Concerning the fixed point property for λ-dendroids

by

J. J. Charatonik (Notre Dame, Ind.)

A metric continuum is said to be a dendroid if it is hereditarily unicoherent and arcwise connected. It follows that it is hereditarily decomposable (see [1], (47), p. 239). A hereditarily unicoherent and hereditarily decomposable continuum is called a λ-dendroid. It is proved in [2], Corollary 2, p. 29 that for every λ-dendroid X there exists a unique decomposition $\mathfrak{D}$ of X (called the canonical decomposition):

(1) $$X = \bigcup \{S_d \mid d \in \Delta(X)\}$$

such that

(i) $\mathfrak{D}$ is upper semicontinuous,

(ii) the elements S_d of $\mathfrak{D}$ are continua,

(iii) the hyperspace $\Delta(X)$ of $\mathfrak{D}$ is a dendroid,

(iv) $\mathfrak{D}$ is the finest possible decomposition among all decompositions satisfying (i), (ii), and (iii).

The elements S_d of $\mathfrak{D}$ are called strata of X. The monotone mapping φ of X onto $\Delta(X)$ defined by

(2) $$\varphi^{-1}(d) = S_d \quad \text{for} \quad d \in \Delta(X)$$

is called canonical.

If A is a space, we denote by 2^A the set of all non-empty closed subsets of A, and $\mathcal{K}(A)$ the family of connected members of 2^A. Thus $\mathcal{K}(\Delta(X))$ is the family of all subcontinua of the dendroid $\Delta(X)$.

Let X be λ-dendroid and let

$$f\colon X \to X$$

be a continuous mapping of X into itself. Put

(3) $$F(d) = \varphi\big(f(\varphi^{-1}(d))\big) \quad \text{for} \quad d \in \Delta(X).$$

Since φ is monotone, $\varphi^{-1}(d) = S_d$ is a continuum, hence $F(d)$ is a continuum and, by definition, it is a subset of $\Delta(X)$. Thus (3) defines a continuum-valued mapping

$$F\colon \Delta(X) \to \mathcal{K}(\Delta(X)).$$

LEMMA 1. *The mapping F is upper semicontinuous.*

Proof. Let

(4) $$\lim_{n\to\infty} d_n = d\,, \quad \text{where} \quad d_n \in \Delta(X)\,.$$

Thus

$$\operatorname*{Ls}_{n\to\infty} \varphi^{-1}(d_n) \subset \varphi^{-1}(d)$$

by the upper semicontinuity of the decomposition $\mathfrak{D}$. So

(5) $$f\big(\operatorname*{Ls}_{n\to\infty} \varphi^{-1}(d_n)\big) \subset f\big(\varphi^{-1}(d)\big)\,.$$

Since

(6) $$f(\operatorname*{Ls}_{n\to\infty} A_n) = \operatorname*{Ls}_{n\to\infty} f(A_n)\,,$$

where A_n are subsets of a compact space and f is a continuous mapping (see [6], Lemma 8.4, p. 23), we have from (5)

$$\operatorname*{Ls}_{n\to\infty} f\big(\varphi^{-1}(d_n)\big) \subset f\big(\varphi^{-1}(d)\big)\,,$$

whence

$$\varphi\Big(\operatorname*{Ls}_{n\to\infty} f\big(\varphi^{-1}(d_n)\big)\Big) \subset \varphi\Big(f\big(\varphi^{-1}(d)\big)\Big)\,.$$

It implies, by (6) again, that

$$\operatorname*{Ls}_{n\to\infty} \varphi\Big(f\big(\varphi^{-1}(d_n)\big)\Big) \subset \varphi\Big(f\big(\varphi^{-1}(d)\big)\Big)\,,$$

i.e.

$$\operatorname*{Ls}_{n\to\infty} F(d_n) \subset F(d)$$

by (3). It shows that the mapping F is upper semicontinuous.

THEOREM 1. *For every continuous mapping f of a λ-dendroid X into itself there exists a stratum of X which intersects its image.*

Proof. Since the continuum-valued mapping F, which was defined by (3), is upper semicontinuous by Lemma 1, hence there exists by Theorem 1 in [7], p. 162 a point d_0 in $\Delta(X)$ such that $d_0 \in F(d_0)$, i.e.

$$d_0 \in \varphi\Big(f\big(\varphi^{-1}(d_0)\big)\Big)$$

by (3). It implies the existence of a point $p \in f\big(\varphi^{-1}(d_0)\big) \subset X$ such that $\varphi(p) = d_0$, i.e. $p \in \varphi^{-1}(d_0)$. Therefore p is a common point for continua $S_{d_0} = \varphi^{-1}(d_0)$ and $f(S_{d_0})$, thus $S_{d_0} \cap f(S_{d_0}) \neq \emptyset$, which finishes the proof.

If X is an arcwise connected λ-dendroid, i.e. if it is a dendroid simply, then all its strata are points (see [2], (2.25), p. 22). In this case the existence of a stratum intersecting its image means the existence of a fixed point. Therefore we have the fixed point theorem for dendroids as a corollary from Theorem 1.

LEMMA 2. *If A_n are subcontinua of a continuum and if*

(7) $$A_n \cap A_{n+1} \neq \emptyset \quad \textit{for} \quad n = 1, 2, \ldots,$$

then $\operatorname*{Ls}_{n\to\infty} A_n$ is a continuum.

Proof. Obviously $\operatorname*{Ls}_{n\to\infty} A_n$ is a compactum. We should prove the connectedness of it only. Let p and q be two distinct points of $\operatorname*{Ls}_{n\to\infty} A_n$. We prove that $I(p,q)$, a continuum irreducible from p to q, is contained in $\operatorname*{Ls}_{n\to\infty} A_n$.

Since $p \in \operatorname*{Ls}_{n\to\infty} A_n$, there are a subsequence of continua A_{n_k} and a sequence of points p_k such that $p_k \in A_{n_k}$ and

(8) $$\lim_{k\to\infty} p_k = p\,.$$

Similarly, q being in $\operatorname*{Ls}_{n\to\infty} A_n$, one can find a subsequence of continua A_{m_k} and a sequence of points q_k such that $q_k \in A_{m_k}$,

(9) $$\lim_{k\to\infty} q_k = q$$

and, moreover, $n_k < m_k$ for $k = 1, 2, \ldots$

Put

(10) $$L_k = A_{n_k} \cup A_{n_k+1} \cup \ldots \cup A_{m_k}\,.$$

Using (7) we see that L_k are continua with

(11) $$p_k \in L_k \quad \text{and} \quad q_k \in L_k\,.$$

Let $\{L_{k_r}\}$ be a convergent subsequence of the sequence $\{L_k\}$, and put

$$L = \operatorname*{Lim}_{r\to\infty} L_{k_r}\,.$$

Thus L is a continuum and $p, q \in L$ by (8), (9), and (11). Hence L contains an irreducible continuum $I(p,q)$:

(12) $$I(p,q) \subset L\,.$$

To state the lemma it is sufficient to prove that

(13) $$L \subset \operatorname*{Ls}_{n\to\infty} A_n\,.$$

So, if x is a point in L, then there exist points $x_r \in L_{k_r}$ with

(14) $$\lim_{r\to\infty} x_r = x\,.$$

Since

$$L_{k_r} = A_{n_{k_r}} \cup A_{n_{k_r}+1} \cup \ldots \cup A_{m_{k_r}}$$

according to (10), hence there is an index j such that $n_{k_r} \leqslant j \leqslant m_{k_r}$ and $x_r \in A_j$. Therefore (14) implies that $x \in \operatorname{Ls}_{n\to\infty} A_n$ which gives (13), whence $I(p,q) \subset \operatorname{Ls}_{n\to\infty} A_n$ by (12). Thus $\operatorname{Ls}_{n\to\infty} A_n$ is connected.

The following lemma is proved in [4], p. 933.

LEMMA 3. *A continuum X has fixed point property with respect to a class $\mathcal{F}$ of continuous mappings of X into itself if and only if for every mapping $f \in \mathcal{F}$ there exists in X a transfinite sequence of subcontinua K_α $(\alpha < \Omega)$ such that*

(15) $$\beta < \alpha \quad \text{implies} \quad K_\alpha \subset K_\beta\,,$$

(16) $$\text{if } \beta < \alpha \text{ and } K_\beta \text{ is not a point, then } K_\alpha \neq K_\beta\,,$$

(17) $$f(K_\alpha) \subset K_\alpha \quad \text{for every} \quad \alpha < \Omega\,.$$

Here we shall assume X to be a λ-dendroid and $\mathcal{F}$ to be the class of all continuous mappings of X into itself. Let $f \in \mathcal{F}$. Denote by $\mathcal{S}(X,f)$ the family of strata S of X which intersect their images under f:

(18) $$S \in \mathcal{S}(X,f) \quad \text{if and only if} \quad S \cap f(S) \neq \emptyset\,.$$

Further, let $\mathcal{S}_1(X,f)$ be the subfamily of $\mathcal{S}(X,f)$ consisting of those elements S of $\mathcal{S}(X,f)$ for which

$$\operatorname{Ls}_{n\to\infty} f^n(S) \neq X\,,$$

(where $f^0(S) = S$ and $f^n(S)$ is the image of S under f^n, the nth iteration of f), and let $\mathcal{S}_2(X,f)$ denote the subfamily of $\mathcal{S}(X,f)$ consisting of those elements S of $\mathcal{S}(X,f)$ for which

$$\overline{\bigcup_{n=0}^{\infty} f^n(S)} \neq X\,.$$

Since $\operatorname{Ls}_{n\to\infty} f^n(S) \subset \overline{\bigcup_{n=0}^{\infty} f^n(S)}$, hence

(19) $$\mathcal{S}_2(X,f) \subset \mathcal{S}_1(X,f) \subset \mathcal{S}(X,f)\,.$$

We see from Theorem 1 and from (18) that $\mathcal{S}(X,f)$ is not empty. But $\mathcal{S}_1(X,f)$ as well as $\mathcal{S}_2(X,f)$ can be empty. As an example it is enough to take an arbitrary monostratiform λ-dendroid X (i.e. such that it consists of only one stratum—see [3]) and as f a continuous surjective (i.e. onto) mapping. An open question is whether there exist a non-monostratiform (in particular, a hereditarily stratified, i.e. containing no monostratiform non-trivial subcontinuum) λ-dendroid X and a continuous surjective mapping f defined on it such that $\mathcal{S}_2(X,f)$ or $\mathcal{S}_1(X,f)$ even is empty.

Let $\mathcal{F}_i$ ($i = 1$ or 2) be the class of continuous mappings of a λ-dendroid X into itself such that for every non-trivial (i.e. different from a point) subcontinuum K of X

(20) $$\text{if } f(K) \subset K, \text{ then } \mathcal{S}_i(K, f|K) \neq \emptyset\,.$$

We see that if $f \in \mathcal{F}_2$ and if a non-trivial subcontinuum K of X contains its image $f(K)$, then $\mathcal{S}_2(K, f|K) \neq \emptyset$ by definition hence $\mathcal{S}_1(K, f|K) \neq \emptyset$ by (19), thus $f \in \mathcal{F}_1$. Therefore

(21) $$\mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F}\,.$$

THEOREM 2. *Every λ-dendroid X has fixed point property with respect to the class $\mathcal{F}_1$ of continuous mappings.*

Proof. According to Lemma 3 we should define a transfinite sequence of continua K_α for which (15)–(17) hold. Admit $K_0 = X$ and assume we have defined continua K_β satisfying (15)–(17) for all $\beta < \alpha$.

Consider two cases. Firstly let $\alpha = \beta + 1$. Define

(22) $$K_\alpha = \operatorname{Ls}_{n\to\infty} f^n(S)\,,$$

where

(23) $$S \in \mathcal{S}_1(K_\beta, f|K_\beta)\,.$$

It implies in particular that $S \in \mathcal{S}(K_\beta, f|K_\beta)$ by (19), i.e. S is a stratum of K_β such that

(24) $$S \cap f(S) \neq \emptyset\,.$$

Since $f^n\big(S \cap f(S)\big) \subset f^n(S) \cap f^{n+1}(S)$, hence we have

$$f^n(S) \cap f^{n+1}(S) \neq \emptyset$$

by (24). Substituting $f^n(S)$ for A_n in Lemma 2 we see from (22) that K_α is a continuum.

By the inductive hypothesis (17) we have $f(K_\beta) \subset K_\beta$, hence

(25) $$f^n(K_\beta) \subset K_\beta \quad \text{for} \quad n = 1, 2, \ldots$$

Since $S \subset K_\beta$ by definition, hence $f^n(S) \subset f^n(K_\beta)$, thus $f^n(S) \subset K_\beta$ by (25), and $\operatorname{Ls}_{n\to\infty} f^n(S) \subset K_\beta$, K_β being a compact set. So we see that (15) holds by (22). Further, (23) implies that

$$\operatorname{Ls}_{n\to\infty} f^n(S) \neq K_\beta$$

by the definition of $\mathcal{S}_1(K_\alpha, f|K_\alpha)$, which gives (16) according to (22). To show (17) consider $f(K_\alpha)$:

(26) $$f(K_\alpha) = f\big(\operatorname{Ls}_{n\to\infty} f^n(S)\big) = \operatorname{Ls}_{n\to\infty} f^{n+1}(S)$$

by equality (6) proved in [6], Lemma 8.4, p. 23. Therefore (26) and (22) imply $f(K_\alpha) = K_\alpha$ and (17) holds true.

Secondly let $\alpha = \lim_{\beta<\alpha} \beta$. Define

(27) $$K_\alpha = \bigcap_{\beta<\alpha} K_\beta .$$

So (15) and (16) are satisfied by the above definition. To see (17) observe that

(28) $$f(K_\alpha) = f(\bigcap_{\beta<\alpha} K_\beta) \subset \bigcap_{\beta<\alpha} f(K_\beta) .$$

Since $f(K_\beta) \subset K_\beta$ by the inductive hypothesis (17), hence

$$\bigcap_{\beta<\alpha} f(K_\beta) \subset \bigcap_{\beta<\alpha} K_\beta$$

and (17) follows by (27) and (28). This completes the proof.

Theorem 2 and the first inclusion in (21) imply

COROLLARY 1. *Every λ-dendroid X has fixed point property with respect to the class $\mathcal{F}_2$ of continuous mappings.*

Describe now some relations between the class $\mathcal{F}_2$ (or $\mathcal{F}_1$) and some other classes of continuous mappings, introduced in [5].

Let X and Y be λ-dendroids and let φ and ψ be their canonical mappings onto dendroids $\Delta(X)$ and $\Delta(Y)$ respectively. A continuous mapping f of X into Y belongs to the class $\mathcal{C}$ provided that it takes every stratum of X into a stratum of Y. In other words, a continuous mapping

$$f\colon X \to Y$$

belongs to $\mathcal{C}$ if and only if for every point $d \in \Delta(X)$ there exists a point $d' \in \Delta(Y)$ such that

$$f\big(\varphi^{-1}(d)\big) \subset \psi^{-1}(d')$$

(see [5], p. 337, where a necessary and sufficient condition is given for f to be in $\mathcal{C}$).

A subclass $\mathcal{C}_h$ of $\mathcal{C}$ is defined as follows: a continuous mapping f of X onto Y is in $\mathcal{C}_h$ provided that f belongs to $\mathcal{C}$ hereditarily. It means that for every subcontinuum K of X the partial mapping $f|K$, which maps K onto $f(K)$, is in $\mathcal{C}$ (see [5], p. 341).

We shall consider now continuous mappings f of X into X i.e. we shall assume that $Y = f(X) \subset X$.

THEOREM 3. *If a λ-dendroid X is hereditarily stratified and if a continuous mapping f of X onto $Y \subset X$ is in $\mathcal{C}_h$, then $f \in \mathcal{F}_2$.*

Proof. Recall that every subcontinuum of a λ-dendroid is a λ-dendroid itself. Assume K is a non-trivial subcontinuum of X with $f(K) \subset K$. According to (20) it ought to be proved that

(29) $$\mathcal{S}_2(K, f|K) \neq \emptyset .$$

Since $f \in \mathcal{C}_h$ by hypothesis, hence $f|K \in \mathcal{C}$. Thus there exists a stratum S of K which contains its image:

$$f(S) \subset S$$

(see [5], Lemma, p. 340). It implies

$$f^{n+1}(S) \subset f^n(S) \quad \text{for} \quad n = 0, 1, 2, \ldots$$

whence

(30) $$\overline{\bigcup_{n=0}^{\infty} f^n(S)} = S .$$

X being hereditarily stratified, we have $S \neq K$, thus

$$\overline{\bigcup_{n=0}^{\infty} f^n(S)} \neq K$$

by (30). Therefore $S \in \mathcal{S}_2(K, f|K)$ according to the definition, and (29) follows.

Denote by $\mathcal{H}$ the class of all homeomorphisms and by $\mathcal{M}$ the class of all continuous monotone mappings of a λ-dendroid X into itself. Obviously $\mathcal{H} \subset \mathcal{M}$ and we have $\mathcal{M} \subset \mathcal{C}_h$ by Property 8 in [5], p. 341. So Theorem 3 and (21) imply

COROLLARY 2. *If a λ-dendroid X is hereditarily stratified, then*

$$\mathcal{H} \subset \mathcal{M} \subset \mathcal{C}_h \subset \mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F} .$$

So we see that Theorem 2 is a generalization of fixed point theorems proved in [4], p. 934 for $\mathcal{M}$ and in [5], p. 343 for $\mathcal{C}_h$. These papers, especially [4], contain also a larger list of references concerning the fixed point property.

It is known (see [5], Proposition 9) that if the λ-dendroid X is arcwise connected (i.e. if X is a dendroid) then class $\mathcal{C}_h$ contains $\mathcal{F}$, and therefore classes $\mathcal{C}_h$, $\mathcal{F}_2$, $\mathcal{F}_1$ and $\mathcal{F}$ coincide. However, it is not so without the arcwise connectedness of X. For hereditarily stratified λ-dendroids the projection (parallel to the x-axis) of $\sin 1/x$ — curve onto its limit segment is a mapping in $\mathcal{C}_h \setminus \mathcal{M}$. The example of a mapping h from an irreducible continuum X into itself described in [4] after Corollary 5 shows that $\mathcal{F}_2 \setminus \mathcal{C}_h \neq \emptyset$. But I do not know whether classes $\mathcal{F}_2$, $\mathcal{F}_1$ and $\mathcal{F}$ are different for hereditarily stratified λ-dendroids.

References

[1] J. J. Charatonik, *On ramification points in the classical sense*, Fund. Math. 51 (1962), pp. 229–252.

[2] — *On decompositions of λ-dendroids*, Fund. Math. 67 (1970), pp. 15–30.

[3] J. J. Charatonik, *An example of a monostratiform λ-dendroid*, Fund. Math. 67 (1970), pp. 75–87.
[4] — *Fixed point property for monotone mappings of hereditarily stratified λ-dendroids*, Bull. Acad. Polon. Sci., Ser. sci. math., astronom. et phys. 16 (1968), pp. 931–936.
[5] — *Remarks on some class of continuous mappings of λ-dendroids*, Fund. Math. 67 (1970), pp. 337–344.
[6] R. Duda, *On biconnected sets with dispersion points*, Rozprawy Matematyczne 37, Warszawa 1964.
[7] L. E. Ward, Jr., *Characterization of the fixed point property for a class of set-valued mappings*, Fund. Math. 50 (1961), pp. 159–164.

INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY
UNIVERSITY OF NOTRE DAME

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Homotopy dependence of fundamental sequences, relative fundamental equivalence of sets and a generalization of cohomotopy groups

by

Stanisław Godlewski (Warszawa)

In order to extend some standard notions of the homotopy theory onto arbitrary compacta K. Borsuk introduced in [4] the notion of the *fundamental sequence*. Replacing maps by fundamental sequences one can obtain generalizations of many standard notions. In such a manner we obtain the notions of homotopy dependence of fundamental sequences (§ 4), relative fundamental domination and relative fundamental equivalence of sets (§ 6) and fundamental skeletons (§ 7). All these notions are generalizations of the notions introduced by K. Borsuk in [1] and [2]. Using the notion of the fundamental skeleton, we define groups $\underline{\pi}_k^n(X)$ which are generalizations of the generalized cohomotopy groups $\pi_k^n(X)$ introduced by K. Borsuk in [3].

§ 1. Basic notions. In [4], [5], and [6] K. Borsuk introduced the basic notions of theory of shape. We recall some of the basic definitions. All spaces considered in this paper are compact and metric, and thus we can assume that they lie in the Hilbert cube Q.

By a *fundamental sequence from X to Y* (notation $\underline{f} = \{f_k, X, Y\}$ or $\underline{f}\colon X \to Y$) we understand an ordered triple consisting of the compacta $X, Y \subset Q$ and of a sequence of maps $f_k\colon Q \to Q$, $k = 1, 2, \ldots$, such that for every neighborhood V of Y there exists a neighborhood U of X such that $f_k|U \simeq f_{k+1}|U$ in V for almost all k.

We say that the fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, X, Y\}$ are *homotopic* (notation $\underline{f} \simeq \underline{g}$) if for every neighborhood V of Y there exists a neighborhood U of X such that $f_k|U \simeq g_k|U$ in V for almost all k. This relation is reflexive, symmetric and transitive and it decomposes all fundamental sequences into *fundamental classes*. The fundamental class with representative $\underline{f}$ is denoted by $[\underline{f}]$ or, precisely, by $[\underline{f}]\colon X \to Y$.

If $f\colon X \to Y$ is a map, then there exists a map $\hat{f}\colon Q \to Q$ such that $\hat{f}(x) = f(x)$ for $x \in X$. Setting $f_k = \hat{f}$ for $k = 1, 2, \ldots$ we get a fundamental