Peripheral and inner points

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In the theory of manifolds there is a well-defined notion of what points are and are not boundary points. A homotopic generalization of this notion has been introduced by Hopf and Pannwitz [7]. Those points corresponding to boundary points are called *labil points* and the remaining points are called *stable points*. Stable points and labil points have been studied, among other places, in [2], and [8].

The purpose of this paper is to study elementary properties of a cohomological definition of boundary and non-boundary points, what we call peripheral and inner points. We also relate these notions to those of the preceding paragraph. In a later paper we apply these properties to topological algebra.

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1. Basic definitions and equivalences. The Alexander cohomology theory will be employed throughout this paper; the coefficient group will be arbitrary unless specified. If X is a topological space, $H^*(X)$ will denote the graded cohomology group of X. Basic definitions, notation, and properties of Alexander cohomology and codimension, the dimension function we employ, may be found in [4] and [9].

DEFINITION 1.1. A point $x \in X$, a topological space, is *marginal* if for any open set U containing x, there exists an open set V containing x and contained in U such that $H^*(X, X\setminus V)$ is trivial.

A. D. Wallace [10] was one of the early investigators of cohomological boundary points. The preceding definition is closely akin to his. The next definition is essentially due to Hofmann and Mostert, although it was inspired by results of Bredon ([3], p. 76).

DEFINITION 1.2. A point $x \in X$, a topological space, is *peripheral* if for any open set U containing x, there exists an open set V containing x

and contained in U such that the homomorphism $i^*\colon H^*(X, X\setminus V) \to H^*(X, X\setminus U)$ induced by the inclusion mapping i is the trivial or zero homomorphism. A point is an *inner* point if it is not peripheral.

THEOREM 1.3. *In a regular space X the following are equivalent*:

(1) *the point x is marginal in X;*

(2) *the point x is marginal in K, a neighborhood of x;*

(3) *for any open set U such that $x \in U$, there exists an open set V containing x contained in U such that the natural homomorphism $H^*(X) \to H^*(X\setminus V)$ is an isomorphism.*

Proof. Assume x is marginal in X and K is a neighborhood of x. Suppose $x \in W$, an open subset in K. Let V and U be open neighborhoods of x in X such that $V^* \subset U \subset U^* \subset W$, and $H^*(X, X\setminus V)$ is trivial. By excision $H^*(U^*, U^*\setminus V)$ is trivial. Again by excision in the subspace K, $H^*(K, K\setminus V)$ is trivial. Hence x is marginal in K.

To show (2) implies (1), one simply reverses the preceding argument. That (1) and (3) are equivalent follows from the exact sequence for pairs.

THEOREM 1.4. *The following are equivalent in a regular space X*:

(1) *the point x is peripheral in X;*

(2) *there exists a basis $\mathcal{U}$ of open neighborhoods of x with the property that given $U \in \mathcal{U}$, there exists an open set V such that $x \in V \subset U$ and the natural map $H^*(X, X\setminus V) \to H^*(X, X\setminus U)$ is trivial;*

(3) *the point x is peripheral in K, a neighborhood of x;*

(4) *for any open set U containing x, there exists an open set V such that $x \in V \subset U$ and*

(a) *the natural mapping $H^*(X) \to H^*(X\setminus V)$ is one-to-one,*

(b) *if $h \in H^p(X\setminus V)$ for any p, then $h|(X\setminus U)$ extends to X.*

Proof. It is easily verified that (1) implies (2) by letting $\mathcal{U}$ be all open sets containing x. Conversely, suppose (2) is satisfied and that W is an open set containing x. Then there exist open sets U and V such that $x \in V \subset U \subset W$, $U \in \mathcal{U}$, and the natural homomorphism $H^*(X, X\setminus V) \to H^*(X, X\setminus U)$ is trivial. Since the natural homomorphism $H^*(X, X\setminus V) \to H^*(X, X\setminus W)$ factors through the preceding homomorphism, it is also trivial. Hence x is peripheral.

That (1) implies (3) follows from an application of the excision axiom similar to that in the proof of the preceding theorem.

To show (3) implies (2), we choose an open set W containing x such that $W^* \subset K^\circ$ and set $\mathcal{U}$ equal to all open sets U such that $x \in U$ and $U^* \subset W$. That $\mathcal{U}$ has the desired properties again follows from the excision axiom.

To show that (1) implies (4) we consider the following commutative diagram:

$$\begin{array}{ccccccc}
H^p(X, X\setminus V) & \longrightarrow & H^p(X) & \longrightarrow & H^p(X\setminus V) & \longrightarrow & H^{p+1}(X, X\setminus V)\\
\downarrow & & \downarrow{\scriptstyle m^*} & & \downarrow{\scriptstyle n^*} & & \downarrow\\
H^p(X, X\setminus U) & \longrightarrow & H^p(X) & \xrightarrow{i^*} & H^p(X\setminus U) & \longrightarrow & H^{p+1}(X, X\setminus U)
\end{array}$$

The rows are exact since they constitute part of the exact sequence for pairs; the vertical homomorphisms are those induced by inclusion. If U is an open neighborhood of x, V can be chosen so that the outside vertical homomorphisms are trivial. This fact, together with the observation that m^* is the identity, implies that the homomorphism $H^p(X) \to H^p(X\setminus V)$ is one-to-one and image $(n^*) \subset$ image (i^*) by diagram chasing.

Conversely, assume that U is an open set containing x and V is as in condition (4). Since $H^p(X) \to H^p(X\setminus V)$ is one-to-one, the homomorphism $H^p(X, X\setminus V) \to H^p(X)$ is trivial, and hence δ^* is onto in the following diagram:

$$\begin{array}{ccccc}
 & & H^{p-1}(X\setminus V) & \xrightarrow{\delta^*} & H^p(X, X\setminus V)\\
 & & \downarrow{\scriptstyle n^*} & & \downarrow\\
H^{p-1}(X) & \xrightarrow{i^*} & H^{p-1}(X\setminus U) & \longrightarrow & H^p(X, X\setminus U)
\end{array}$$

Since also image $(n^*) \subset$ image (i^*), it follows by diagram chasing that $H^p(X, X\setminus V) \to H^p(X, X\setminus U)$ is trivial.

DEFINITION 1.5. Let $x \in X$, a topological space. The open set U *surrounds* x if $x \in U$ and for any open neighborhood V of x such that $V \subset U$, the natural homomorphism $H^*(X, X\setminus V) \to H^*(X, X\setminus U)$ is non-trivial.

THEOREM 1.6. *The following are equivalent for a regular space X*:

(1) *the point x is an inner point of X;*

(2) *there exists an open set U which surrounds x;*

(3) *the point x is inner in K, a neighborhood of x.*

Proof. The equivalence of (1) and (2) follows directly from the definition of inner. The equivalence of (1) and (3) again follows from excision, or can be deduced directly from Theorem 1.4.

The following theorem is sometimes useful even though it is quite trivial. We leave the proof to the reader.

THEOREM 1.7. *Let $x \in X$, a topological space. If V and U are open sets, $x \in V \subset U$, and if U surrounds x, then V surrounds x.*

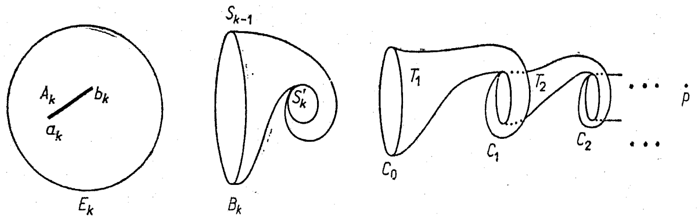
In the remainder of this section we compare the two notions of marginal and peripheral with each other and with the notion of stability.

THEOREM 1.8. *Let $x \in X$, a topological space. If x is marginal, then x is peripheral.*

Proof. Let U be an open neighborhood of x. There exists an open neighborhood V of x such that $V \subset U$ and $H^*(X, X\backslash V)$ is trivial. Hence the natural homomorphism $H^*(X, X\backslash V) \to H^*(X, X\backslash U)$ is trivial.

The following example shows that the converse of Theorem 1.8 is not true in general.

EXAMPLE 1.9. For each positive integer k let E_k be a 2-cell with distinguished points a_k and b_k in the interior. Let A_k be an arc in the interior of E_k with initial point a_k and final point b_k. Denote by B_k the identification space obtained from E_k by identifying a_k with b_k. Let S_{k-1} be the image of the boundary of E_k in the space B_k and S'_k the image of A_k in B_k. Suppose that $h_k\colon S_k \to S'_k$ is a homeomorphism for each integer $k \geqslant 2$. The space X' is obtained from the disjoint union of the B_k by identifying x in S_k with $h_k(x)$ in S'_k. Let X be the one-point compactification of X' and denote the compactifying point by p. Denote by T_k the image of B_k in X, by C_{k-1} the image of S_{k-1} in X, by L_k the union $\bigcup_{i=1}^{k} T_i$, and by R_k the subspace $(\bigcup_{i=k+1}^{\infty} T_i)^*$. Note that $L_k \cap R_k = C_k$.



We show that p is a peripheral point of X but p is not marginal in X.

Since A_k is a strong deformation retract of E_k, we conclude S'_k is a strong deformation retract of B_k. By piecing finitely many homotopies together, we conclude that for each $k \geqslant 1$, R_k is a deformation retract of X and C_k is a deformation retract of L_k. Hence the natural homomorphisms $H^*(X) \to H^*(R_k)$ and $H^*(L_k) \to H^*(C_k)$ are isomorphisms for each $k \geqslant 1$. By piecing together countably many homotopies, we conclude that the point p is a deformation retract of X, and hence X is acyclic.

To see that p is peripheral, we let U be an open set about p. Since X is acyclic, any open set V, $p \in V \subset U$, has the property that $H^*(X) \to H^*(X\backslash V)$ is one-to-one. We now choose such a V that satisfies (b) of Theorem 1.4 (4). Choose k so that $R_k \subset U$. Set $V = R_k\backslash C_k$ and $V' = R_{k+1}\backslash C_{k+1}$ so that $X\backslash V = L_k$ and $X\backslash V' = L_{k+1}$. The inclusion map $C_k \to L_{k+1}$ can be factored through E_{k+1} in a natural way $C_k \to E_{k+1} \to L_{k+1}$. Hence $H^*(L_{k+1}) \to H^*(C_k)$ is the zero homomorphism. In the following commutative diagram with homomorphisms i, j, and k induced by inclusions,

$$\begin{array}{ccc} H^*(C_k) & \xleftarrow{\ i\ } & H^*(L_{k+1}) \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & H^*(L_k) & \end{array}$$

k is an isomorphism, i is zero and hence j is zero. Thus $H^*(X\backslash V') \to H^*(X\backslash V)$ is zero and thus so is $H^*(X\backslash V') \to H^*(X\backslash U)$. Thus for $h \in H^*(X\backslash V')$ we have that $h|(X\backslash U) = 0$ extends to X.

We now indicate that p is not marginal. Let U be a small open set about p. Let $m = \max\{k\colon L_k \subset X\backslash U\}$; then $C_m \subset X\backslash U$. The cohomology classes of C_m extend to any closed superset which fails to contain T_{m+1}. Hence $X\backslash U$ has non-trivial cohomology in dimension 1, and thus the homomorphism $H^*(X) \to H^*(X\backslash U)$ is not an isomorphism. Since U was arbitrary, it follows from Theorem 1.3 that p is not marginal.

We now compare the concepts of peripherality and marginality to the concept of a labil point.

DEFINITION 1.10. A point p in X is *labil* if for each open set U about p there is a continuous function $F\colon X \times I \to X$ satisfying:

(1) $F(x, 0) = x$ for each x in X,

(2) $F(x, t) = x$ for each x in $X\backslash U$ and t in I,

(3) $F(x, t) \in U$ for each x in U and t in I,

(4) $F(x, 1) \neq p$ for each x in X.

THEOREM 1.11. *If a point p of locally compact Hausdorff X is labil, then p is a peripheral point of X.*

Proof. Let U be an open set about p with U^* compact and $F\colon X \times I \to X$ be a continuous function satisfying (1)–(4) of Definition 1.10 for p and U. Then

$$F(X \times \{1\}) = F(X\backslash U \times \{1\}) \cup F(U^* \times \{1\}) = X\backslash U \cup F(U^* \times \{1\})$$

which is the union of two closed sets and is hence closed. Set $V = X\backslash F(X \times \{1\})$. Define $F_1\colon X \to X$ by $F_1(x) = F(x, 1)$ for each $x \in X$ and let $F'_1\colon (X, X\backslash U) \to (X, X\backslash V)$ and $F''_1\colon (X, X\backslash U) \to (X\backslash V, X\backslash V)$ be defined by F_1. Then the inclusion map $i\colon (X, X\backslash U) \to (X, X\backslash V)$ is homotopic to F'_1 and thus $F_1'^* = i^*$. Thus the following diagram commutes:

$$\begin{array}{ccc} H^*(X, X\backslash V) & \xrightarrow{\ i^*\ } & H^*(X, X\backslash U) \\ & {}_{j^*}\searrow \quad \nearrow^{F_1''^*} & \\ & H^*(X\backslash V, X\backslash V) & \end{array}$$

where j is an inclusion map. Since $H^*(X\setminus V, X\setminus V) = 0$ it follows that $i^* = 0$ and hence p is peripheral.

In order to show that the converse of Theorem 1.11 is not true we need a couple of results. If X is a topological space, then the cone of X, denoted CX, is the identification space $X\times I/X\times\{1\}$. The vertex of CX is the image of $(x, 1)\in X\times I$ in CX.

THEOREM 1.12. *Let X be a topological space and p be the vertex of CX. Then p is marginal if and only if X is acyclic.*

Proof. Identify X with $X\times\{0\}$ in CX. Suppose that p is marginal. Then by Theorem 1.3 there exists an open set V about p, $V\cap X=\emptyset$, so that the inclusion map induces an isomorphism $H^*(CX)\to H^*(CX\setminus V)$. Now X is a retract of $CX\setminus V$ and hence the inclusion map induces an onto homomorphism $H^*(CX\setminus V)\to H^*(X)$. Since CX is acyclic it follows that X is also.

Suppose that X is acyclic. Let U be an open set about p. Choose an open set V about p so that $V\subset U$ and $CX\setminus V$ has X as a deformation retract, e.g., let V to be the image of $X\times(t, 1]$ in CX. Hence $CX\setminus V$ is acyclic and hence the inclusion map induces an isomorphism $H^*(CX)\to H^*(CX\setminus V)$. Thus by Theorem 1.3, p is marginal.

THEOREM 1.13 (Kuperberg [8]). *If X is a compact Hausdorff space the vertex of CX is labil if and only if X is contractible.*

EXAMPLE 1.14. In view of the above results we choose a compact space X which is acyclic but not contractible, for example let $X=\{(x,y)\colon y=\sin(1/x),\ 0<x<1\}^*$. It follows that the vertex of CX is marginal but stable.

2. Existence of inner points.

The following concept of a roof in cohomology was suggested to us by P. Conner. It includes the roofs as defined by A. D. Wallace [10] and also satisfies a cohomology addition theorem and a "small roof" theorem which are included for completeness.

DEFINITION 2.1. Let A be a closed subset of a topological space X and G a proper subgroup of $H^q(A)$. A *G-roof* is a closed set R containing A satisfying:

(a) if $i\colon A\to R$ is the inclusion map then $i^*H^q(R)\subset G$;

(b) if S is a closed proper subset of R, $A\subset S$, and $j\colon A\to S$ is the inclusion map then $j^*H^q(S)\not\subset G$.

THEOREM 2.2. *Let A be a closed subset of a compact Hausdorff space X and G a proper subgroup of $H^q(A)$. If $i\colon A\to X$ is the inclusion map and $i^*H^q(X)\subset G$, then there is a G-roof in X.*

Proof. This result follows easily from an application of the Hausdorff maximality principle and the continuity of the Alexander theory.

THEOREM 2.3. *Let A be a closed subset of a compact Hausdorff space X with G a proper subgroup of $H^q(A)$. If R_1 and R_2 are distinct G-roofs then $H^{q+1}(R_1\cup R_2)\neq 0$.*

Proof. Since R_1 and R_2 are distinct there is an h in $H^q(A)\setminus G$ and h' in $H^q(R_1\cap R_2)$ so that $h'|A=h$. Consider the Mayer–Vietoris sequence:

$$\to H^q(R_1)\oplus H^q(R_2)\xrightarrow{I} H^q(R_1\cap R_2)\xrightarrow{\Delta} H^{q+1}(R_1\cup R_2)\to$$

Suppose $h' = I(k_1, k_2) = k_1|R_1\cap R_2 - k_2|R_1\cap R_2$.

Then $h = h'|A = k_1|A - k_2|A$ is in G. This is a contradiction and hence $\Delta(h')\neq 0$.

THEOREM 2.4. *Let X be a compact Hausdorff space and A a closed subset of X. Let G be a proper subgroup of $H^q(A)$, R a G-roof, and U an open subset of $R\setminus A$. If $i\colon U^*\setminus U\to U^*$ is the inclusion map and $K=i^*H^q(U^*)$, then U^* is a K-roof.*

Proof. Clearly K is a subgroup of $H^q(U^*\setminus U)$. To see that K is proper let $h\in H^q(A)\setminus G$ and $h'\in H^q(R\setminus U)$ so that $h'|A=h$. Consider the following diagram induced by inclusion maps:

$$\begin{array}{ccccc} H^q(R) & \longrightarrow & H^q(R\setminus U) & \longrightarrow & H^q(A)\\ \downarrow & & \downarrow & & \\ H^q(U^*) & \longrightarrow & H^q(U^*\setminus U) & & \end{array}$$

Suppose $h'' = h'|(U^*\setminus U)$ extends to $g\in H^q(U^*)$.

The Mayer–Vietoris sequence

$$\to H^q(R)\to H^q(U^*)\oplus H^q(R\setminus U)\xrightarrow{I} H^q(U^*\setminus U)\to$$

gives that $I(g, h')=0$ and hence h' extends to R. This is a contradiction and thus h'' does not extend to U^* and K is proper.

Let S be a closed proper subset of U^* that contains $U^*\setminus U$. For $j\colon U^*\setminus U\to S$ the inclusion map suppose that $j^*H^q(S)\subset K$. There exists $h\in H^q(A)\setminus G$ so that h extends to $h'\in H^q\big(S\cup(R\setminus U)\big)$. Let $h''=h'|(R\setminus U)$, $k'=h'|S$, and $k''=h''|(U^*\setminus U)$. By assumption there exists $k\in H^q(U^*)$ so that $k|(U^*\setminus U)=k''$. By the above Mayer–Vietoris argument it follows that h'' extends to R which is a contradiction. Thus U^* is a K-roof.

THEOREM 2.5. *If X is an n-dimensional compact Hausdorff space, A is a closed subspace of X, K is a proper subgroup of $H^{n-1}(A)$, and R is a K-roof, then x in $R\setminus A$ is an inner point of X.*

Proof. Suppose that x in $R\setminus A$ is peripheral. Let U be an open set about x so that $U\cap A=\emptyset$. Choose an open set V, $x\in V\subset U$, so that the inclusion map induces $H^*(X)\to H^*(X\setminus V)$ one-to-one and h in $H^*(X\setminus V)$ implies that $h|(X\setminus U)$ extends to X. Consider the following commutative diagram with horizontal Mayer–Vietoris sequences and vertical homomorphisms induced by inclusions:

$$\begin{array}{ccccccc}
H^{n-1}(R)\oplus H^{n-1}(X) & \xrightarrow{I} & H^{n-1}(R) & \longrightarrow & H^{n}(X) & \xrightarrow{J} & H^{n}(R)\oplus H^{n}(X)\\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'}\\
H^{n-1}(R)\oplus H^{n-1}(X\setminus V) & \xrightarrow{I'} & H^{n-1}(R\setminus V) & \xrightarrow{\Delta} & H^{n}(X\setminus V) & \xrightarrow{J'} & H^{n}(R)\oplus H^{n}(X\setminus V)\\
\downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l'} & & & & \\
H^{n-1}(R)\oplus H^{n-1}(X\setminus U) & \xrightarrow{I''} & H^{n-1}(R\setminus U) & & & & \\
 & & \downarrow{\scriptstyle k} & & & & \\
 & & H^{n-1}(A) & & & &
\end{array}$$

Choose $h\in H^{n-1}(A)\setminus K$ so that h extends to $h'\in H^{n-1}(R\setminus V)$. Since $\operatorname{cd}X=n$ and by the way V was chosen, i is an isomorphism. Since J has as one component an identity map, J is one-to-one; the homomorphism i' is the direct sum of one-to-one maps and is one-to-one. It follows that J' is one-to-one and hence that Δ is zero. Thus I' is onto and h' extends to (h_1, h_2) in $H^{n-1}(R)\oplus H^{n-1}(X\setminus V)$. By the way that V was chosen $\big(h_1, h_2|(X\setminus U)\big)$ extends to (h_1, h_2') in $H^{n-1}(R)\oplus H^{n-1}(X)$. Now $kI''l(h_1, h_2)$ $=kl'I'(h_1, h_2)=h$ and hence $kI''\big(h_1, h_2|(X\setminus U)\big)=h$. Then it follows $kI''lj(h_1, h_2')=h$ and hence $kl'j'I(h_1, h_2')=h$. This implies that $I(h_1, h_2')$ is an extension of h to R. This is a contradiction. Thus x is an inner point of X.

The above theorem yields the next result about the existence of inner points in locally compact finite dimensional Hausdorff spaces. With restrictions on the coefficient ring this result follows from [3], p. 76, and [5].

THEOREM 2.6. *If X is a finite dimensional locally compact Hausdorff space, then the set of inner points in X are dense in X.*

Proof. Suppose that $\operatorname{cd}X=n$. Let U be an open set in X. Then $\operatorname{cd}U=m\leqslant n$. Thus there is a compact subset B of U so that $\operatorname{cd}B=m$ [4]. Choose an open set V in X so that $B\subset V\subset V^*\subset U$ and V^* is compact. It follows that $\operatorname{cd}V^*=\operatorname{cd}V=m$. By the definition of codimension [4] there is a closed subset A of B and $h\in H^{m-1}(A)$ so that, for $i\colon A\to B$ the inclusion map, $h\notin i^*H^{m-1}(B)=K$. Then K is a proper subgroup of $H^{m-1}(A)$. Choose a K-roof R and it follows from the previous theorem that $x\in R\setminus A$ is an inner point of V^*. Since $x\in B\subset V$, by Theorem 1.6 x is an inner point of X.

3. Elementary properties of peripheral and inner points. If X is a metric space, $x\in X$, and ε is a positive real number, $N(x;\varepsilon)$ will denote the set of points of X whose distance to x is less than ε. A set is F_σ if it is the countable union of closed sets and G_δ if it is the countable intersection of open sets.

THEOREM 3.1. *Let X be a metric space. The inner points are F_σ, or equivalently, the peripheral points are G_δ.*

Proof. We set $F_n=\{x\in X\colon N(x;1/n) \text{ surrounds } x\}$ for each positive integer n. It follows from Theorems 1.6 and 1.7 that $\bigcup F_n$ is exactly the set of inner points.

We show $F_n^*\subset F_{n+1}$. Let $\{x_k\}$ be a sequence in F_n converging to x. Let V be an open neighborhood of x contained in $N\big(x;1/(n+1)\big)$. There exists an integer m such that $x_m\in V$ and $N\big(x;1/(n+1)\big)\subset N(x_m;1/n)$. The following triangle induced by inclusion mappings is commutative:

$$\begin{array}{ccc}
H^*(X, X\setminus V) & \xrightarrow{j^*} & H^*\big(X, X\setminus N(x_m;1/n)\big)\\
 {\scriptstyle i^*}\searrow & & \nearrow\\
 & H^*\big(X, X\setminus N(x;1/(n+1))\big) &
\end{array}$$

Since $N(x_m, 1/n)$ surrounds x_m, j^* is not trivial. Hence i^* is not trivial. Since V was arbitrary, $N\big(x;1/(n+1)\big)$ surrounds x; thus $x\in F_{n+1}$. Thus the inner points are equal to $\bigcup F_n^*$.

As a corollary to this theorem, we note that if the peripheral points are dense, then the inner points are a set of first category. Doyle and Hocking [6] give an example of a tree (a one-dimensional Peano continuum with no simple closed curves) in which the endpoints are dense. Borsuk and Jaworowski have shown that endpoints in trees are labil [2], pp. 161, 162; hence by Theorem 1.11, they are peripheral.

THEOREM 3.2. *Let X be a compact Hausdorff space of codimension n. If the peripheral points form a closed set, then they have codimension m where $m\leqslant n-1$.*

Proof. Suppose the peripheral points P have codimension n. Then there exists a compact subset A of P such that $H^{n-1}(P)\to H^{n-1}(A)$ is not onto. Let K be the image of $H^{n-1}(P)$ in $H^{n-1}(A)$ and let R be a K-roof contained in P. By Theorem 2.5 $x\in R\setminus A$ is an inner point; this is a contradiction since $R\subset P$.

If the coefficient group is a field, we obtain the following theorem concerning Cartesian products:

THEOREM 3.3. *Let X and Y be compact, Hausdorff spaces. If x and y are inner points of X and Y resp., then (x, y) is an inner point of $X\times Y$. If x or y is peripheral, then (x, y) is peripheral.*

Proof. Suppose that x and y are both inner. Then there exists an open set U_1 which surrounds x and an open set U_2 which surrounds y.

We show $U_1\times U_2$ surrounds (x,y). Let V_1 be an open neighborhood of x contained in U_1, and V_2 an open neighborhood of y contained in U_2. There exist non-negative integers p and q such that $H^p(X, X\setminus V_1)\to H^p(X, X\setminus U_1)$ and $H^q(Y, Y\setminus V_2)\to H^q(Y, Y\setminus U_2)$ are non-trivial. Since we are assuming the coefficient group is a field, by the Kunneth formula [9], p. 360, the horizontal homomorphisms in the following commutative square are isomorphisms:

$$\begin{array}{ccc}
\sum\limits_{i+j=p+q} H^i(X, X\setminus V_1)\otimes H^j(Y, Y\setminus V_2) & \to & H^{p+q}\big(X\times Y, (X\setminus V_1)\times Y\cup X\times (Y\setminus V_2)\big)\\
\downarrow & & \downarrow\\
\sum\limits_{i+j=p+q} H^i(X, X\setminus U_1)\otimes H^j(Y, Y\setminus U_2) & \to & H^{p+q}\big(X\times Y, (X\setminus U_1)\times Y\cup X\times (Y\setminus U_2)\big)
\end{array}$$

Again, since the coefficients are a field, the homomorphism $H^p(X, X\setminus V_1)\otimes H^q(Y, Y\setminus V_2)$ into $H^p(X, X\setminus U_1)\otimes H^q(Y, Y\setminus U_2)$ is non-trivial; hence the left vertical homomorphism is non-trivial. Hence the right vertical homomorphism is non-trivial. Since for any open set V containing (x,y) and contained in $U_1\times U_2$, there exist open sets V_1 and V_2 such that $(x,y)\in V_1\times V_2\subset V$, we conclude that (x,y) is inner.

A similar argument shows that if x or y is peripheral, then (x,y) is peripheral.

We note that by passing to compact neighborhoods and employing Theorems 1.5 and 1.6 the preceding theorem generalizes to products of locally compact spaces. A similar result also obtains for locally compact fiber bundles by making use of the locally trivial structure.

The next theorem is due to K. Hofmann and is included here with his kind permission. Note that the contrapositive gives a condition for the existence of peripheral points.

THEOREM 3.4. *Let X be a compact Hausdorff space, T a topological space with distinguished point* 1, *and $\Gamma\colon T\times X\to X$ a continuous function with $\Gamma(1, x) = x$ for each x in X. If p is an inner point of X there is an open set U containing* 1 *so that if t is in the component of* 1 *in U, denoted $C_0(U)$, then $p\in\Gamma(\{t\}\times X)$.*

Proof. Suppose that the conclusion is false. For $t\in T$, we let $\Gamma_t\colon X\to X$ be the function defined by $\Gamma_t(x) = \Gamma(t, x)$. Since p is inner in X there is an open set W that surrounds p. Choose an open set V containing p so that $V^*\subset W$. Then $\Gamma_1(X\setminus W)\subset X\setminus V^*$. For $x\in X\setminus W$ choose an open set V_x containing x and U_x containing 1 so that for $t\in U_x$ we have $\Gamma_t(V_x)\subset X\setminus V^*$. Since $X\setminus W$ is compact there is a finite set $\{V_{x_1}, \dots, V_{x_k}\}$ that covers $X\setminus W$. Set $U = \bigcap_{i=1}^{k} U_{x_i}$. Then for $t\in U$ we have $\Gamma_t(X\setminus W)\subset X\setminus V^*\subset X\setminus V$. By the assumption that the conclusion of the theorem is false there is $t\in C_0(U)$ so that $p\notin \Gamma_t(X)$. Thus $V' = V\cap X\setminus\Gamma_t(X)$ is an open set about p and is contained in W. By the generalized homotopy theorem, if i and j are inclusion maps, the following diagram commutes:

$$\begin{array}{ccc}
H^*(X, X\setminus V') & \xrightarrow{\ \Gamma_t^* = i^*\ } & H^*(X, X\setminus W)\\
 & {\scriptstyle j^*}\searrow \qquad \nearrow{\scriptstyle \Gamma_t^*} & \\
 & H^*(X\setminus V', X\setminus V') &
\end{array}$$

Since $H^*(X\setminus V', X\setminus V') = 0$ it must be that $i^* = 0$, but this is contrary to the fact that W surrounds p. Thus the proof is complete.

DEFINITION 3.5. Let X be a metric space. The space X is *uniformly inner at x* if there exists an open set U containing x and an $\varepsilon > 0$ such that if $y\in U$, then $N(y;\varepsilon)$ surrounds y.

THEOREM 3.6. *If each point of a locally compact metric space X is inner, then the set of uniformly inner points is open and dense.*

Proof. We again set

$$F_n = \{x\in X\colon\ N(x; 1/n)\ \text{surrounds}\ x\}.$$

We saw in the proof of Theorem 3.1 that $X = \bigcup F_n^*$. If V is an open subset of X, there exists an integer m such that $(F_m^*)^\circ\cap V\neq\emptyset$ by Baire's theorem. Since $F_m^*\subset F_{m+1}$, we conclude that $F_{m+1}^\circ\cap V\neq\emptyset$. Taking $U = F_{m+1}^\circ\cap V$ and $\varepsilon = 1/(m+1)$, we conclude that X is uniformly inner at each point of U. Hence the uniformly inner points are dense. It follows easily from the definition that they are open.

The next theorem states that the property of being uniformly inner is a local property.

THEOREM 3.7. *Suppose X and Y are locally compact metric, U is an open neighborhood of x in X, V is an open neighborhood of y in Y, and h is a homeomorphism from U onto V taking x to y. If X is uniformly inner at x, then Y is uniformly inner at y.*

Proof. By passing to smaller neighborhoods if necessary we may assume that h is defined on U^*, U^* is compact, and there exists $\varepsilon > 0$ such that $N(x';\varepsilon)$ surrounds x' for all $x'\in U$. Then V^* is also compact; hence there exists $\delta_1 > 0$ such that if the distance between y_1 and y_2 in V^* is less than δ_1, then the distance between $h^{-1}(y_1)$ and $h^{-1}(y_2)$ is less than ε. Pick δ_2 such that $N(y; 2\delta_2)\subset V$. Let $\delta = \min\{\delta_1, \delta_2\}$ and $W = N(y;\delta)$.

To show Y is uniformly inner at y, we show $N(y';\delta)$ surrounds y' for any $y'\in W$. Since $\delta < \delta_2$, we have $N(y';\delta)\subset V$. Since $\delta<\delta_1$, $h^{-1}\big(N(y';\delta)\big)\subset N\big(h^{-1}(y');\varepsilon\big)$. By Theorem 1.7, $h^{-1}\big(N(y';\delta)\big)$ surrounds $h^{-1}(y')$. Since h

is a homeomorphism and by use of excision techniques as in Theorems 1.3, 1.4, and 1.6, we conclude that $N(y'; \delta)$ surrounds y'.

The next theorem is an analog for uniformly inner points of Theorem 3.4.

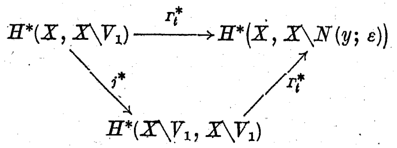
THEOREM 3.8. *Let X be a compact, metric space, Γ a continuous mapping from $T \times X$ into X, and $1 \in T$ such that Γ_1 is the identity map on X. If X is uniformly inner at p, then there exists an open set U containing* 1, *an open set V containing p, such that if $t \in C_0(U)$, then $V \subset \Gamma_t(X)$.*

Proof. There exist an open neighborhood W of p and $\varepsilon > 0$ such that if $y \in W$, then $N(y; \varepsilon)$ surrounds y. Choose δ such that $N(p; \delta) \subset W$ and $\delta < \varepsilon/2$; choose an open set V containing p such that $V^* \subset N(p; \delta)$. As in the proof of Theorem 3.4, there exists an open set U containing 1 such that $\Gamma_t(X \setminus N(p; \delta)) \subset X \setminus V$ for all $t \in U$.

Suppose there exists $t \in C_0(U)$, $y \in V$, such that $y \notin \Gamma_t(X)$. We set $V_1 = V \cap (X \setminus \Gamma_t(X))$; then $y \in V_1$. If $z \in N(p; \delta)$, then

$$d(y, z) \leqslant d(y, p) + d(p, z) < 2\delta < \varepsilon .$$

Hence $X \setminus N(y; \varepsilon) \subset X \setminus N(p; \delta)$; thus for all $s \in C_0(U)$, $\Gamma_s(X \setminus N(y; \varepsilon)) \subset X \setminus V \subset X \setminus V_1$. Let j be the injection of $(X \setminus V_1, X \setminus V_1)$ into $(X, X \setminus V_1)$. The following triangle is commutative:



By the generalized homotopy lemma (see [4]) the horizontal Γ_t^* induced by Γ_t is equal to Γ_1^*. Since Γ_1 is just the injection of $(X, X \setminus N(y; \varepsilon))$ into $(X, X \setminus V_1)$ and since $H^*(X \setminus V_1, X \setminus V_1)$ is trivial, we conclude the injection induces a trivial homomorphism. This contradicts the fact $N(y; \varepsilon)$ surrounds y.

If a space X has all uniformly inner points, we get the following stability condition on X.

COROLLARY 3.9. *Suppose X and T satisfy the hypotheses of the preceding theorem with the added assumption that all points of X are uniformly inner. Then there exists an open set U containing* 1 *such that if $t \in C_0(U)$, then $\Gamma_t(X) = X$.*

Proof. Choose for each $x \in X$ an open set U_x containing 1, an open set V_x containing x, such that if $t \in C_0(U_x)$ then $\Gamma_t(X) \supset V_x$.

By taking a finite subcover of $\{V_x\}_{x \in X}$, intersecting the corresponding U_x, we get the desired open set U.

A space X is said to be *locally homogeneous* if given $x, y \in X$, there exist open neighborhoods U and V of x and y resp. and a homeomorphism from U onto V which carries x to y. Note that any homogeneous space is locally homogeneous.

THEOREM 3.10. *Let X be a locally homogeneous locally compact metric space. If X has an inner point, then all points of X are uniformly inner. In particular, if X is finite dimensional, then all points of X are uniformly inner.*

Proof. Suppose X has an inner point x and $y \in X$. Let $h\colon (U, x) \to (V, y)$ be a homeomorphism of open neighborhoods. By Theorem 1.6 x is inner in U, and hence y is inner in V. Again by Theorem 1.6, y is inner in X. Hence all points of X are inner. It then follows from Theorem 3.6 that X has an uniformly inner point, and from Theorem 3.7 that all points are uniformly inner.

If X is finite dimensional, then X has an inner point by Theorem 2.6.

As a corollary of this theorem, we note that all points of Euclidean space are uniformly inner. Thus by Theorem 3.7 if a point in a metric space has an Euclidean neighborhood, then the point is uniformly inner.

Corollary 3.9 and Theorem 3.10 imply if X is a finite-dimensional, locally homogeneous, compact metric space and $F\colon X \times I \to X$ is a homotopy of X, then $F_t(X) = X$ for all t in some neighborhood of 0 if F_0 is the identity. Bing and Borsuk [1] have raised the question whether a retract of an n-cell can be locally homogeneous. If one could show such retracts admitted a homotopy F such that F_0 was the identity and the range of F_t was proper when $t \neq 0$, then it would follow that none of them could be locally homogeneous.

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A class of topologies with T_1-complements

by

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1. Introduction. Let Λ be the family of all T_1 topologies definable on an arbitrary set X. When $\tau_1 \in \Lambda$ and $\tau_2 \in \Lambda$, $\tau_1 < \tau_2$ if every set in τ_1 is in τ_2. Under this order, Λ is a complete lattice. The greatest element of Λ is the discrete topology, 1, and the least element is the cofinite topology $C = \{U\colon U = \emptyset$ or $X - U$ is finite$\}$.

Recently several papers have been published dealing with the structure of the lattice Λ. An example [17] was given to show that Λ is not a complemented lattice, unless X is a finite set. In [19], a T_1-complement for the reals with the usual topology is constructed. This result was generalized in [1] to yield the fact that every T_1 space with a countable dense metric subspace has a T_1-complement. For other results on the lattice Λ, see [4].

The main purpose of this paper is to show that the construction used in [19] can be made to do much more than has been previously realized. It turns out to be quite an interesting exercise to see how much of the construction in [19] can be jettisoned. Now it appears that large classes of nice topological spaces have T_1-complements. For example, it can be proved that every first axiom Hausdorff space has a T_1-complement and that every locally compact Hausdorff space has a T_1-complement. Actually the theorems deduced here are quite a bit stronger than these statements. Another result of [1] is extended to show that there is a large class of spaces (X, T) and that T is one of three mutually T_1-complementary topologies on the set X. Furthermore, it is shown that every T_1 space is an open and closed subspace of a T_1 space that has a T_1-complement.

Lastly, some questions are raised. I am indebted to Roger Countryman for an interesting conversation on the properties of Fréchet spaces and symmetrizable spaces.

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