[7] H. Hopf and E. Pannwitz, *Über stetige deformationen von Komplexen in sich*, Math. Annalen 108 (1933), pp. 433–465.
[8] W. Kuperberg, *Stable points of a polyhedron*, Fund. Math. 59 (1966), pp. 43–48.
[9] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.
[10] A. D. Wallace, *On the structure of topological semigroups*, Bull. Amer. Math. Soc. 61 (1955), pp. 95–112.

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A class of topologies with T_1-complements

by

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1. Introduction. Let Λ be the family of all T_1 topologies definable on an arbitrary set X. When $\tau_1 \in \Lambda$ and $\tau_2 \in \Lambda$, $\tau_1 < \tau_2$ if every set in τ_1 is in τ_2. Under this order, Λ is a complete lattice. The greatest element of Λ is the discrete topology, 1, and the least element is the cofinite topology $C = \{U\colon U = \emptyset \text{ or } X-U \text{ is finite}\}$.

Recently several papers have been published dealing with the structure of the lattice Λ. An example [17] was given to show that Λ is not a complemented lattice, unless X is a finite set. In [19], a T_1-complement for the reals with the usual topology is constructed. This result was generalized in [1] to yield the fact that every T_1 space with a countable dense metric subspace has a T_1-complement. For other results on the lattice Λ, see [4].

The main purpose of this paper is to show that the construction used in [19] can be made to do much more than has been previously realized. It turns out to be quite an interesting exercise to see how much of the construction in [19] can be jettisoned. Now it appears that large classes of nice topological spaces have T_1-complements. For example, it can be proved that every first axiom Hausdorff space has a T_1-complement and that every locally compact Hausdorff space has a T_1-complement. Actually the theorems deduced here are quite a bit stronger than these statements. Another result of [1] is extended to show that there is a large class of spaces (X, T) and that T is one of three mutually T_1-complementary topologies on the set X. Furthermore, it is shown that every T_1 space is an open and closed subspace of a T_1 space that has a T_1-complement.

Lastly, some questions are raised. I am indebted to Roger Countryman for an interesting conversation on the properties of Fréchet spaces and symmetrizable spaces.

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2. T_1-complements. In this section, we show that several of the most important classes of topological spaces have T_1-complements.

DEFINITION. Suppose (X, T) is a T_1 topological space. A T_1-complement T' for T is a *Steiner complement* iff there exist disjoint T'-open sets B_0 and B_1 in X such that

(1) $B_0 \cup B_1 = X$,

(2) for $i = 0, 1$, B_i contains an infinite subset A_i such that every T'-open set containing points of A_i contains a cofinite subset of B_i, and A_i is T'-closed,

(3) every T-open set containing a cofinite subset of B_i contains points of A_j, $j \neq i$.

LEMMA 1. *Suppose the nonempty T_1 space $(Z, \tau) = M \cup N$ such that*

(i) *M and N are disjoint and τ-open,*

(ii) *$\tau|N$ is discrete,*

(iii) *$\tau|M$ has a Steiner complement.*

Then τ has a T_1-complement τ'. Furthermore, if the Steiner complement $(\tau|M)'$ is compact on cofinite subsets of M, then τ' is compact on cofinite subsets of Z.

Proof. If N is empty, the result is obvious. So we assume N is nonempty. Then define τ' as follows. The $(\tau|M)'$-open sets that miss A_0 form a τ'-base at all points of M except those points in A_0. The cofinite subsets of $N \cup B_0$ are taken to form a τ'-base for the points in $N \cup A_0$. It is then clear that $(\tau'|M) = (\tau|M)'$ so that $\sup\{\tau, \tau'\} = 1$. If $U \in \tau \cap \tau'$ and $U \neq \emptyset$, then since no nonempty element in τ' is contained in N, $U \cap M \in C_M$, the cofinite topology on M. But then $U \cap M$ must contain points of A_0 so that U is cofinite in $N \cup B_0$. Thus $\inf\{\tau, \tau'\} = C_Z$. Since every cofinite subset of Z contains points of A_0, it is clear that if $(\tau|M)'$ is compact on cofinite subsets of M, then τ' is compact on cofinite subsets of Z.

DEFINITION. A topological space (X, T) is *splitable* iff X contains an infinite family of pairwise disjoint nonempty open sets.

DEFINITION. A topological space (X, T) is a *DN-space* iff for each x in X, there is at least one net in $X - \{x\}$ that converges to x and has the property that its range is a discrete subspace of X.

LEMMA 2. *If (X, T) is a T_1, splitable DN-space, then T has a T_1-complement T' that is compact on cofinite subsets of X.*

Proof. Pick a maximal infinite (hence dense) family of pairwise disjoint nonempty open sets in X and define $Y = \bigcup\{U_i\colon i \in I\}$ to be the union of the elements in this family. Partition the index set I into two disjoint infinite subsets, I_0 and I_1. For each i in I, pick e_i in U_i and

a net S_{ia} in $U_i - \{e_i\}$ that converges to e_i and whose range is a discrete subspace. We shall find it convenient in what follows to identify a net with its range. Suppose

$$A_j = \bigcup\{S_{ia}\colon i \in I_j\}, \quad j = 0, 1,$$

$$B_j = A_j \cup [(Y - A_k) \cap (\bigcup\{U_i\colon i \in I_k\})], \quad i, k = 0, 1; j \neq k,$$

$$E_j = \{e_i\colon i \in I_j\}, \quad j = 0, 1,$$

$$D = \{d_a\colon a \in \Omega\} = Y - (A_0 \cup A_1 \cup E_0 \cup E_1).$$

Define $(T|Y)'$ to be the topology generated by

(i) $\{x\}$, $x \in E_0 \cup E_1$,

(ii) C_Y, the cofinite topology on Y,

(iii) B_i, $i = 0, 1$,

(iv) C_a, $a \in \Omega$,

when C_a is defined as follows. If $d_a \in D$, then there is an $i(a)$ such that $d_a \in U_{i(a)}$. Let C_a be $\{d_a\} \cup \{e_j\colon j \in I \text{ and } j \neq i(a)\}$. Then B_0 and B_1 are disjoint sets whose union is Y. Since $C_Y \subset (T|Y)'$, it follows that $(T|Y)'$ is a T_1-topology.

Now, it is easy to see that $\sup\{T|Y, (T|Y)'\} = 1$. Y has three kinds of points; those in $E_0 \cup E_1$, in $A_0 \cup A_1$ and in D. Points in $E_0 \cup E_1$ are isolated by $(T|Y)'$. If $x \in A_0$, then there is an i such that $x \in S_{ia}$. Since the range of S_{ia} is a T-discrete subspace, there is a T-open set V such that $V \subset U_i$ and $V \cap S_{ia} = \{x\}$. Thus $V \cap B_0 = \{x\}$ and a similar argument holds for points in A_1. If $d_a \in D$, then there is an i such that $d_a \in U_i$, and clearly $U_i \cap C_a = \{d_a\}$.

Finally, we must show that $\inf\{T|Y, (T|Y)'\} = C_Y$. Suppose $U \in T|Y \cap (T|Y)'$ and $U \neq \emptyset$. Suppose there is a d_a such that $d_a \in U$. Let us assume that $d_a \in B_0$. Then since $U \in (T|Y)'$, U must contain a cofinite subset of $C_a \cap B_0$ since these are the basic open sets at d_a. But this implies that U contains a point of E_1. If $U \in T|Y$ and U contains a point of E_1, then U must contain a point of A_1. The $(T|Y)'$-open sets that contain points of A_1 are cofinite in B_1. Thus U must be cofinite in B_1 and hence contain a point of E_0. Then, as above, U must contain a point of A_0 and therefore be cofinite in B_0.

It is clear that the same kind of argument will work if we assume $d_a \in B_1$. Furthermore, if we assume that U contains a point of E_1 (E_0) or A_1 (A_0), we just enter the above argument at the appropriate point and continue from there. Thus, the $\inf\{T|Y, (T|Y)'\} = C_Y$.

Since every cofinite subset of Y contains points of both A_0 and A_1, it is easily seen that $(T|Y)'$ is compact on cofinite subsets of Y. Since Y is T-dense in X, the result follows from Theorem 6 of [18].

The argument given for Lemma 2 is essentially that used in [19], with most (hopefully) of the irrelevant details omitted. There is nothing immutable about the proofs given so far. As a matter of fact, there are a number of variations available for both Lemma 1 and Lemma 2. Note that the T' of Lemma 2 is a Steiner complement.

DEFINITION. Suppose (X, T) is a topological space and N is the set of isolated points in X. Then $(X - \mathrm{Cl}\, N)$ is the *open kernel* of X.

THEOREM 1. *Suppose (X, T) is a T_1 space whose open kernel is empty or a splitable DN-space. Then T has a T_1-complement T' that is compact on cofinite subsets of X.*

Proof. If the open kernel, M, of X is empty, the result follows from Theorem 6 of [18]. If M is nonempty, and N is the set of isolated points, then $M \cup N$ is dense in X and the result follows from Lemmas 1 and 2 and Theorem 6 of [18].

REMARK 1. Actually, by Theorem 6 of [18], it follows that any T_1 space which has a dense subspace satisfying the hypotheses of Theorem 1 has a T_1-complement that is compact on cofinite subsets. The same is true of the corollaries now to be stated.

It is easily seen that the hypotheses of Theorem 1 are satisfied by large classes of important topological spaces.

COROLLARY 1. *Every Fréchet Hausdorff space has a T_1-complement that is compact on cofinite subsets.*

Proof. By definition (see [7] and [8]), a Fréchet space has the property that a point is in the closure of a set iff there is a sequence in the set converging to the point. Thus, every first axiom space is a Fréchet space. Every sequence in a Hausdorff space that converges to a point not in the range of the sequence clearly is a discrete subspace. It is well-known that every infinite Hausdorff space is splitable ([16], p. 88). One may now repeat the proof of Theorem 1.

Certain well-known Hausdorff spaces are as badly non-Fréchet as possible.

EXAMPLE 1. A compact Hausdorff space with no dense Fréchet subspace.

If N denotes the positive integers, then $\beta N - N$ is a compact Hausdorff space. Suppose X is a subspace of $\beta N - N$ that is not discrete. We claim that X is not a Fréchet space. By hypothesis, there is a point of X that is a limit point of $X - \{x\}$. If X is a Fréchet space, then there is a sequence $\{x_i\}_{i\geqslant 1} \subset (X - \{x\})$ that converges to x in X and hence in βN. Then $\{x_i\}_{i\geqslant 1} \cup \{x\}$ would be a closed countable subset of βN, contradicting a result of Čech [13]. But no discrete subspace of $\beta N - N$ can be dense in $\beta N - N$ since $\beta N - N$ is dense-in-itself ([15], p. 414).

COROLLARY 2. *Every locally compact Hausdorff space has a T_1-complement that is compact on cofinite subsets.*

Proof. Suppose (X, T) is a locally compact Hausdorff space. It is clear, by previous arguments, that it will suffice to consider the open kernel, M, of X. M is also locally compact and Hausdorff. If $M = \emptyset$, then the result is obvious, so assume $M \neq \emptyset$. Let $\{U_i\}_{i\leqslant 1}$ be an infinite maximal (hence dense) family of pairwise disjoint open sets in M. Then each U_i is locally compact and we may pick an open set V_i such that $V_i \subset \mathrm{Cl}\, V_i \subset U_i$ and $\mathrm{Cl}\, V_i$ is compact. Since M has no isolated points, neither does V_i. Thus V_i is an infinite Hausdorff space and one may pick a countably infinite discrete subspace $\{e_{ij}\}_{j\geqslant 1}$ of V_i. By the compactness of $\mathrm{Cl}\, V_i$, $\{e_{ij}\}_{j\geqslant 1}$ has a cluster point e_i in $\mathrm{Cl}\, V_i$ and a subnet $\{A_{i_\alpha}\colon \alpha \in K_i\}$ converging to e_i. Since the subnet has range a subset of $\{e_{ij}\}_{j\geqslant 1}$, the subnet has a range that is a discrete subspace, even though the net may not be a sequence. Now by the argument of Lemma 2, $(M, T|M)$ has a T_1-complement that is compact on cofinite subsets.

Example 1 shows that Corollaries 1 and 2 are really different results, since it is trivial to construct Fréchet Hausdorff spaces that aren't locally compact.

Let us consider one more class of spaces with T_1-complements.

DEFINITION. A topological space (X, T) is *symmetrizable* iff there is a real-valued function d defined on X^2 such that

(i) for any x, y in X, $d(x, y) = d(y, x) \geqslant 0$,

(ii) for any x, y in X, $d(x, y) = 0$ iff $x = y$,

(iii) for any $P \subset X$, $P = \mathrm{Cl}\, P$ iff for any x in $X - P$, $d(x, P) > 0$.

If, in addition, we require

(iv) for any $P \subset X$, $x \in \mathrm{Cl}\, P$ iff $d(x, P) = 0$,

then (X, T) is said to be *semi-metrizable*.

COROLLARY 3. *Every Hausdorff symmetrizable space has a T_1-complement that is compact on cofinite subsets.*

Proof. The open kernel of a symmetrizable space is symmetrizable, and it is easily seen that a symmetrizable dense-in-itself Hausdorff space is a DN-space. The result then follows directly from Theorem 1.

It turns out that Corollary 3 is not subsumed by Corollaries 1 and 2.

EXAMPLE 2. A symmetrizable Hausdorff space with no dense subspace that is either Fréchet or locally compact.

Let X be the plane and define d as follows. Suppose $x = (a_1, b_1)$ and $y = (a_2, b_2)$ are in X. Then $d(x, y)$ is ordinary Euclidean distance iff x any y can be joined by either a horizontal or vertical line (i.e., iff either $a_1 = a_2$ or $b_1 = b_2$), and $d(x, y) = 1$ otherwise. The topology T induced by this symmetric clearly contains the usual plane topology τ

and fails to be first axiom. Thus the whole space fails to be a Fréchet space ([3], p. 129).

Now, every T-dense subset must be τ-dense and it is easy to see that if D is a T-dense subset, $x \in D$ and A_x is the set of points in D that can't be joined to x by either a horizontal or vertical line, then in $(D, T|D)$, $x \in \mathrm{Cl}\, A_x$, but no sequence in A_x converges to x. Thus D is not a Fréchet space.

Finally, no dense subset is locally compact. For suppose D is a dense locally compact subset. Then D must be open ([11], p. 163) and $(D, T|D)$ is a regular space with all points G_δ's since $T \supset \tau$. If $x \in D$, pick an open set V_x such that $x \in V_x$ and $\mathrm{Cl}_D V_x$ is compact. But then $\mathrm{Cl}_D V_x$ is a compact Hausdorff space such that all of its points are G_δ's. It follows easily that $\mathrm{Cl}_D V_x$ must be first axiom. Since x was arbitrary in D and V_x is open in X, D is a dense first axiom subspace. This contradicts the preceding paragraph.

3. Other uses of the Steiner construction. As we have now seen, it is possible to use the construction of [19] to verify that many topological spaces have T_1-complements. However, we still haven't exhausted the uses for this construction. The following result is presented as an aid to disposing of conjectures concerning spaces that might not have T_1-complements.

THEOREM 2. *Every T_1 space (X, T) is an open-closed subspace of a T_1 space (X^*, T^*) of the same cardinal, weight and dimension such that T^* has a T_1-complement that is compact on cofinite subsets.*

Proof. We may assume that X has at least $\aleph_0$ points. The main idea here is that we don't really need the DN-property on the whole space. It will suffice for the space to be DN at one point in each of an infinite family of pairwise disjoint open sets, as we saw in the proof of Corollary 2.

Thus, let X^* be the topological free union of X and the rationals with the usual topology. The rationals can be written as the union of infinitely many pairwise disjoint open intervals. Add X to one of the open intervals and proceed as in Lemma 2, being sure to pick the e_i's and $s_{i\alpha}$'s in the rationals.

The construction in [19] made it appear that resolvability (see [6] and [9]) was crucial for the existence of T_1-complements. However, the arguments given here have not used resolvability. It may, however, still be that there is some connection.

COROLLARY 4. *There are irresolvable dense-in-themselves spaces that have T_1-complements.*

Proof. This is an immediate consequence of Theorem 2.

It is possible to use the methods developed so far to extend a major result of [1] in several ways. An argument was given in [1] to show the existence of three mutually T_1-complementary topologies on a countably infinite set. The proof given here works on a large class of topological spaces, the cardinality of the set involved is unimportant and it is not necessary (to answer a question of S. C. Armentrout) to pick smaller and smaller subspaces as we define more and more topologies.

THEOREM 3. *Suppose (X, T) is a T_1 DN-space that can be expressed as the union of infinitely many pairwise disjoint open sets. Then there exist topologies T_a and T_b on X such that every distinct pair from $\{T, T_a, T_b\}$ are T_1-complements.*

Proof. We may assume that X is the union of exactly $\aleph_0$ pairwise disjoint open sets. Split this family into subfamilies as follows. If M and N are countably infinite pairwise disjoint sets indexed by the set of all integers, there is a 1-1 function from $M \cup N$ onto the family of pairwise disjoint open sets. Let P_M and P_N denote the "even elements" of M and N respectively, and let Q_M and Q_N denote the "odd elements" of M and N. It will be helpful to the reader at this point to draw two rows of pairwise disjoint circular discs and label the top row the M-row, the bottom row the N-row and alternate columns between P-columns and Q-columns. From now on we shall use the notation MP to mean all points in the open sets indexed by P_M, that is, all points in the M-row that are also in the P-columns. NP, MQ and NQ have the corresponding meanings. Similarly P will denote all points in the P-columns and Q all points in the Q-columns.

We now proceed with the definition of T_a and T_b. Pick a point in each open set. The notation e_{mi} will denote the point picked in the set indexed by the i in M. This gives sets (with obvious definitions) E_{MP}, E_{MQ}, E_{NP}, E_{NQ}, E_M, E_N, E_P, E_Q and E. For each e_{mi}, pick a net S_{mi} such that if U_{mi} is the open set associated with i in M

(i) the first element of S_{mi} is e_{ni},

(ii) every other element of S_{mi} is in $U_{mi} - \{e_{mi}\}$,

(iii) S_{mi} T-converges to e_{mi},

(iv) the range of S_{mi} is a discrete subspace.

For each e_{ni}, pick a net S_{ni} such that

(v) the first element of S_{ni} is $e_{m,i+1}$,

(vi) every other element of S_{ni} is in $U_{ni} - \{e_{ni}\}$,

(vii) S_{ni} T-converges to e_{ni},

(viii) the range of S_{ni} is a discrete subspace.

Now, make the following definitions.

$$A_{pa} = \bigcup \{S_{mi}\colon\ i \in P_M\}, \qquad A_{qa} = \bigcup \{S_{mi}\colon\ i \in Q_M\},$$

$$A_{pb} = \bigcup \{S_{ni}\colon\ i \in P_N\}, \qquad A_{qb} = \bigcup \{S_{ni}\colon\ i \in Q_N\}.$$

Furthermore,

$$B_{pa} = A_{pa} \cup [(X - A_{qa}) \cap Q], \qquad B_{qa} = A_{qa} \cup [(X - A_{pa}) \cap P],$$
$$B_{pb} = A_{pb} \cup [(X - A_{qb}) \cap (MP \cup NQ)],$$
$$B_{qb} = A_{qb} \cup [(X - A_{pb}) \cap (MQ \cup NP)].$$

Notice that

$$E_{NP} \subset A_{pa} \subset P, \qquad E_{NQ} \subset A_{qa} \subset Q, \tag{1}$$
$$E_{MQ} \subset A_{pb} \subset (NP \cup MQ), \qquad E_{MP} \subset A_{qb} \subset (NQ \cup MP)$$

and

$$E_{NP} \cup E_{MQ} \subset B_{pa}, \qquad E_{NQ} \cup E_{MP} \subset B_{qa}, \tag{2}$$
$$E_Q \subset B_{pb}, \qquad E_P \subset B_{qb}.$$

Furthermore

$$B_{pa} \cup B_{qa} = X, \qquad B_{pa} \cap B_{qa} = \emptyset, \tag{3}$$
$$B_{pb} \cup B_{qb} = X, \qquad B_{pb} \cap B_{qb} = \emptyset.$$

If $x \notin E \cup A_{pa} \cup A_{qa}$, we make the following observations and definitions.

(4) If $x \in P$, then $x \in B_{qa}$ and $C_{xa} = \{x\} \cup E_{MP}$.
If $x \in Q$, then $x \in B_{pa}$ and $C_{xa} = \{x\} \cup E_{MQ}$.

If $x \notin E \cup A_{pb} \cup A_{qb}$, we make the following observations and definitions.

(5) If $x \in MP \cup NQ$, then $x \in B_{pb}$ and $C_{xb} = \{x\} \cup E_{NQ}$.
If $x \in NP \cup MQ$, then $x \in B_{qb}$ and $C_{xb} = \{x\} \cup E_{NP}$.

Finally we can define T_a and T_b. T_a is the topology generated by singletons in E_M, cofinite subsets of B_{pa} and B_{qa}, $\{C_{xa}: x \notin E \cup A_{pa} \cup A_{qa}\}$.

Notice that every T_a-open set containing a point of A_{pa} (A_{qa}) is cofinite in B_{pa} (B_{qa}), and that if $x \notin E \cup A_{pa} \cup A_{qa}$, then every T_a-open set containing x contains infinitely many points of E_M. T_b is the topology generated by singletons in E_N, cofinite subsets of B_{pb} and B_{qb}, $\{C_{xb}: x \notin E \cup A_{pb} \cup A_{qb}\}$.

Every T_b-open set containing a point of A_{pb} (A_{qb}) is cofinite in B_{pb} (B_{qb}), and if $x \notin E \cup A_{pb} \cup A_{qb}$, then every T_b-open set containing x contains infinitely many points of E_N.

By an argument like that given in Lemma 2, one easily sees that T and T_a are T_1-complements. Similarly T and T_b are T_1-complements.

Thus, it will suffice to show that T_a and T_b are T_1-complements. First let us verify that $\sup\{T_a, T_b\} = 1$. Clearly we may assume that $x \notin E$. Suppose $x \in MP$. Then by (5), $x \in B_{pb}$ and by (4) either $x \in A_{pa}$ or $x \in B_{qa}$. If $x \in A_{pa}$, then by (1), (2), (3) and (5), $C_{xb} \cap B_{pa} = \{x\}$. If $x \in B_{qa}$, then $C_{xa} \cap C_{xb} = \{x\}$. Similarly one can dispose of the cases $x \in MQ$, $x \in NP$ and $x \in NQ$.

Lastly, we must prove that $\inf\{T_a, T_b\} = C_X$. Suppose $U \in T_a \cap T_b$ and $U \neq \emptyset$. It is clear that every nonempty T_a-open set intersects E_M and every nonempty T_b-open set intersects E_N. If $U \cap E_{MP} \neq \emptyset$, then since $E_{MP} \subset A_{qb}$, U must contain a cofinite subset of B_{qb}. But $E_{NP} \subset B_{qb}$ and $E_{NP} \subset A_{pa}$ which implies U must contain a cofinite subset of B_{pa}. Since $E_{MQ} \subset B_{pa}$ and $E_{MQ} \subset A_{pb}$, U must contain a cofinite subset of B_{pb}. Thus U is cofinite in $B_{pb} \cup B_{qb} = X$. A similar argument holds for the other cases.

It is clear that every Hausdorff DN-space has an open dense subspace that satisfies the hypotheses of Theorem 3. This result also implies that every infinite set carries three mutually T_1-complementary topologies.

4. Questions. At present, there are few known examples ([4], [17]) of a T_1 space with no T_1-complement. The theorems of this paper show that if a space is to fail to have a T_1-complement, it must have some fairly bad features. There is a class of spaces that incorporate certain properties which seem to insure that if these spaces have T_1-complements, they cannot be described by constructions like the ones used here.

DEFINITION. A topological space that is dense-in-itself and has the property that every dense subset is open is called an *MI-space* ([9], p. 322).

REMARK 2. *If (X, T) is a Hausdorff MI-space, then*

(i) (X, T) *is not resolvable,*

(ii) *no dense subspace of (X, T) is locally compact, in fact all compact sets are finite,*

(iii) (X, T) *has the property that no "non-trivial" convergent net in the space has a subnet whose range is a discrete subspace, so that no subspace whatsoever is a DN-space,*

(iv) *no dense subspace of (X, T) is Fréchet, in fact all Fréchet subspaces are totally isolated (have no limit points in X),*

(v) *no dense subspace of (X, T) is symmetrizable, in fact all symmetrizable subspaces are totally isolated,*

(vi) *we may assume (X, T) is countable, connected and thus not regular.*

Proof. Hewitt ([9], p. 322) established (i) and Kirch [12] proved (ii). Lemma 4 of [12] shows that X can't be locally countably compact.

It is clear that no totally isolated subspace is a DN-space. Thus, if Y is a DN-subspace of X, then $T\text{-}\operatorname{Int} Y \neq \emptyset$ ([9], p. 325). Then $T\text{-}\operatorname{Int} Y$ is a DN-subspace that in also a Hausdorff MI-space ([9], p. 325). Thus it will suffice to show that (X, T) itself is not a DN-space. Suppose $x \in X$ and $\{S_\alpha: \alpha \in A\}$ is a net in $X - \{x\}$ converging to x. It is clear that every discrete subspace of a dense-in-itself space must have a void interior. But every subset of an MI-space that has a void interior must be totally isolated and hence closed. Thus, x is not a limit point of any discrete subspace of $\{S_a: a \in A\}$ and therefore no cofinal subset of $\{S_a: a \in A\}$ is a discrete subspace.

Every Fréchet subspace with a non-isolated point would have a non-trivial convergent net whose range was a discrete subspace. Thus every Fréchet subspace is totally isolated. Clearly (v) also follows from (iii).

The properties mentioned in (vi) follow from known results. For example, see [2] or [5] (Part 1, p. 138, 139, 155).

QUESTION 1. *Does every Hausdorff MI-space have a T_1-complement?*

For a particularly explicit example of an *MI*-space, see [14].

It is clear from Remark 2 that the results of this paper don't answer this question. In view of (iii) of Remark 2 and the fact that the Steiner method appears to require *DN*-ness at at least $\aleph_0$ points (E_0 and E_1), it seems likely that if the answer to this question is affirmative, some new technique will have to be developed.

In [1], it was proved that if a T_1 space (X, T) has a countable dense metric subspace, then T has a T_1-complement. Is Corollary 1 for countable sets really an improvement on the result?

QUESTION 2. *Does every countable Fréchet (first axiom) space have a dense metrizable subspace?*

Note that by Remark 2, there are countable connected Hausdorff spaces such that every metrizable subspace is totally isolated. It is also known [10] that there exist countable regular Hausdorff dense-in-themselves spaces with no dense metric subspaces.

References

[1] B. A. Anderson and D. G. Stewart, *T_1-complements of T_1 topologies*, Proc. Amer. Math. Soc., 23 (1969), pp. 77–81.
[2] D. R. Anderson, *On connected irresolvable Hausdorff spaces*, Proc. Amer. Math. Soc. 16 (1965), pp. 463–466.
[3] A. V. Arhangel'skii, *Mappings and spaces*, Russian Math. Surveys, 21 (1966), pp. 115–162.
[4] R. W. Bagley, *On the characterization of the lattice of topologies*, J. London Math. Soc. 30 (1955), pp. 247–249.
[5] N. Bourbaki, *General Topology*, Addison–Wesley, 1966.
[6] J. G. Cedar, *On maximally resolvable spaces*, Fund. Math. 55 (1964), pp. 87–93.
[7] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), pp. 107–115.
[8] — *Spaces in which sequences suffice II*, Fund. Math. 61 (1967), pp. 51–56.
[9] E. Hewitt, *A problem of set-theoretic topology*, Duke J. Math. 10 (1943), pp. 309–333.
[10] M. Katětov, *On topological spaces containing no disjoint dense subsets*, Math. Sb. N.S. 21 (63), (1947), pp. 3–12.
[11] J. L. Kelley, *General Topology*, Van Nostrand, 1955.

[12] M. R. Kirch, *A class of spaces in which compact sets are finite*, Amer. Math. Monthly, 76 (1969), p. 42.
[13] J. Novak, *On the cartesian product of two compact spaces*, Fund. Math. 40 (1953), pp. 106–112.
[14] K. Padmavally, *An example of a connected irresolvable Hausdorff space*, Duke J. Math. 20 (1953), pp. 513–520.
[15] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke J. Math. 23 (1956), pp. 409–419, 633.
[16] W. Sierpiński, *General Topology* (second edition), Toronto 1952.
[17] A. K. Steiner, *Complementation in the lattice of T_1-topologies*, Proc. Amer. Math. Soc. 17 (1966), pp. 884–885.
[18] A. K. Steiner and E. F. Steiner, *Topologies with T_1-complements*, Fund. Math. 61 (1967), pp. 23–28.
[19] — *A T_1-complement for the reals*, Proc. Amer. Math. Soc. 19 (1968), pp. 177–179.

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