On some combinatorial problems involving large cardinals*

by

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§ 1. Introduction. Notation. We are going to use the usual notations of set theory. Cardinals are identified with initial ordinals. $\alpha, \beta, \gamma, \delta$ denote cardinals $\xi, \zeta, \eta, \varphi, \nu, \sigma, \tau, \varrho$ denote ordinals. $|A|$ is the cardinality of the set A. α^+ is the smallest, cardinal greater than α. ω_ξ is the sequence of infinite cardinals, $\omega_0 = \omega$. i, j, r, s, l, k denote non-negativ integers. We put

$$[A]^\alpha = \{X \subset \alpha\colon\ |X| = \alpha\}, \quad [A]^{<\alpha} = \{X \subset \alpha\colon\ |X| < \alpha\}.$$

For the convenience of the reader we recall here the definition of some of the partition symbols defined in earlier papers of P. Erdös, R. Rado and of the author.

Definition 1. The ordinary partition symbol. (See [3], 3.1.). $\alpha \to (\beta_\nu)^\delta_{\nu<\gamma}$ denotes that the following statement is true: Whenever $[\alpha]^\delta = \bigcup_{\nu<\gamma} I_\nu$ then there are $A \subset \alpha$, $\nu < \gamma$ such that $|A| = \beta_\nu$, $[A]^\delta \subset I_\nu$.

Here and for all other symbols to be defined $\not\to$ denotes the negations of the corresponding statements respectively.

$$\alpha \to (\beta)^\delta_\gamma \text{ denotes } \alpha \to (\beta_\nu)^\delta_{\nu<\gamma} \text{ if } \beta_\nu = \beta \text{ for } \nu < \gamma.$$

We are going to use for the symbols some other more or less self explanatory abbreviations defined in detail in [3].

Definition 2. The Ramsey symbol. (See [3], 3.2). $\alpha \to (\beta)^{<\omega}_\gamma$ means that the following statement is true. Whenever $[\alpha]^r = \bigcup_{\nu<\gamma} I^r_\gamma$ for every $r < \omega$ then there are $A \subset \alpha$ and $f \in {}^\omega\gamma$ such that $|A| = \beta$ and $[A]^r \subset I^r_{f(r)}$ for every $r < \omega$.

* §§ 1–4 of this paper contain the detailed proofs of the results stated by the author in his talk given at the Symposium on Models of Axiomatic Theories held in Warszawa, August 25th–September 2nd, 1968.

DEFINITION 3. The polarized partition symbol. (See [3], 3.3.). Let $r = r_0 + \ldots + r_{s-1}$.

$$\begin{pmatrix}\alpha_0\\ \ldots\\ \alpha_{s-1}\end{pmatrix} \to \begin{pmatrix}\beta_{0,\nu}\\ \ldots\\ \beta_{s-1,\nu}\end{pmatrix}^{r_0,\ldots,r_{s-1}}_{\nu<\gamma}$$

means that the following statement is true: Whenever $[\alpha_0]^{r_0} \times \ldots \times [\alpha_{s-1}]^{r_{s-1}} = \bigcup_{\nu<\gamma} I_\nu$ then there exist sets $A_i \subset \alpha_i$ for $i < s$ and $\nu < \gamma$ such that $[A_0]^{r_0} \times \ldots \times [A_{s-1}]^{r_{s-1}} \subset I_\nu$ and $|A_i| = \beta_{i,\nu}$ for $i < s$.

In [3] we have investigated in detail the first two symbols, and the special case $r_0 = r_1 = 1$, $s = 2$, $\gamma = 2$ of the third symbol. The polarized partition symbol has never been investigated in full generality. We mention that a surprising number of set theoretical problems are equivalent to different special cases of this symbol. As a matter of fact even the ordinary partition symbol $\alpha \to (\beta_\nu)^\delta_{\nu<\gamma}$ can be written in the form $\begin{pmatrix}1\\ \alpha\end{pmatrix} \to \begin{pmatrix}1\\ \beta_\nu\end{pmatrix}^{1,\delta}_{\nu<\gamma}$.

As a common generalization of the symbol defined in Definition 2 and of a special case of the symbol defined in Definition 3 one can define the following new symbol:

DEFINITION 4. $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}^{1,<\omega}_{\gamma}$ denotes that the following statement is true: Assume that for every $r < \omega$, $\alpha_0 \times [\alpha_1]^r = \bigcup_{\nu<\gamma} I^r_\nu$. Then there are $A_0 \subset \alpha_0$, $A_1 \subset \alpha_1$, $f \in {}^\omega\gamma$ such that $|A_0| = \beta_0$, $|A_1| = \beta_1$ and $A_0 \times [A_1]^r \subset I^r_{f(r)}$ for $r < \omega$.

The definition of this symbol was suggested by F. Galvin.

We started to discuss with Paul Erdös the special case $r_0 = 1$, $r_1 = 2$, $s = 3$ of the polarized partition symbol more than ten years ago. We obtained the following results:

$$\begin{pmatrix}\omega_1\\ \omega\end{pmatrix} \to \begin{pmatrix}1 & \omega\\ 3, & \omega\end{pmatrix}^{1,2} \quad\text{and}\quad \begin{pmatrix}\omega\\ \omega_1\end{pmatrix} \to \begin{pmatrix}1 & \omega\\ 3, & \omega\end{pmatrix}^{1,2}.$$

We could not replace the 3's by 4 that time, and in our paper [5] we stated (using a different notation) $\begin{pmatrix}\omega_1\\ \omega\end{pmatrix} \to \begin{pmatrix}1 & \omega\\ 4, & \omega\end{pmatrix}^{1,2}$; $\begin{pmatrix}\omega\\ \omega_1\end{pmatrix} \to \begin{pmatrix}1 & \omega\\ 4, & \omega\end{pmatrix}^{1,2}$ as open problems. See problems 60 and 59 of [5] respectively. Recently we collected our results on the special case $r_0 = 1$, $r_1 = 2$, $s = 3$ of the polarized partition symbol and proved a number of new results, which will be published in our forthcoming paper [4]. During our work on this paper we have proved that if $\alpha \geqslant \omega$ is 0,1-measurable then $\begin{pmatrix}\alpha^+\\ \alpha\end{pmatrix} \to \begin{pmatrix}1 & \alpha\\ \alpha, & \alpha\end{pmatrix}^{1,2}$

holds. This certainly furnishes a solution of the first problem mentioned above. After having communicated this result to F. Galvin he proved the following theorem:

$$\begin{pmatrix}\omega_1\\ \omega\end{pmatrix} \to \begin{pmatrix}\omega\\ \omega\end{pmatrix}^{1,r}_{\gamma} \quad \text{for every } \gamma < \omega, \quad \text{and for every } r.$$

He mentioned that his proof breaks down for 0,1-measurable cardinals greater than ω, but he conjectured that $\begin{pmatrix}\alpha^+\\ \alpha\end{pmatrix} \to \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}^{1,<\omega}_{\gamma}$ will hold for 0,1-measurable cardinals $\alpha > \omega$ and for $\gamma < \alpha$.

This is clearly a generalization of a theorem of P. Erdös and the author proved in [2] which states that $\alpha \to (\alpha)^{<\omega}_{\gamma}$ provided $\alpha > \omega$ is 0,1-measurable and $\beta < \alpha$. In § 2 I will give a proof of Galvin's conjecture. I think that just like the old theorem $\alpha \to (\alpha)^{<\omega}_{\gamma}$ it is bound to have a number of model theoretic applications. Using similar arguments in § 3 I am going to prove the following theorem:

If $\alpha \geqslant \omega$ *is* 0,1-*measurable then* $\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix} \to \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \vee \begin{pmatrix}1 & \alpha\\ \alpha, & \alpha\end{pmatrix}^{1,2}$ *holds for every* $\beta < \alpha$.

This obviously gives a solution of the second problem mentioned above. In § 4 I am going to give some remarks rather than theorems. I do not even know fore sure if the remarks are new or known. In § 5 I am going to prove a theorem I have obtained recently.

§ 2. Proof of Galvin's conjecture.

THEOREM 1. *Assume* $\alpha > \omega$, α *is* 0,1-*measurable*, $\gamma < \alpha$. *Then*

$$\begin{pmatrix}\alpha^+\\ \alpha\end{pmatrix} \to \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}^{1,<\omega}_{\gamma}.$$

Note that the statement of Theorem 1 is obviously weaker than the statement that α is 0,1-measurable. I do not know if it is really stronger than $\alpha \to (\alpha)^{<\omega}_{\gamma}$.

THEOREM 2. *Assume* $\alpha \geqslant \omega$, α *is* 0,1-*measurable. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha\end{pmatrix} \to \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}^{1,r}_{\gamma}$$

holds for every r.

For $\alpha = \omega$ Theorem 2 is due to F. Galvin. For $\alpha > \omega$ it is a corollary of Theorem 1. Note that with an obvious modification the proof of Theorem 1 would yield a proof of Theorem 2 for $\alpha = \omega$ as well. We will only give the proof of Theorem 1.

The proof is a simple combination of the proof of $\alpha \to (\alpha)^{<\omega}_{\gamma}$ given in [2] and of the proof of the ramification lemma in [3].

Proof of Theorem 1. For every $\xi<\alpha^+$ and for every r let $[\alpha]^r = \bigcup_{\nu<\gamma} I_\nu^{\xi,r}$ be a disjoint r-partition of type γ of α. By Definition 4 it is sufficient to prove the following statement:

(1) There exist sets $A\subset\alpha$, $B\subset\alpha^+$, $|A|=|B|=\alpha$ and an $f\in{}^\omega\gamma$ such that

$$[A]^r \subset \bigcap_{\xi\in B} I^{\xi,r}_{f(r)} \quad \text{for every } r<\omega .$$

Let μ be a non-trivial 0,1-valued α-complete measure on all subsets of α. For every $\xi<\alpha^+$ and for every r we define a sequence $[\alpha]^{r-i} = \bigcup_{\nu<\gamma} I_\nu^{\xi,r,i}$ of disjoint $r-i$-partitions of α for $i\leqslant r$ by induction on i as follows.

(2) $I_\nu^{\xi,r,0}=I_\nu^{\xi,r}$ for $\nu<\gamma$. Assume $0\leqslant i<r$ and the disjoint $r-i$-partition $[\alpha]^{r-i}=\bigcup_{\nu<\gamma} I_\nu^{\xi,r,i}$ has already been defined.

Let $X\in[\alpha]^{r-i-1}$. Put $X\in I_\nu^{\xi,r,i+1}$ for the uniquely determined $\nu<\gamma$ for which $\mu(\{y\in\alpha-X\colon X\cup\{y\}\in I_\nu^{\xi,r,i}\})=1$.

(3) For every $\xi<\alpha^+$, let f_ξ be the uniquely determined function $f_\xi\in{}^\omega\gamma$ for which $f_\xi(r)=\nu$ iff $0\in I_\nu^{\xi,r,r}$ for $r<\omega$.

Considering that α is measurable $\gamma^\omega<\alpha^+$, hence there are α^+ ξ's for which $f_\xi=f$ for an $f\in{}^\omega\gamma$. Hence we may assume in the proof that $f_\xi=f$ for every $\xi<\alpha^+$.

Put briefly

(4) $I^{\xi,r,i}_{f(r)}=I_{\xi,r,i}$ for every $\xi<\alpha^+$, $0\leqslant i\leqslant r$,
$H(X,\xi,r,i)=\{y\in\alpha-X\colon X\cup\{y\}\in I_{\xi,r,i-1}$ for $X\in[\alpha]^{r-i}\}$, $0<i\leqslant r$.

By (2) and (3) we have

(5) $X\in I_{\xi,r,i}$ iff $\mu(H(X,\xi,r,i))=1$ for $X\in[\alpha]^{r-i}$, $0<i\leqslant r$ for every $\xi<\alpha^+$ and for every r, and $0\in I_{\xi,r,r}$ for every $\xi<\alpha^+$.

To prove (1) we will prove

(6) There exist sets $A\subset\alpha$, $B\subset\alpha^+$ such that $|A|=|B|=\alpha$ and $[A]^r\subset\bigcap_{\xi\in B} I_{\xi,r,0}$ for every $r<\omega$.

Now we prove the following lemma:

(7) Let $A'\subset\alpha$, $B'\subset\alpha^+$, $S\subset\alpha^+$, $|A'|<\alpha$, $|B'|<\alpha$.

(⁎) Assume that for every $X\in[A']^s$ for every $r\geqslant s$ and for every $\xi\in S\cup B'$ $r-i=s$ implies that $X\in I_{\xi,r,i}$. Then there exist a subset $T\subset\alpha-A'$, ($|T|\leqslant\alpha$) and $b_\varrho\in S$, $S_\varrho\subset S$ for $\varrho\in T$ satisfying the following conditions:

(i) $b_\varrho\in S_\varrho-B'$ for $\varrho\in T$.

(ii) $S-B'=\bigcup_{\varrho\in T} S_\varrho$.

(iii) If we put $A'_\varrho=A'\cup\{\varrho\}$, $B'_\varrho=B'\cup\{b_\varrho\}$ then for every $\varrho\in T$ condition (⁎) is satisfied replacing A', B', S by A'_ϱ, B'_ϱ, S_ϱ respectively.

To prove (7) for every $\xi\in S-B'$ put

$$U_\xi=\bigcap_{s<\omega}\ \bigcap_{0<i<\omega}\ \bigcap_{X\in[A']^s}\ \bigcap_{\zeta\in B'\cup\{\xi\}} H(X,\zeta,s+i,i)\,.$$

Considering that $\alpha>\omega$, $|A'|<\alpha$, $|B'|<\alpha$ and that μ is α-complete it follows from (5) that $\mu(U_\xi)=1$ for every $\xi\in S-B'$. Let $T=\{\varrho\in\alpha-A'\colon \bigvee\xi(\xi\in S-B'\wedge\varrho\in U_\xi)\}$. Put $S_\varrho=\{\xi\in S-B'\colon\ \varrho\in U_\xi\}$ for $\varrho\in T$. Then $S_\varrho\subset S-B'$, $T\subset\alpha-A'$, $S_\varrho\neq 0$ for $\varrho\in T$. Let b_ϱ be an arbitrary element of S_ϱ. Considering $\mu(U_\xi)=1$, $U_\xi\cap(\alpha-A')\neq 0$ hence there is a $\varrho\in T$ such that $\xi\in S_\varrho$ i.e. $S-B'=\bigcup_{\varrho\in T}S_\varrho$. Thus conditions (i) and (ii) are satisfied.

Considering that $S_\varrho\cup B_\varrho\subset S\cup B'$ to prove that (iii) holds it is sufficient to prove that $X\in[A']^s$, $\xi\in S_\varrho\cup B_\varrho$ imply that $X\cup\{\varrho\}\in I_{\xi,r,i}$ for every $\varrho\in T$, $r\geqslant s+1$, $r-i=s+1$.

Then by the definition of U_ξ and considering that $S_\varrho\neq 0$, $\varrho\in H(X,\xi,s+i+1,i+1)$ for every $\xi\in S_\varrho\cup B'$, $i<\omega$, hence by (4) $X\cup\{\varrho\}\in I_{\xi,r,i}$. This proves (7).

Using the fact that by (5) $A'=0$, $B'=0$, $S=\alpha^+$ satisfy (⁎), in the set α^+ we can build up inductively a ramification system of length α as defined in Lemma 1 [3].

Let $\mathbf{F}$ be the set of all triples (A',B',S) satisfying condition (⁎) of (7). Put $(A'_0,B'_0,S_0)\prec(A'_1,B'_1,S_1)$ iff $A'_0\subsetneq A'_1$, $B'_0\subsetneq B'_1$, $S_1\subset S_0$, $B'_1\subset S_0$. Let $\tau_\varrho=(A'_\varrho,B'_\varrho,S_\varrho)$ for $\varrho<\varphi$ for some φ such that $\tau_\varrho\prec\tau_\sigma$ for $\varrho<\sigma<\varphi$. Put $A'=\bigcup_{\varrho<\varphi}A'_\varrho$, $B'=\bigcup_{\varrho<\varphi}B'_\varrho$, $S=\bigcap_{\varrho<\varphi}S_\varrho$. Then obviously $(A',B',S)\in\mathbf{F}$. Using the facts that $(0,0,\alpha^+)\in\mathbf{F}$, (7) holds, and that by the measurability of α $\alpha^{\mathfrak{S}}<\alpha^+$ Lemma 1 [3] yields us that there exists a sequence $\{\tau_\varrho\}_{\varrho<\alpha}=(A'_\varrho,B'_\varrho,S_\varrho)$, $\tau_\varrho\in\mathbf{F}$, $\tau_\varrho\prec\tau_\sigma$ for $\varrho<\sigma<\alpha$. Then $A=\bigcup_{\varrho<\alpha}A'_\varrho$, $B=\bigcup_{\varrho<\alpha}B'_\varrho$ are such that $A\subset\alpha$, $B\subset\alpha^+$, $|A|=|B|=\alpha$ and $X\in[A]^s$, $\xi\in B$ implies that $X\in I_{\xi,r,i}$ for every $r\geqslant s$, $r-i=s$. Thus A, B satisfy (6) and this proves the theorem.

Remark. A perhaps slightly simpler way to prove Theorem 1 would be the following. By Scott's theorem (see e.g. T. 2012 [5]) α has a normal measure μ. Then by Rowbottom's generalization of $\alpha\to(\alpha)_\gamma^{<\omega}$ (see eg. T 2036 [5]) for each $\xi<\alpha^+$ there exists $f_\xi\in{}^\omega\gamma$ and $T_\xi\subset\alpha$ such that $\mu(T_\xi)=1$ and $[T_\xi]^r\subset I^{\xi,r}_{f_\xi(r)}$ for $r<\omega$. Then again there are α^+, ξ for which f_ξ equals to the same f so we may again assume that $f_\xi=f$ for every $\xi<\alpha^+$. To finish the proof one has to prove that then there is a set $B\subset\alpha^+$, $|B|=\alpha$ for which $|\bigcap_{\xi\in B}T_\xi|=\alpha$.

The last statement is essentially the same as $\binom{\alpha^+}{\alpha}\to\binom{\alpha}{\alpha}^{1,1}_2$ for measurable α and I can not prove it without using the same ramification argument as I have used in the original proof. I do not know if a generalization of Rowbottom type of Theorem 1 is true or not i.e. I do not know if in (1) $|A| = \alpha$ can be replaced by $\mu(A) = 1$, for normal measures μ.

§ 3. Proof of Theorems 3 and 4.

THEOREM 3. *Let* $\alpha \geqslant \omega$, α *0,1-measurable*, $\beta < \alpha$. *Then*

$$\binom{\alpha}{\alpha^+} \rightarrow \binom{\alpha}{\beta} \vee \binom{1}{\alpha}, \binom{\alpha}{\alpha}^{1,2}.$$

As it was already mentioned, in our paper [4] with P. Erdös we give a number of other results on this symbol. These results show that in some way this theorem is best possible. By Definition 3 Theorem 3 is equivalent to the following statement.

Assume $\alpha \geqslant \omega$, α *is 0,1-measurable and* $\beta < \alpha$. *Let* $[\alpha^+]^2 = I_0^\xi \cup I_1^\xi$ *be a 2-partition of type 2 of* α^+ *for every* $\xi < \alpha$. *Then one of the following conditions* (i), (ii), (iii) *holds.*

(i) *There are* $A \subset \alpha$, $B \subset \alpha^+$, $|A| = \alpha$, $|B| = \beta$ *such that*

$$[B]^2 \subset \bigcap_{\xi \in A} I_0^\xi.$$

(ii) *There are* $\xi \in A$, $B \subset \alpha^+$, $|B| = \alpha$ *such that*

$$[B]^2 \subset I_0^\xi.$$

(iii) *There are* $A \subset \alpha$, $B \subset \alpha^+$, $|A| = |B| = \alpha$ *such that*

$$[B]^2 \subset \bigcap_{\xi \in A} I_1^\xi.$$

Instead of Theorem 3 we are going to prove the following slightly stronger Theorem 4 which can not be expressed in terms of the polarized partition symbol.

THEOREM 4. *Let* $\alpha \geqslant \omega$ *be 0,1-measurable and* $\beta < \alpha$. *Assume that* $[\alpha^+]^2 = I_0^\xi \cup I_1^\xi$ *for every* $\xi < \alpha$. *Then one of the following conditions* (i), (ii), (iii) *holds.*

(i) *There are* $A \subset \alpha$, $B \subset \alpha^+$, $|A| = \alpha$, $|B| = \beta$ *such that*

$$[B]^2 \subset \bigcap_{\xi \in A} I_0^\xi.$$

(ii) *There are* $A \subset \alpha$, $B \subset \alpha^+$, $|A| < \alpha$, $|B| = \alpha^+$ *such that*

$$[B]^2 \subset \bigcup_{\xi \in A} I_0^\xi.$$

(iii) *There are* $A \subset \alpha$, $B \subset \alpha^+$, $|A| = |B| = \alpha$ *such that*

$$[B]^2 \subset \bigcap_{\xi \in A} I_1^\xi.$$

Theorem 4 is really stronger than Theorem 3 since (ii) implies (ii)′ because $\alpha^+ \rightarrow (\alpha)^2_\delta$ holds provided α is strongly inaccessible and $\delta < \alpha$ (see. e.g. [3]).

Proof of Theorem 4. We assume that (i) and (ii) are false and we prove that then (iii) holds.

Let μ be a non-trivial 0,1-valued α-complete measure on all subsets of α. For each $X \in [\alpha^+]^2$ let

(1) $T(X) = \{\xi \in \alpha\colon\ X \in I_1^\xi\}$.

Put $I_i = \{X \in [\alpha^+]^2\colon\ \mu(T(X)) = i\}$, $i = 0, 1$. Then $[\alpha^+]^2 = I_0 \cup I_1$. α being strongly inaccessible we have $\alpha^+ \rightarrow (\beta, \alpha^+)^2$ for every $\beta < \alpha$. By the assumption that (i) is false there is no $Y \subset \alpha^+$, $|Y| = \beta$ such that $[Y]^2 \subset I_0$ hence there is a $Z \subset \alpha^+$, $|Z| = \alpha^+$ such that $[Z]^2 \subset I_1$ i.e. $\mu(T(X)) = 1$ for every $X \in [Z]^2$.

Thus we may assume that

(2) $\mu(T(X)) = 1$ for every $X \in [\alpha^+]^2$.

We prove the following lemma:

(3) Let $A' \subset \alpha$, $B' \subset \alpha^+$, $S \subset \alpha^+$ such that $|A'|$, $|B'| < \alpha$ satisfying the following condition:

$$(**) \qquad [B']^2 \cup [B', S] \subset \bigcap_{\xi \in A'} I_1^\xi.$$

Here $[B', S]$ denotes the set $\{\{x, y\} \in [\alpha^+]^2\colon\ x \in B' \wedge y \in S\}$.

Then there exist a $T \subset S - B'$, $|T| \leqslant \alpha$ and $a_\varrho \in \alpha - A'$, $S_\varrho \subset S - (B' \cup T)$ for $\varrho \in T$ satisfying the following conditions:

$$\text{(i)} \qquad S - B' = T \cup \bigcup_{\varrho \in T} S_\varrho.$$

(ii) If we put $A'_\varrho = A' \cup \{a_\varrho\}$, $B'_\varrho = B' \cup \{\varrho\}$ then condition (**) is satisfied substituting A', B', S by $A'_\varrho, B'_\varrho, S_\varrho$ respectively.

To prove (3) put $S - B' = S'$. Let $\sigma \in S'$. Put

(4) $U_\sigma = (\alpha - A') \cap \bigcap_{X \in [B \cup \{\sigma\}]^2} T(X)$.

Then by (2) U_σ is non empty.

(5) Let $\xi(\sigma) \in U_\sigma$ for $\sigma \in S'$ and $Z_\xi = \{\sigma \in S'\colon\ \xi(\sigma) = \xi\}$ for $\xi \in \alpha - A'$. Then $S' = \bigcup_{\xi \in \alpha - A'} Z_\xi$ and the Z_ξ are disjoint. For each $\xi \in \alpha - A'$ let T_ξ be a subset of Z_ξ maximal with respect to the property $[T_\xi]^2 \subset \bigcup_{\zeta \in A' \cup \{\xi\}} I_0^\zeta$.

By the assumption that (ii) is false we have

(6) $|T_\xi| \leqslant \alpha$ for every $\xi \in \alpha - A'$. Let $\sigma \in Z_\xi - T_\xi$. By the maximality of T_ξ there exists a $\varrho \in T_\xi$ such that $\{\varrho, \sigma\} \notin \bigcup_{\zeta \in A' \cup \{\xi\}} I_1^\zeta$. Put

(7) $S_\varrho = \{\sigma \in Z_\xi - T_\xi\colon\ \{\varrho, \sigma\} \in \bigcap_{\zeta \in A' \cup \{\xi\}} I_1^\zeta\}$, $T = \bigcup_{\xi \in \alpha - A'} T_\xi$, $a_\varrho = \xi$ for $\varrho \in T_\xi$. Then $T \subset S - B' = S'$, $S_\varrho \subset S' - T$ for $\varrho \in T$, $\bigcup_{\varrho \in T_\xi} S_\varrho = Z_\xi - T_\xi$ for $\xi \in \alpha - A'$, hence $S - B' = T \cup \bigcup_{\varrho \in T} S_\varrho$, and, by (6), $|T| \leqslant \alpha$.

Let $\varrho \in T$. Then $[B'_\varrho]^2 \cup [B'_\varrho, S_\varrho] = [B']^2 \cup [B', S_\varrho \cup \{\varrho\}] \cup [\{\varrho\}, S_\varrho]$. If $\varrho \in T$ then $\varrho \in T_\xi$ for a unique ξ and $a_\varrho = \xi$ for this ξ. It follows from (4) and (5) that

$$[B']^2 \subset \bigcap_{\zeta \in A' \cup \{\xi\}} I_1^\zeta, \qquad [B', Z_\xi] \subset \bigcap_{\zeta \in A' \cup \{\xi\}} I_1^\zeta$$

and then $S_\varrho \cup \{\varrho\} \subset Z_\xi$ implies $[B', S_\varrho \cup \{\varrho\}] \subset \bigcap_{\zeta \in A' \cup \{\xi\}} I_1^\zeta$. $[\{\varrho\}, S_\varrho] \subset \bigcap_{\zeta \in A' \cup \{\xi\}} I_1^\zeta$ follows from (7). Hence we have $[B'_\varrho]^2 \cup [B'_\varrho, S_\varrho] \subset \bigcap_{\zeta \in A'_\varrho} I_1^\zeta$ and T; a_ϱ, S_ϱ for $\varrho \in T$ satisfy the requirements of lemma (3).

Now just like in the proof of Theorem 1 let $\mathbf{F}$ be the set of all triples $\tau = (A', B', S)$ satisfying the requirements of $(\ast\ast)$. Let again $\precsim$ be the partial ordering defined on $\mathbf{F}$ by the stipulation $\tau_0 \precsim \tau_1$ iff $A'_0 \not\subset A'_1$, $B'_0 \not\subset B'_1$, $S_1 \subset S_0$, $B'_1 \subset S_0$. Again we obviously have that if for some φ $\{\tau_\varrho\}_{\varrho < \varphi} \subset \mathbf{F}$ is such that $\tau_\varrho \precsim \tau_\sigma$ for $\varrho < \sigma < \varphi$ then (A', B', S) satisfies the condition $(\ast\ast)$ of (3) where $A' = \bigcup_{\varrho < \varphi} A'_\varrho$, $B' = \bigcup_{\varrho < \varphi} B'_\varrho$ and $S = \bigcap_{\varrho < \varphi} S_\varrho$. Considering again that $(0, 0, \alpha^+) \in \mathbf{F}$, (3) holds and that $\alpha^{\underline{\alpha}} < \alpha^+$ the ramification argument of Lemma 1 ([3]) gives us the existence of a chain of length α i.e. $\{\tau_\varrho\}_{\varrho < \alpha} \subset \mathbf{F}$, $\tau_\varrho \precsim \tau_\sigma$ for $\varrho < \sigma < \alpha$. Then $A = \bigcup_{\varrho < \alpha} A'_\varrho$, $B = \bigcup_{\varrho < \alpha} B'_\varrho$ obviously satisfy the requirements of (iii). This proves the theorem.

§ 4. Remarks.

4.1. Following the notations of [10] we say that $i(\alpha, \beta)$ holds if α carries an α-complete nontrivial β-saturated ideal. $k(\alpha)$ holds if $i(\alpha, \alpha)$ but $i(\alpha, \beta)$ is false for $\beta < \alpha$. A Souslin α-tree is a tree (ramification system) of power α such that every chain and antichain is of power $< \alpha$. It is well-known that if α is weakly compact then there are no Souslin α-trees (see e.g. T 1234 [10] or [9]).

It would be interesting to know if for a strongly inaccessible not weakly compact α there are Souslin α-trees. (The consistency of this statement is proved, see T. 1235 [10]). On the other hand it is a known open problem if there are strongly inaccessible α's for which $k(\alpha)$ holds (see P 2054 [10]).

The following trivial remark gives a connection.

Remark 1. If $k(\alpha)$ holds for a strongly inaccessible α then there is a Souslin α-tree.

Proof. By a theorem of A. Levy (T. 2053, [10]) α is not weakly compact. Then by a well-known result of [9] there is a tree $(A, \precsim)$ of power α not containing a chain of power α, such that for each level $\xi < \alpha$ the elements of A of order ξ form a set of power less than α.

Since $k(\alpha)$ holds there is an α-complete-non-trivial α-saturated ideal I in A. Put $A' = \{x \in A\colon \{y \in A\colon x \precsim y\} \notin \mathrm{I}\}$. Considering that each level of A contains an element of A' $|A'| = \alpha$ and $(A', \precsim)$ does not contain an antichain of power α because I is α-saturated.

To put Remark 1 in a more interesting form one can say that either every strongly inaccessible α for which $i(\alpha, \alpha)$ holds is 0,1-measurable or there exists a Souslin α-tree for some strongly inaccessible α.

J. de Groot asked the following: Is it true that the number of open subsets of a Hausdorff space is always a power of 2? Obviously if α is strongly inaccessible the usual ordered set corresponding to a Souslin α-tree would exhibit a counterexample. In our paper [7] with I. Juhász we proved a number of partial results on de Groot's problem which certainly imply that at least assuming GCH the answer is affirmative except possibly if the cardinality of the space is inaccessible and not weakly compact see e.g. [7]. Thus a Souslin α-tree seems to be a typical possible counterexample, and in fact by Prikry's result T. 1235 [10] already mentioned it is consistent that de Groot's conjecture fails.

Added in proof: This paper was written before I learned about Jensen's general result that $V = L \Rightarrow$ (Weak compactness $\Leftrightarrow$ Souslin hypothesis).

4.2. In our paper [2] with P. Erdös we have defined the symbol $(\alpha, \beta, <\omega) \to \gamma$ to denote the following statement. Whenever $f \in {}^{[\alpha]^{<\omega}}P(\alpha)$ is such that $f(X) \cap X = 0$ and $|f(X)| < \beta$ for $X \in [\alpha]^{<\omega}$ then there exists $A \subset \alpha$, $|A| = \gamma$ such that $f(A) \cap A = 0$. i.e. every set-mapping of type $<\omega$ and of order at most β defined on a set of power α has a free set of power α. We proved that if $\alpha > \omega$ is 0,1-measurable then $(\alpha, \beta, <\omega) \to \alpha$ holds for $\beta < \alpha$, but this is anyway a corollary $\alpha \to (\alpha)_\gamma^{<\omega}$ for $\gamma < \alpha$. I want to mention that the set-mapping theorem is true under weaker assumptions too.

Remark 2. Assume $\alpha > \omega$ and $i(\alpha, \gamma)$ holds for some $\gamma < \alpha$. Then $(\alpha, \beta, <\omega) \to \alpha$ holds for every $\beta < \alpha$.

I do not know if the condition $i(\alpha, \gamma)$ can be replaced by $i(\alpha, \alpha)$. Though I have not seen this statement anywhere this is probably well-known since the corollary of it, that there is no Jonsson algebra of power α if $i(\alpha, \gamma)$ holds, is well-known. As to the proof of it the old proof given in [2] applies using the following trivial remark. If I is an α-complete non-trivial $\delta^+_1 < \alpha$ saturated ideal in α, and $\mathbf{F} \subset \mathbf{P}(\alpha) - \mathrm{I}$ $|\mathbf{F}| = \delta^+$ then there is an $\mathbf{F}' \subset \mathbf{F}$, $|\mathbf{F}'| = \delta^+$ with $\bigcap \mathbf{F}' \neq 0$. Note that using Rowbottom's idea one can prove the existence of a free set not in I provided I is normal. I omit the details.

4.3. The following symbols can be defined.

DEFINITION 5. (i) $\alpha \to [\beta]^r_{\gamma,\delta}$ holds iff whenever $[\alpha]^r = \bigcup_{\nu<\gamma} I_\nu$ then there are $B \subset \alpha$, $D \subset \gamma$ such that $|B| = \beta$, $|D| = \delta$ and $[B]^r \subset \bigcup_{\nu \in D} I_\nu$ (see p. 158 [3]).

(ii) $\alpha \to [\beta]^{<\omega}_{\gamma,\delta}$ holds iff whenever $[\alpha]^r = \bigcup_{\nu<\gamma} I^r_\nu$ for $r < \omega$ then there are $B \subset \alpha$, $D \subset \gamma$ such that $|B| = \beta$, $|D| = \delta$ and $[B]^r \subset \bigcup_{\nu \in D} I^r_\nu$ for $r < \omega$.

Using again the old proof given in [2] one can prove

Remark 3. Assume $i(\alpha, \delta^+)$ for some $\alpha > \omega$, $\delta^+ < \alpha$, then $\alpha \to [\alpha]^{<\omega}_{\delta^+,\delta}$.

Using Rowbottom's improvement one can even prove the existence of a subset $A \subset \alpha$, $A \notin$ I satisfying the requirements of Definition 5 (ii) provided I is a δ^+-saturated normal ideal in α.

I omit the obvious details. To indicate that theorems of this type might be usefull I mention that analyzing Rowbottom's proof for T. 2032 [10] it is easy to see that the existence of an α for which $\alpha \to [\omega_1]^{<\omega}_{\omega_1,\omega}$ holds implies that there are only countably many constructible reals hence by Remark 3 the same conclusion follows e.g. from the existence of a real valued measurable α. (This inference is not very usefull since it trivially follows from a more general theorem of R. Solovay see T. 2050 [10]).

On the other hand there are many open problems for the symbols defined in Definition 5 which seem to be interesting e.g. Assume $\alpha > \omega$, $i(\alpha, \omega_1)$ does then $\alpha \to [\alpha]^2_{\alpha,\omega}$ or $\alpha \to [\alpha]^2_{3,2}$ hold. These should be compared with problems 15, 16, 17 of [5].

Added in proof: After the paper was written K. Kunen proved that the existence of an α for which $i(\alpha, \alpha)$ implies that there are only countably many constructible reals. See K. Kunen, *Some applications of iterated ultrapowers in set theory*, Ann. Math. Logic (appear).

4.4. In papers with P. Erdös, R. Rado and E. C. Milner we often needed the following for reference purposes.

Remark 4. Assume $V = L$. Then $\alpha^{++} \not\to [\alpha^+]^2_{\alpha^+,\alpha}$ for $\alpha \geqslant \omega$.

This follows from the following simple facts. Rowbottom's proof mentioned several times yields that $V = L$ implies $\alpha^{++} \not\to [\alpha^+]^{<\omega}_{\alpha^+,\alpha}$ for $\alpha \geqslant \omega$ and from the fact that $\alpha^{++} \to [\alpha^+]^2_{\alpha^+,\alpha}$ implies $\alpha^{++} \to [\alpha^+]^{<\omega}_{\alpha^+,\alpha}$ for $\alpha \geqslant \omega$.

This can be seen as follows: Assume $[\alpha^{++}]^r = \bigcup_{\nu<\gamma} I^r_\nu$ is a disjoint r-partition of type γ of α^{++} for $2 \leqslant r < \omega$. For each $\xi < \alpha^{++}$ let f_ξ be a one-to-one mapping of ξ onto an ordinal $\leqslant \alpha^+$. Define the 2-partition $[\alpha^{++}]^2 = \bigcup_{\nu<\alpha^+} I_\nu$ of type α^+ of α^{++} as follows.

Let $[\zeta, \xi\} \in [\alpha^{++}]^2$, $\zeta < \xi$. Put $\{\zeta, \xi\} \in I_\mu$ for the minimal μ for which

$$\Big\{\nu\colon \bigvee X\Big(|X| = r \wedge \zeta, \xi \in X \wedge \bigwedge \varrho\big(\varrho \in X - \{\zeta, \xi\} \Rightarrow f_\xi(\varrho) < f_\xi(\zeta)\big) \wedge X \in I^r_\nu\Big)\Big\} \subset \mu .$$

Then $[\alpha^{++}]^2 = \bigcup_{\nu<\alpha^+} I_\nu$; and $A \subset \alpha^{++}$, $D \subset \alpha^+$, $[A]^2 \subset \bigcup_{\nu \in D} I_\nu$ imply that $[A]^r \subset \bigcup_{\nu \in D} I^r_\nu$ for $2 \leqslant r < \omega$.

4.5. Finally I want to mention a problem which I have formulated from a problem of F. Galvin. The chromatic number of a graph is the least cardinal β for which the set of vertices is the union of β sets not containing edges. Galvin's original question was the following. Let **G** be a graph of chromatic number $> \omega$. Does there exist two disjoint subsets of vertices such that the subgraphs spanned by these subsets are both of chromatic number $> \omega$. Let **G** be a graph of chromatic number $> \omega$ with α vertices and let I be the ideal of all those subsets which spann a subgraph of chromatic number $\leqslant \omega$. If α were the first cardinal for which the answer to Galvin's question were not affirmative I would be an ω_1-complete non-trivial prime ideal in α and this would obviously contradict to known indescribability principles. I want to state an improved version of the problem.

Let **G** be a graph of chromatic number $> \omega$. Does there exist a sequence $\{g_\nu\}_{\nu<\omega_1}$ of pairwise disjoint subsets of the set of vertices, such that for each $\nu < \omega_1$, g_ν defines a subgraph of chromatic number $> \omega$?

Obviously if a counterexample exists its cardinality is very large.

§ 5. Generalization of some results of Kiesler-Tarski [9].

Let C^*_0 be the class of cardinals satisfying the condition

(i) There is an α-complete field B of subsets of α, $[\alpha]^{<\alpha} \subset B$, α-generated by $\leqslant \alpha$ elements of it such that there is no proper α-complete α-saturated ideal I in B, $[\alpha]^{<\alpha} \subset$ I.

The class C_0 of [9] can be defined by the following condition.

(i)′ There is an α-complete field B of subsets of α, $[\alpha]^{<\alpha} \subset B$, α-generated by $\leqslant \alpha$ elements of it such that there is no proper α-complete 2-saturated ideal I in B, $[\alpha]^{<\alpha} \subset I$.

Consider the following statements.

(ii) There is an α-complete field B of sets α-generated by at most α elements of it and an α-complete proper ideal $\mathrm{I} \subset B$ such that I can not be extended to a proper α-complete α-saturated ideal of B.

(ii)′ Is the statement obtained from (ii) if α-saturated is replaced by 2-saturated.

Let $(S, \lessdot)$ be a partially ordered set. We say that it is an α-tree, if it is a ramification system as defined in [9]; $|S| = \alpha$ each element of S has level $< \alpha$ and on each level there are fewer than α elements.

(iii) There is an α-tree $(S, \prec)$ such that each subset $S' \subset S$ of power α contains an antichain of power α.

(iii)′ There is an α-tree $(S, \prec)$ such that it contains no chain of power α (in other words this means each subset $S' \subset S$ of power α contains an antichain of power 2).

(iv) There is an ordered set $(S, \prec)$ of power α not containing increasingly or decreasingly well ordered subsets of power α and satisfying the following condition: If $S' \subset S$, $|S'| = \alpha$, S' does not have the α-Souslin property (i.e. there are α pairwise disjoint intervals with endpoints in S').

(iv)′ is (iv) without the second condition.

It is known from [9] that (i)′ (ii)′ (iv)′ are equivalent for every $\alpha \geqslant \omega$ and (iii)′ is equivalent to these statements for strongly inaccessible α. The implications were proved by different people. For detailed references see [9].

I am going to prove

THEOREM 5. *Assume α is strongly inaccessible. Then* (i), (ii), (iii) *and* (iv) *are equivalent.*

It is obvious that (i) ... (iv) imply (i)′ ... (iv)′ respectively. In our paper with G. Fodor [6] we stated in Theorem 5 that (i) holds for a large class of strongly inaccessible α's. We did not give a proof since this can be carried out with a routine modification of the proof of Theorem 3 of [6] or better with a modification of the proofs of Fremlin, Jensen, Solovay (T. 2049 [10]). Obviously $\alpha \in \boldsymbol{C}_0^*$ implies that $i(\alpha, \alpha)$ is false and it is not known if $\alpha \in \boldsymbol{C}_0^*$ and $\alpha \in \boldsymbol{C}_0$ are equivalent for strongly inaccessible α. (See the problem of [10], mentioned in 4.1 of this paper).

Proof of Theorem 5.

(1) (i) $\Rightarrow$ (ii) is obvious.

(2) (ii) $\Rightarrow$ (iii), ((ii)′ $\Rightarrow$ (iii)′ is a theorem of Erdös and Tarski).

Though the proof is very similar to the proof of the Erdös–Tarski theorem I give the details since I think that the formulation of (iii) is the main point of the theorem. Assume (iii) is false, and let B be an α-complete field of sets α-generated by at most α-elements of it, and I an α-complete proper ideal in B. By the assumptions we may assume $|B| = \alpha$. Let $B = \{A_\xi\}_{\xi<\alpha}$ be a well-ordering of B, let $B_\xi = \{A_\zeta\}_{\zeta<\xi}$; $\xi < \alpha$. For each $\xi < \alpha$ let $p_\xi \in \bigcup B - \bigcup (B_\xi \cap \mathrm{I})$. Let $\prec$ be the usual partial ordering on ${}^{\underset{\smile}{\alpha}}2$. Let $f_\xi \in {}^{\xi}2$ be defined by the stipulation $f_\xi(\zeta) = 0$ iff $p_\xi \notin A_\zeta$. Let S be the set $\{f \in {}^{\underset{\smile}{\alpha}}2\colon \bigvee \xi (\xi < \alpha \wedge f \prec f_\xi)\}$.

Then α being strongly inaccessible $(S, \prec)$ is an α-tree. By the assumption that (iii) is false there is $S' \subset S$, $|S'| = \alpha$ such that S' contains no antichain of power α. We define $J \subset B$ by the stipulation $A_\zeta \in J$ iff $f(\zeta) = 0$ for all but fewer than $f \in S'$. It is obvious that $\mathrm{I} \subset J$, and J is an α-complete proper ideal in B. Let $D \subset \alpha$ be such that $A_\zeta \notin J$, $A_\zeta \cap A_{\zeta'} = 0$ for $\zeta \neq \zeta' \in D$. Then by the definition of J, for each $\zeta \in D$ there exist η, ξ; $f \in S'$, $\zeta < \eta \leqslant \xi$ such that

(3) $f = f_\xi \restriction \eta$ and $f(\zeta) = 1$ i.e. $p_\xi \in A_\zeta$. Let now $\zeta \neq \zeta' \in D$ and let η, ξ, f; η', ξ', f' the ordinals and functions satisfying (3) for ζ and ζ' respectively. We may assume $\eta \leqslant \eta'$. Then $\zeta < \eta'$, $p_\xi \in A_\zeta$, $p_{\xi'} \in A_{\zeta'}$, hence A_ζ, $A_{\zeta'}$ being disjoint $p_{\xi'} \notin A_\zeta$ hence $f_\xi(\zeta) = f(\zeta) = 1$, $f_{\xi'}(\zeta) = f'(\zeta) = 0$, f and f' are incomparable in $(S', \prec)$. For each $\zeta \in D$ let f^ζ be a function satisfying (3). Then $\{f^\zeta\}_{\zeta \in D}$ is an antichain of power $|D|$ in $(S', \prec)$ hence $|D| < \alpha$ and J is α-saturated. This proves (2).

(4) (iii) $\Rightarrow$ (iv) ((iii)′ $\Rightarrow$ (iv)′ is a theorem of Hanf).

I am going to follow mine proof of Hanf's theorem given in [8] when outlining the proof of (4).

Let $(S, \prec)$ be an α-tree satisfying the conditions of (iii). For each $x \in S$ let $\xi(x) = \operatorname{typ}\{y \prec x\colon y \in S\}$, for each $\zeta \leqslant \xi(x)$ let $x|\zeta$ be the unique element of S with $\xi(x|\zeta) = \zeta$, $x|\zeta \prec x$. Put $S_\xi = \{x \in S\colon \xi(x) = \xi\}$ and let $\prec_\xi$ be an arbitrary ordering of S_ξ. Define the ordering $\prec^*$ of S by the following stipulation. Suppose $x, x' \in S$, $x \prec^* x'$ if and only if $x \prec x'$ or $x \not\prec x'$ and $x|\xi_0 \prec_{\xi_0} x'|\xi_0$ where ξ_0 is the first ordinal for which $x|\xi_0 \neq x'|\xi_0$. We prove that $(S, \prec^*)$ satisfies (iv). It has been proved in [8] that $(S, \prec^*)$ is an ordered set and satisfies (iv)′ provided $(S, \prec)$ satisfies (iii)′. Hence $(S, \prec^*)$ satisfies the first condition of (iv). Let $S' \subset S$, $|S'| = \alpha$. Put

$$S'' = \{x \in S\colon \bigvee Y_x, Z_x (Y_x \in S' \wedge Z_x \in S' \wedge Y_x \prec^* Z_x \wedge x \prec Y_x \wedge x \prec Z_x)\}.$$

Obviously $|S''| = \alpha$. By (iii) there is an $S''' \subset S''$, $|S'''| = \alpha$ such that the elements of S''' are pairwise incomparable in $(S, \prec)$. Then $\{(Y_x, Z_x)\}_{x \in S'''}$ form a system of power α of pairwise disjoint intervals with endpoints in S'.

(5) (iv) $\Rightarrow$ (i). Assume that (i) is false and let $(S, \prec)$ be an ordered set satisfying the condition

(6) S contains no increasingly or decreasingly well-ordered subset of power α.

Let B be the α-complete field α-generated by the intervals of S. (Then by (6) B contains $[\alpha]^{<\alpha}$ and all convex subsets of S.) Let I be an α-complete α-satureted ideal in B, $[\alpha]^{<\alpha} \subset \mathrm{I}$. Let $<$ be a well-ordering of S. For each $x \in S$ let U_x be the maximal convex set such that $z \in U_x$ implies $x < z$, $x \prec z$. Let $S' = \{x \in S\colon U_x \notin \mathrm{I}\}$. We prove $|S'| = \alpha$. Assume $|S'| < \alpha$. Then α being strongly inaccessible S is the union of S'

and of fewer than α-convex sets not containing elements of S'. Hence there would be a convex set K, $K \cap S' = 0$, $K \notin I$. Using (6) it is easy to see that there is a $T \subset K$, $|T| < \alpha$ such that $K = T \cup \bigcup_{x \in T} U_x$ which is a contradiction. Hence $|S'| = \alpha$. It follows from (6) that there is an $S'' \subset S'$, $|S''| = \alpha$ such that S'' is densely ordered by $\precsim$ i.e. if $x \precsim y$; $x, y \in S''$ then there is a $z \in S''$, $x \precsim z \precsim y$. We prove that $x \precsim y \in S''$ implies $(x, y) \notin I$. Let $x, y \in S''$, $x \precsim y$. Let z_0 be the minimal element of $(x, y) \cap S''$ and let z_1 be the minimal element of $(x, z_0) \cap S''$ in the well ordering $<$. Then $U_{z_1} \subset (x, z_0) \subset (x, y)$ $U_{z_1} \notin I$ hence $(x, y) \notin I$. It follows that S'' does not contain the end points α-pairwise disjoint intervals hence (iv) is false as well. This proves (5).

Note that in Remark 1 we really proved (iii) $\Rightarrow$ (i) and Theorem 5 yields the slightly stronger conclusion that for every strongly inaccessible α $\alpha \in \boldsymbol{C}_0 \wedge \alpha \notin \boldsymbol{C}_0^*$ $\Rightarrow$ there is a Souslin α-tree. Finally I want to mention that $\alpha \not\to (\alpha)^2_2$ is known to be equivalent with $\alpha \in \boldsymbol{C}_0$. I could not find a corresponding result for $\alpha \in \boldsymbol{C}_0^*$. I suspect that $\alpha \not\to [\alpha]^2_\alpha$ might be the right statement.

$\alpha \to [\alpha]^2_\alpha$ is defined as follows. Whenever $[\alpha]^2 = \bigcup_{\nu<\alpha} I_\nu$ then there are $A \subset \alpha$, $\nu_0 < \gamma$, $|A| = \alpha$ such that $[A]^2 \subset \bigcup_{\nu<\gamma, \nu \neq \nu_0} I_\nu$. Let $\alpha \Rightarrow [\alpha]^2_\alpha$ be the following weaker statement. Whenever $[\alpha]^2 = \bigcup_{\nu<\alpha} I_\nu$ then there are $A \subset \alpha, f \in {}^\alpha\alpha$ such that $|A| = \alpha$ and for every $\sigma < \tau \in A$ $\{\sigma, \tau\} \in \bigcup_{\nu<\gamma, \nu \neq f(\sigma)} I_\nu$. In [1] we have proved that $2^\alpha = \alpha^+$ implies $\alpha^+ \not\Rightarrow [\alpha^+]^2_{\alpha^+}$ and it is easy to see that $\alpha \not\Rightarrow [\alpha]^2_\alpha$ implies $\alpha \in \boldsymbol{C}_0^*$.

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