n(X_0 \subset X)$ of the group $\pi^n(X_0)$ (see [3], p. 617). K. Borsuk has proved that if X_1 and X_2 are homotopic k-skeletons of X, where $k < 2n-1$, then the groups $\pi^n(X_1 \subset X)$ and $\pi^n(X_2 \subset X)$ are isomorphic (see [3], p. 617). The abstract group isomorphic to all groups $\pi^n(X_1 \subset X)$ where X_1 is a homotopic k-skeleton of X (where $k < 2n-1$), is denoted by $\pi^n_k(X)$ (see [3], p. 619).

Now, let X_0 be a closed subset of a compactum X with $\mathrm{Fd}(X_0) < 2n-1$. Then by Theorem (5.28) there exists the n-th cohomotopy group $\pi^n(X_0)$ of X_0. Therefore we can define the group $\pi^n(X_0 \subset X)$ as above.

Let us prove a theorem analogous to Borsuk's Theorem mentioned above ([3], p. 617).

(8.1) THEOREM. *Let X_1 and X_2 be two strong fundamental k-skeletons of X, where $k < 2n-1$. Then the group $\pi^n(X_1 \subset X)$ is isomorphic to the group $\pi^n(X_2 \subset X)$.*

In [3] (p. 617) K. Borsuk has proved the following algebraic

(8.2) LEMMA. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Abelian groups and $\Phi\colon \mathfrak{A}\to\mathfrak{B}$ and $\Psi\colon \mathfrak{B}\to\mathfrak{A}$ be homomorphisms. Let A and B be two subsets of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, such that*

$$\Psi\Phi(x) = x \quad \text{for} \quad x \in A\,, \qquad \Phi\Psi(y) = y \quad \text{for} \quad y \in B\,.$$

Let Φ_A denote the partial homomorphism of Φ considered on the subgroup (A) of $\mathfrak{A}$ generated by the set A and let Ψ_B denote the partial homomorphism of Ψ considered on the subgroup (B) of $\mathfrak{B}$ generated by the set B. Then Φ_A is an isomorphism of (A) onto (B) and Ψ_B is the isomorphism inverse to Φ_A.

Proof of Theorem (8.1). By (7.5) $X_1 \underset{F}{=} X_2$ in X. Then there exist fundamental sequences $\underline{\alpha}_1\colon X_1\to X_2$ and $\underline{\alpha}_2\colon X_2\to X_1$ such that $\underline{i}_1 \simeq \underline{i}_2\underline{\alpha}_1$ and $\underline{i}_2 \simeq \underline{i}_1\underline{\alpha}_2$. By (5.30) the fundamental sequences $\underline{\alpha}_1$ and $\underline{\alpha}_2$ induce the homomorphisms

$$\underline{\alpha}_1^{\#}\colon \pi^n(X_2)\to\pi^n(X_1)\,, \quad \text{and} \quad \underline{\alpha}_2^{\#}\colon \pi^n(X_1)\to\pi^n(X_2)\,,$$

respectively. By (6.20)

$$\underline{\alpha}_2^{\#}\underline{\alpha}_1^{\#}([\varphi]) = [\varphi] \quad \text{for} \quad [\varphi] \in [S^{X_2\subset X}]\,,$$

$$\underline{\alpha}_1^{\#}\underline{\alpha}_2^{\#}([\psi]) = [\psi] \quad \text{for} \quad [\psi] \in [S^{X_1\subset X}]\,.$$

Hence by Lemma (8.2) the homomorphism $\underline{\alpha}_1^{\#}$ maps isomorphically the group $\pi^n(X_2\subset X)$ onto the group $\pi^n(X_1\subset X)$. Thus, the proof of Theorem is finished.

It follows from Theorem (8.1) that the algebraic structure of the group $\pi^n(X_1\subset X)$ does not depend on the choice of the strong fundamental k-skeleton X_1 of X. The abstract group, isomorphic with all groups $\pi^n(X_1\subset X)$, where X_1 is a strong fundamental k-skeleton of X (where $k < 2n-1$), we denote by $\underline{\pi}_k^n(X)$.

It follows by (7.6) that

(8.3) *If a compactum X has a homotopic k-skeleton (where $k < 2n-1$), then the group $\underline{\pi}_k^n(X)$ is isomorphic to the group $\pi_k^n(X)$.*

It follows by (5.28) and (7.4) that if $\mathrm{Fd}_{\mathrm{rel}\,X}(X) < 2n-1$, then there exists the nth cohomotopy group $\pi^n(X)$ of X. Hence by (7.7) we obtain the following

(8.4) COROLLARY. *If $k = \mathrm{Fd}_{\mathrm{rel}\,X}(X) < 2n-1$, then the group $\underline{\pi}_k^n(X)$ is isomorphic to the n-th cohomotopy group $\pi^n(X)$ of X.*

§ 9. Problems. In § 7 we have shown that $\mathrm{Fd}_{\mathrm{rel}X}(X) \geqslant \mathrm{Fd}(X)$ (see (7.4)). The author does not known any example of a compactum X satisfying the condition $\mathrm{Fd}_{\mathrm{rel}X}(X) > \mathrm{Fd}(X)$.

(9.1) PROBLEM. *Is true that $\mathrm{Fd}_{\mathrm{rel}X}(X) = \mathrm{Fd}(X)$ for every compactum X?*

(9.2) PROBLEM. *Let $\mathrm{Fd}(X) \leqslant k$. Does there exists a (strong) fundamental k-skeleton of X?*

(9.3) PROBLEM. *Let $k = \mathrm{Fd}(X) < 2n-1$ and suppose that the compactum X has a strong fundamental k-skeleton. Is it true that the group $\underline{\pi}_k^n(X)$ is isomorphic to the n-th cohomotopy group $\pi^n(X)$ of X?*

If the answer to problem (9.1) is positive, then by (7.7) the answer to problem (9.2) is also positive.

A positive answer to problem (9.1) would give also, by (8.4), a positive answer to problem (9.3).

(9.4) PROBLEM. *Let $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. Is it true that the existence of a (strong) fundamental k-skeleton of X implies the existence of a (strong) fundamental k-skeleton of Y?*

(9.5) PROBLEM. *Let $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and let compacta X and Y both have strong fundamental k-skeletons. Is it true that the group $\underline{\pi}_k^n(X)$ (for $k < 2n-1$) is isomorphic to the group $\underline{\pi}_k^n(Y)$?*

In [2] K. Borsuk has proved (see p. 612) that for an arbitrary ANR-set X satisfying the condition (Δ) and for every $k = 0, 1, 2, \ldots$ there exist a homotopic k-skeleton of X.

(9.6) PROBLEM. *Is it true that for an arbitrary compactum X satisfying the condition (Δ) and for every $k = 0, 1, 2, \ldots$ there exists a (strong) fundamental k-skeleton of X?*

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