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On the strong local simple connectivity of the decomposition spaces of toroidal decompositions*

by

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1. Introduction. A topological space X is *strongly locally simply connected* if and only if each point of X has arbitrarily small simply connected open neighborhoods.

An upper semicontinuous decomposition G of E^3 is a *toroidal* decomposition of E^3 if and only if there is a sequence $M_0, M_1, M_2, \ldots$ of compact 3-manifolds-with-boundary in E^3 such that (1) for each i, $M_{i+1} \subset \operatorname{Int} M_i$ and each component of M_i is a solid torus (cube with one handle) and (2) g is a non-degenerate element of G if and only if g is a non-degenerate component of $\bigcap_{i=0}^{\infty} M_i$.

The main result of this paper is that the decomposition spaces of a certain class of pointlike toroidal decompositions of E^3 are not strongly locally simply connected. We shall now state in greater detail the main result of this paper.

A toroidal decomposition G of E^3 is *simple* if and only if, in addition to the conditions described previously, (1) M_0 is a solid torus T_0 and (2) if $i = 0, 1, 2, \ldots$ and T_a is any component of M_i, T_a is polyhedral and the components of M_{i+1} in T_a form a chain of solid tori circling T_a. We denote by m_a the number of solid tori in this chain, and by n_a the number of times the chain circles T_a.

Let G be a pointlike simple toroidal decomposition of E^3 such that for each index a, $m_a < 2n_a$. It follows from the results of [4] and essentially from [13] that in this case, the associated decomposition space is topologically distinct from E^3. In this paper, we shall establish the following stronger result:

For each such decomposition G of E^3, the associated decomposition space is not strongly locally simply connected.

In Section 3 of [7], Bing describes an interesting toroidal decomposition G of E^3 such that the associated decomposition space E^3/G is

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topologically distinct from E^3. One corollary of the main result of this paper is that E^3/G is not strongly locally simply connected. In [4] there is described a toroidal decomposition of E^3 into tame arcs and one-point sets such that the associated decomposition space is not homeomorphic to E^3. It follows from the results of this paper that this decomposition space also fails to be strongly locally simply connected. Another corollary deals with the (m, n)-spaces studied by Sher [13]. An (m, n)-space is the decomposition space of a certain type of simple toroidal decomposition of E^3. In the notation above, these have the property that for each index a, $m_a = m$ and $n_a = n$. Sher proved [13] that if $m < 2n$, an (m, n)-space is topologically distinct from E^3. It follows from the main result of this paper that if X is an (m, n)-space associated with a pointlike decomposition and $m < 2n$, then X is not strongly locally simply connected.

There are now several results concerning local topological properties of decomposition spaces. It is shown in [1] that the dogbone decomposition space described by Bing [6] is not strongly locally simply connected.

We define a topological space X to be *locally peripherally spherical* provided each point of X has arbitrarily small neighborhoods whose boundaries are 2-spheres (Lambert [8] uses the term "locally spherical"). Bing proved that the toroidal decomposition of Section 3 of [7] was not locally peripherally spherical. Lambert [8] has studied locally peripherally spherical spaces of toroidal decompositions. Lambert [9] has shown that the dogbone decomposition space [6] is not locally peripherally spherical. In [2], we study decomposition spaces in which each point has arbitrarily small compact, locally connected, simply connected neighborhoods.

2. Terminology and notation. If G is an upper semicontinuous decomposition of E^3, then E^3/G denotes the associated decomposition space and P denotes the projection map from E^3 onto E^3/G. The union of all the non-degenerate elements of G is denoted by H_G.

A continuum K in E^3 is *pointlike* if and only if $E^3 - K$ is homeomorphic to the complement in E^3 of any one-point subset. By a *pointlike decomposition* of E^3 is meant an upper semicontinuous decomposition of E^3 into pointlike continua.

Suppose that n is a positive integer. The statement that M is an *n-manifold* means that M is a separable metric space, each point of which has open n-cell neighborhood. The statement that M is an n-manifold-with-boundary means that M is a separable metric space, each point of which has an n-cell neighborhood. Suppose that M is an n-manifold-with-boundary. If x is a point of M, then x is an *interior point* of M if and only if x has an open n-cell neighborhood. The *interior* of M, denoted by Int M, is the set of all interior points. The *boundary* of M, Bd M, is $M - \operatorname{Int} M$.

Suppose that T is a solid torus in E^3, and J is a simple closed curve on Bd T. J is *trivial* on Bd T if and only if J bounds a disc on Bd T. J is *meridional* on Bd T if and only if J is non-trivial on Bd T but J bounds a disc in T. J has *non-zero longitudinal component* on Bd T if and only if J is neither trivial nor meridional on Bd T. A *meridional disc* in T is a disc D such that Bd D is a meridional simple closed curve on Bd T and such that Int $D \subset$ Int T.

Suppose that T is a polyhedral solid torus in E^3 and D is a polyhedral disc such that (1) Bd D and T are disjoint and (2) D and Bd T are in relative general position. Each component of $D \cap$ Bd T is a simple closed curve. D intersects Bd T *non-trivially* if and only if some component of $D \cap$ Bd T is non-trivial on Bd T. D intersects Bd T *meridionally* if and only if some component of $D \cap$ Bd T is meridional on Bd T.

If T is a solid torus, then we denote the universal covering space of T by T^*, and the associated projection map by φ.

We use "$\sim$" to mean "is homotopic to". "Cl" denotes closure.

3. Toroidal decompositions. Suppose T is a solid torus. By a *chain* of solid tori in T we shall mean a set $\{T_1, T_2, \ldots, T_m\}$ of mutually disjoint solid tori in Int T such that (1) $m \geqslant 2$, (2) for each i, T_i lies in a 3-cell in T, and (3) for each i, if T_i^* and T_{i+1}^* are adjacent copies of T_i and T_{i+1}, respectively, in T^*, then T_i^* and T_{i+1}^* are linked (relative to the integers) in T^*.

If T is a polyhedral solid torus $\{T_1, T_2, \ldots, T_m\}$ is a chain of solid tori in T, it is possible to define a *winding number* of $\{T_1, T_2, \ldots, T_m\}$ in T. This was done for chains of a certain type by Sher in [13]. A treatment suitable for our purposes can be obtained by slight modifications of Sher's treatment, or by using the universal covering space T^* of T. In either case, one depends heavily on an extension of Theorem 3 of [6] to the case of linking relative to the integers.

If, in the notation of the preceding paragraph, the winding number of $\{T_1, T_2, \ldots, T_m\}$ is n, we shall say that $\{T_1, T_2, \ldots, T_m\}$ *circles T n times*.

The definitions of toroidal decomposition of E^3 and of *simple* toroidal decomposition of E^3 are given in Section 1. We remark that there is no loss in generality in assuming that, for any toroidal decomposition, the defining sets $M_0, M_1, M_2, \ldots$ are polyhedral. If they are not polyhedral, we may, by [5], replace them by polyhedral ones.

If G is any simple toroidal decomposition of E^3, we adopt a notational scheme which we shall now describe and which will be followed from the beginning of Section 5 to the end of the paper. Suppose than that G is a simple toroidal decomposition of E^3. There is a sequence $M_0, M_1, M_2, \ldots$ of compact 3-manifolds-with-boundary as described in Section 1. Let T_0 denote M_0; T_0 is a polyhedral solid torus. Let $T_1, T_2, \ldots$, and T_{m_0} denote the components of M_1. $\{T_1, T_2, \ldots, T_{m_0}\}$ is a chain of polyhedral solid tori in T_0, and $m_0 \geqslant 2$. Let n_0 denote the number of times the chain $\{T_1, T_2, \ldots, T_{m_0}\}$ circles T_0.

Suppose j is a positive integer. If t is a component of M_j, then t will be denoted by $T_{i_1 i_2 \ldots i_j}$ where for certain positive integers $m_0, m_{i_1}, m_{i_1 i_2}, \ldots$, and $m_{i_1 i_2 \ldots i_{j-1}}$, we have that $1 \leqslant i_1 \leqslant m_0$, $1 \leqslant i_2 \leqslant m_{i_1}$, $1 \leqslant i_3 \leqslant m_{i_1 i_2}, \ldots$, and $1 \leqslant i_{j-1} \leqslant m_{i_1 i_2 \ldots i_{j-1}}$. The components of M_{j+1} in $T_{i_1 i_2 \ldots i_j}$ will be denoted by $T_{i_1 i_2 \ldots i_j 1}, T_{i_1 i_2 \ldots i_j 2}, \ldots$, and $T_{i_1 i_2 \ldots i_j m_{i_1 i_2 \ldots i_j}}$ where $\{T_{i_1 i_2 \ldots i_j 1}, T_{i_1 i_2 \ldots i_j 2}, \ldots, T_{i_1 i_2 \ldots i_j m_{i_1 i_2 \ldots i_j}}\}$ is a chain of polyhedral solid tori in $T_{i_1 i_2 \ldots i_j}$ and $m_{i_1 i_2 \ldots i_j} \geqslant 2$. Let $n_{i_1 i_2 \ldots i_j}$ denote the number of times this chain circles $T_{i_1 i_2 \ldots i_j}$.

The statement that a is an *index* means that either $a = 0$ or for some positive integer n, $a = i_1 i_2 \ldots i_n$ where $1 \leqslant i_1 \leqslant m_0$ and for $k = 2, 3, \ldots$, or $n-1$, $1 \leqslant i_k \leqslant m_{i_1 i_2 \ldots i_{k-1}}$. If a is the index $i_1 i_2 \ldots i_n$, then ai denotes $i_1 i_2 \ldots i_n i$, aij denotes $i_1 i_2 \ldots i_n ij$, and so on. Hence, if a is any index, there is a solid torus T_a and a chain $\{T_{a1}, T_{a2}, \ldots, T_{am_a}\}$ of solid tori circling T_a n_a times.

In [13], Sher defines (m, n)-spaces. There are the decomposition spaces of certain toroidal decompositions of E^3. The decompositions of E^3 considered by Sher are simple toroidal decompositions satisfying certain additional conditions. If m and n are integers such that $m \geqslant 2$ and $n \geqslant 0$ then by an (m, n)-*space* is meant the decomposition space E^3/G of a simple toroidal decomposition G of E^3 satisfying additional conditions specified in [13] and such that for each index a, $m_a = m$ and $n_a = n$.

Sher proved in [13] that if X is an (m, n)-space for which $m < 2n$, then X is not homeomorphic to E^3. It follows by the results of Section 3 of [4] that if G is any simple toroidal decomposition of E^3 such that for each index a, $m_a < 2n_a$, then E^3/G is not homeomorphic to E^3.

Throughout this paper, indices are to be computed cyclically.

4. Preliminary lemmas on tori. We need the following slight extension of Theorem 1 of [7].

LEMMA 0. *Suppose that, in E^3, T is a polyhedral solid torus and D is a polyhedral disc such that* (1) BdD *and* BdT *are disjoint, and* (2) D *and* BdT *are in relative general position. If J is a component of $D \cap$* BdT, *then either* (1) J *bounds a disc on* BdT, (2) J *circles* BdT *exactly once meridionally and no times longitudinally, or* (3) J *circles* BdT *exactly once longitudinally.*

Proof. Suppose J is a component of $D \cap \mathrm{Bd}\,T$ such that J does not bound a disc on BdT. By an argument in the proof of Theorem 1 of [7], J bounds a polyhedral disc D' such that IntD' and BdT are disjoint. If $\mathrm{Int}\,D' \subset \mathrm{Int}\,T$, then J circles BdT exactly one time meridionally and no times longitudinally. If $\mathrm{Int}\,D' \subset E^3 - T$, then T has an unknotted centerline and hence T is unknotted. Consequently, J circles BdT exactly one time longitudinally. Since each component of $D \cap \mathrm{Bd}\,T$ that does not bound a disc on BdT is parallel to J, Lemma 0 follows.

LEMMA 1. *Suppose that in E^3, T is a polyhedral solid torus and D is a polyhedral disc such that* (1) BdD *and* T *are disjoint,* (2) D *and* BdT *are in relative general position, and* (3) *there is a component J of $D \cap$* BdT *such that J has non-zero longitudinal component. If U is a neighborhood of T, then there is a disc D' such that* (1) $\mathrm{Bd}\,D' = \mathrm{Bd}\,D$, (2) $D' - U = D - U$, *and* (3) D' *and* T *are disjoint.*

Proof. By Lemma 0, if L is a component of $(\mathrm{Bd}\,T) \cap D$, either L bounds a disc on BdT, or L circles BdT exactly once longitudinally, or L circles BdT exactly once meridionally and no times longitudinally. Therefore J circles T exactly once longitudinally. Since the components of $(\mathrm{Bd}\,T) \cap D$ are either trivial on BdT or parallel to J, there is a polygonal simple closed curve on $(\mathrm{Bd}\,T) - D$ and parallel to J. There is, therefore, a polygonal centerline J_0 of T lying in Int$T$ and disjoint from D. Let T_0 be a polyhedral tubular neighborhood of J_0 lying in Int$T$ and disjoint from D. There is a piecewise linear homeomorphism h from E^3 onto E^3 such that (1) $h[T_0] = T$ and (2) if either $x \in E^3 - U$, or $x \in \mathrm{Bd}\,D$, $h(x) = x$. Let D' denote $h[D]$. It is easy to see that D' satisfies the conclusion of Lemma 1.

LEMMA 2. *Suppose that T is a polyhedral solid torus in E^3, $m \geqslant 2$, $n \geqslant 0$, and $\{T_1, T_2, \ldots, T_m\}$ is a chain of polyhedral solid tori in* Int$T$ *circling T n times. Suppose D is a polyhedral meridional disc in T such that if $j = 1, 2, \ldots$, or m, D and* BdT_j *are in relative general position. Let T^* denote the universal covering space of T. If $m < 2n$, there exist an integer i such that $i \leqslant i \leqslant m$ and consecutive copies D_1 and D_2 of D in T^* such that either* (1) *some copy T_i^* of T_i in T^* intersects both D_1 and D_2 meridionally or* (2) *there exist adjacent copies T_i^* of T_i and T_{i+1}^* of T_{i+1}, both in T^*, such that* (a) *T_i^* intersects D_1 meridionally and* (b) *T_{i+1}^* intersects D_2 meridionally.*

Proof. First we shall show that there exists a polyhedral meridional disc E in T such that (1) $\mathrm{Bd}\,E = \mathrm{Bd}\,D$ and (2) if $j = 1, 2, \ldots$, or m, then (a) E and BdT_j are in relative general position and (b) each component of $E \cap \mathrm{Bd}\,T_j$ is meridional on BdT_j and is a component of $D \cap \mathrm{Bd}\,T_j$.

We first eliminate curves of intersection of D with $\mathrm{Bd}\,T_1, \mathrm{Bd}\,T_2, \ldots$, and $\mathrm{Bd}\,T_m$ that are trivial on $\mathrm{Bd}\,T_1, \mathrm{Bd}\,T_2, \ldots$, and $\mathrm{Bd}\,T_m$, respectively. Suppose there is a component L of $D \cap \mathrm{Bd}\,T_1$ such that $L \sim 0$ on $\mathrm{Bd}\,T_1$.

Then L bounds a disc on $\mathrm{Bd}\,T_1$ and there exists a component L_0 of $D\cap \mathrm{Bd}\,T_1$ such that L_0 bounds a disc Δ_0 on $\mathrm{Bd}\,T_1$ such that $\mathrm{Int}\,\Delta_0$ misses D. Let Δ_0' be the disc on D bounded by L_0. We can replace Δ_0' by Δ_0, adjust the resulting disc slightly, and obtain a polyhedral meridional disc D'' in T such that (1) $\mathrm{Bd}\,D''=\mathrm{Bd}\,D$, (2) $D''\cap \mathrm{Bd}\,T_1$ has fewer components than $D\cap \mathrm{Bd}\,T_1$, and (3) if $j=1,2,\ldots$, or m, each component of $D''\cap \mathrm{Bd}\,T_j$ is a component of $D\cap \mathrm{Bd}\,T_j$. A continuation of this process yields a polyhedral meridional disc D^1 in T such that (1) $\mathrm{Bd}\,D^1=\mathrm{Bd}\,D$, (2) each component of $D^1\cap \mathrm{Bd}\,T_1$ is non-trivial on $\mathrm{Bd}\,T_1$, and (3) if $j=1,2,\ldots$, or m, each component of $D^1\cap \mathrm{Bd}\,T_j$ is a component of $D\cap \mathrm{Bd}\,T_j$.

The process described above be repeated using D^1 and $\mathrm{Bd}\,T_2$, yielding a disc D^2 having properties analogous to those described above. Additional repetitions yield a polyhedral meridional disc E^0 in T such that (1) $\mathrm{Bd}\,E^0=\mathrm{Bd}\,D$ and (2) if $j=1,2,\ldots$, or m, each component of $E^0\cap \mathrm{Bd}\,T_j$ is non-trivial on $\mathrm{Bd}\,T_j$ and is a component of $D\cap \mathrm{Bd}\,T_j$.

We now eliminate curves of intersection of E^0 with $\mathrm{Bd}\,T_1, \mathrm{Bd}\,T_2, \ldots$, and $\mathrm{Bd}\,T_m$ that have non-zero longitudinal component on $\mathrm{Bd}\,T_1, \mathrm{Bd}\,T_2, \ldots$, and $\mathrm{Bd}\,T_m$, respectively. Suppose there is a component of $E^0\cap \mathrm{Bd}\,T_1$ which has non-zero longitudinal component. By Lemma 1, there is a polyhedral disc E^1 such that $\mathrm{Bd}\,E^1=\mathrm{Bd}\,E^0$, E^1 and T_1 are disjoint, and, except in a neighborhood of $T_2\cup T_3\cup\ldots\cup T_m$ that misses T_1, $E^1=E^0$. Repetition of this process yields a polyhedral meridional disc E in T such that (1) $\mathrm{Bd}\,E=\mathrm{Bd}\,D$, (2) if $j=1,2,\ldots$, or m, (a) E and $\mathrm{Bd}\,T_j$ are in relative general position and (b) each component of $E\cap \mathrm{Bd}\,T_j$ is a component of $D\cap \mathrm{Bd}\,T_j$ and is meridional on $\mathrm{Bd}\,T_j$.

Suppose now that E^* is a copy of E in T^* and for some j, $1\leqslant j\leqslant m$, T_j^* is a copy of T_j in T^*. If E^* and T_j^* intersect, each component of $E^*\cap T_j^*$ is meridional on T_j^*.

Suppose that Lemma 2 is false. Consider $n+1$ adjacent copies $E_0, E_1, \ldots$, and E_n of E in T^*. Let T_1^* be a copy of T_1 in T^*, T_2^* a copy of T_2 in T^* linked with T_1^*, T_3^* a copy of T_3 in T^* linked with $T_2^*, \ldots, T_m^*$ a copy of T_m in T^* linked with T_{m-1}^*, and T_1^{**} a copy of T_1 in T^* linked with T_m^*. We regard T^* as a cylinder in E^3 and suppose that if $0\leqslant i<j\leqslant n$, then E_i is to the left of E_j.

We note that if $j=0,1,\ldots$, or n, there is a copy D_j of D in T^* such that (1) $\mathrm{Bd}\,D_j=\mathrm{Bd}\,E_j$ and (2) if $i=1,2,\ldots$, or m, each component of $E_j\cap \mathrm{Bd}\,T_i^*$ is a component of $D_j\cap \mathrm{Bd}\,T_i^*$.

Case 1. T_1^* intersects E_0. As we noted above, T_1^* intersects E_0 meridionally. First, T_1^* lies wholly to the left of E_1. For if not, then T_1^* intersects E_1. But this implies that T_1^* intersects both D_0 and D_1 meridionally. Since Lemma 2 is false, this is impossible. Second, T_2^* lies wholly to the left of E_1. For if not, T_2^* intersects E_1. This implies that T_1^* intersects D_0 meridionally and T_2^* intersects D_1 meridionally. Since Lemma 2 is false, this is impossible.

If T_3^* intersects E_1, then it does not intersect E_2 and it follows, by an argument similar to that above, that T_4^* lies wholly to the left of E_2. If T_3^* does not intersect E_1, it lies wholly to the left of E_1 and T_4^* lies wholly to the left of E_2. See Figure 1.

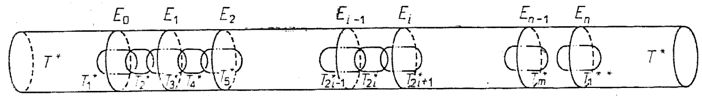

Fig. 1

Let this process continue. Since $m<2n$, it follows that either (a) T_m^* lies wholly to the left of E_{n-1} or (b) T_m^* intersects E_{n-1}. Now consider T_1^{**}. Since $\{T_1, T_2, \ldots, T_m\}$ circles T n times, T_1^{**} intersects E_n. Now if (a) above holds, then T_1^{**} intersects both E_{n-1} and E_n. This implies that T_1^{**} intersects both D_{n-1} and D_n meridionally, but since Lemma 2 is false, this is impossible. If (b) holds, then T_m^* intersects D_{n-1} meridionally and T_1^{**} intersects D_n meridionally, but this is also impossible. Hence Case 1 leads to a contradiction.

Case 2. T_1^* lies between E_0 and E_1. By an argument similar to that used in Case 1, we find that T_1^* lies wholly to the left of E_1, T_2^* either intersects E_1 or lies wholly to the left of E_1, T_3^* lies wholly to the left of E_2, T_4^* either intersects E_2 or lies wholly to the left of $E_2, \ldots$, and,

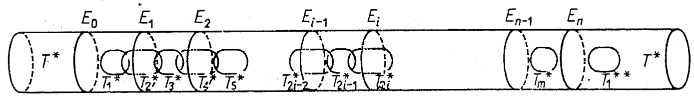

Fig. 2

since $m<2n$, T_m^* lies wholly to the left of E_n. See Figure 2. Now since $\{T_1, T_2, \ldots, T_m\}$ circles T n times, T_1^{**} lies to the right of E_n. This is impossible since T_m^* and T_1^{**} are linked. Hence Case 2 leads to a contradiction.

The remaining cases are similar to Cases 1 and 2. The supposition that Lemma 2 is false leads to a contradiction, and hence Lemma 2 holds.

5. Preliminary lemmas on toroidal decompositions. Throughout the remainder of the paper, we shall use the notation described in Section 3.

Suppose M is a solid torus. By a *homotopy centerline* of M we mean any loop in M homotopic in M to a core of M.

LEMMA 3. *Suppose that α is an index, Δ_1 and Δ_2 are disjoint polyhedral meridional discs in T_α, and for each index β, $m_\beta < 2n_\beta$. Then some element g of G lying in T_α intersects both Δ_1 and Δ_2.*

Proof. It is clear that each homotopy centerline of T_α intersects both Δ_1 and Δ_2. By induction and the proof of Lemma 2 of [4], it follows that there is a sequence $i_1, i_2, i_3, \ldots$ of integers such that for each j, each homotopy centerline of $T_{\alpha i_1 i_2 \ldots i_j}$ intersects both Δ_1 and Δ_2. Hence for each j, $T_{\alpha i_1 i_2 \ldots i_j}$ intersects both Δ_1 and Δ_2. It follows that if

$$g = \bigcap \{T_{\alpha i_1 i_2 \ldots i_j}\colon\ j = 1, 2, 3, \ldots\},$$

g is an element of G lying in T_α and intersecting both Δ_1 and Δ_2.

Suppose that α is an index. T_α^* denotes the universal covering space of T_α. Each of $T_{\alpha 1}, T_{\alpha 2}, \ldots,$ and $T_{\alpha m_\alpha}$ lifts homeomorphically into T_α^*. It follows that if g is an element of G lying in T_α, i is an integer such that $g \subset T_{\alpha i}$, and $T_{\alpha i}^*$ is any copy of $T_{\alpha i}$ in T_α^*, then there is a copy g^* of g in $T_{\alpha i}^*$.

LEMMA 4. *Suppose that α is an index, D is a polyhedral meridional disc in T_α, and if $r = 1, 2, \ldots,$ or m_α, $\mathrm{Bd}\, T_{\alpha r}$ and D are in relative general position. Suppose that for some integer i, there are distinct copies D_1 and D_2 of D in the universal covering space T_α^* of T_α such that D_1 and D_2 intersect $T_{\alpha i}^*$ meridionally. Suppose finally that for each index β, $m_\beta < 2n_\beta$. Then there is an element g of G in $T_{\alpha i}$ such that g^* intersects both D_1 and D_2.*

Proof. If $j = 1$ or 2, let D_j^0 be a component of $T_{\alpha i}^* \cap D_j$ such that one boundary curve μ_j of D_j^0 is meridional on $\mathrm{Bd}\, T_{\alpha i}^*$ and every other boundary curve of D_j^0 is trivial on $\mathrm{Bd}\, T_{\alpha i}^*$. By filling in boundary curves of D_1^0 and D_2^0 distinct from μ_1 and μ_2, respectively, with discs on $\mathrm{Bd}\, T_{\alpha i}^*$ and adjusting slightly, we may construct disjoint polyhedral meridional discs Δ_1 and Δ_2 in $T_{\alpha i}^*$ such that if $j = 1$ or 2, (1) $\mathrm{Bd}\, \Delta_j = \mathrm{Bd}\, D_j$ and (2) if $k = 1, 2, \ldots,$ or $m_{\alpha i}$, and $T_{\alpha i k}^*$ is a copy of $T_{\alpha i k}$ in $T_{\alpha i}^*$, then $T_{\alpha i k}^* \cap \Delta_j = T_{\alpha i k}^* \cap D_j$.

Since for each index β, $m_\beta < 2n_\beta$, it follows by Lemma 3 that there is an element g of G in $T_{\alpha i}$ such that g^* intersects both Δ_1 and Δ_2. Hence g^* intersects both D_1 and D_2.

LEMMA 5. *Suppose that α is an index, U is an open set in E^3 which is a union of elements of G, and D is a polyhedral meridional disc in T_α such that (1) $D \cap (\bigcup_{i=1}^{m_\alpha} T_{\alpha i})$ lies in a punctured disc D_0 lying in $D \cap U$ and (2) for each i, D and $\mathrm{Bd}\, T_{\alpha i}$ are in relative general position. Suppose that, in the universal covering space T_α^* of T_α, there is an integer i such that $T_{\alpha i}^*$ intersects adjacent copies D_1 and D_2 of D in T_α^* meridionally. Then there is a loop γ_α in $T_\alpha \cap U$ such that $\gamma_\alpha \not\sim 0$ in T_α.*

Proof. By Lemma 4, there is an element g of G in $T_{\alpha i}$ such that g^* intersects both D_1 and D_2. Let x and y be points of $g^* \cap D_1$ and $g^* \cap D_2$, respectively. Notice that $\varphi(x)$ and $\varphi(y)$ belong to D_0. Let x' be the point of D_2 that corresponds to x.

Observe that since g^* intersects D_1, g intersects D. Further, since $D \cap T_{\alpha i} \subset U$, g intersects U. Since U is a union of elements of G, $g \subset U$.

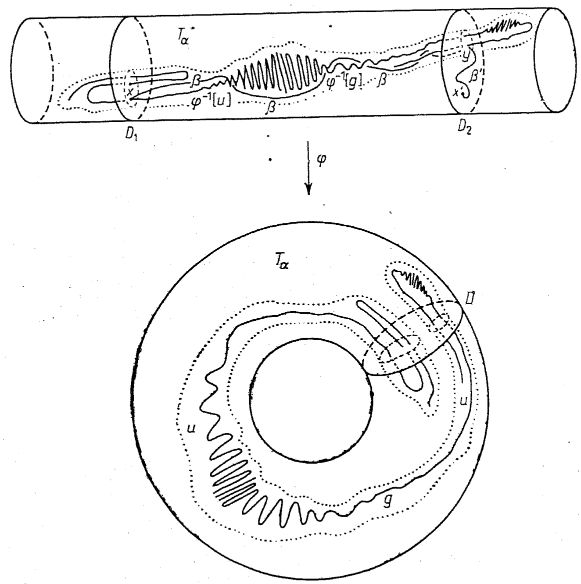

Fig. 3

There is, then, an arc β in T_α^* from x to y such that $\varphi[\beta] \subset U$. There is also an arc β' in D_2 such that β' joins x' and y, and $\varphi[\beta'] \subset D_0$. See Figure 3. It follows that if $\gamma = \varphi[\beta \cup \beta']$, then $\gamma \subset T_\alpha \cap U$ and $\gamma \not\sim 0$ in T_α.

Suppose M is a polyhedral 2-manifold-with-boundary in E^3 and Δ is a polyhedral singular disc in E^3 such that (1) Δ and $\mathrm{Bd}\, M$ are disjoint, (2) $\mathrm{Bd}\, \Delta$ and M are disjoint, and (3) Δ and M are in relative general position. Let Δ^0 be a 2-simplex. Since Δ is a polyhedral singular disc, there is a defining map f from Δ^0 onto Δ such that at each point of ${}^{-1}[\Delta \cap M]$, f is locally a homeomorphism. It follows that each component of $f^{-1}[\Delta \cap M]$ is a simple closed curve. The statement that γ is a *curve of intersection* of Δ with M means that for some component γ_0

of $f^{-1}[\Delta \cap M]$, $\gamma = f[\gamma_0]$. If γ is a curve of intersection of Δ with M, then γ is *trivial* on M if and only if $f|\gamma_0$ is homotopic to 0 on M.

Suppose now that N is a polyhedral solid torus in E^3, Δ is a polyhedral singular disc such that $\operatorname{Bd}\Delta$ and $\operatorname{Bd}N$ are disjoint, and Δ and $\operatorname{Bd}N$ are in relative general position. Suppose γ is a curve of intersection of Δ with $\operatorname{Bd}N$, let f be a defining map for Δ as above, and suppose γ_0 is a component of $f^{-1}[\Delta \cap \operatorname{Bd}N]$ such that $\gamma = f[\gamma_0]$. Then γ is *meridional* on $\operatorname{Bd}N$ if and only if $f|\gamma_0$ is homotopic on $\operatorname{Bd}N$, to a non-zero multiple of some meridional simple closed curve on $\operatorname{Bd}N$. We shall say that γ *has non-zero longitudinal component* on $\operatorname{Bd}N$ if and only if γ is homotopic, on $\operatorname{Bd}N$, to a curve which is a non-zero multiple of a longitude times some multiple of a meridian. Equivalently, γ has non-zero longitudinal component on $\operatorname{Bd}N$ if and only if γ is neither trivial on $\operatorname{Bd}N$ nor meridional on $\operatorname{Bd}N$.

LEMMA 6. *Suppose that α is an index, U is a simply connected open set in E^3, and D is a polyhedral meridional disc in T_α such that* (1) $D \cap (\bigcup_{i=1}^{m_\alpha} T_{\alpha i})$ *lies in a punctured disc D_0 lying in $D \cap U$ and* (2) *for each i, D and $\operatorname{Bd}T_{\alpha i}$ are in relative general position. Suppose that j is an integer such that there exist adjacent copies D_1 and D_2 of D in T_α^*, adjacent copies $T_{\alpha j}^*$ and $T_{\alpha(j+1)}^*$ of $T_{\alpha j}$ and $T_{\alpha(j+1)}$, respectively, in T_α^*, and loops γ_j and γ_{j+1} in $T_{\alpha j}$ and $T_{\alpha(j+1)}$, respectively, such that* (1) $\gamma_j \not\sim 0$ *in $T_{\alpha j}$ and $\gamma_{j+1} \not\sim 0$ in $T_{\alpha(j+1)}$,* (2) *D_1 intersects $T_{\alpha j}^*$ meridionally and D_2 intersects $T_{\alpha(j+1)}^*$ meridionally, and* (3) *each of γ_j and γ_{j+1} lies in U. Then there is a loop γ in $T_\alpha \cap U$ such that $\gamma \not\sim 0$ in T_α.*

Proof. Let Δ be a polyhedral singular disc in U bounded by γ_j and in general position relative to $\operatorname{Bd}T_\alpha$.

Suppose that there is a curve of intersection γ of Δ with $\operatorname{Bd}T_\alpha$ such that γ has non-zero longitudinal component. Then $\gamma \subset U \cap T_\alpha$ and $\gamma \not\sim 0$ in T_α.

Suppose then that each curve of intersection of Δ with $\operatorname{Bd}T_\alpha$ is either trivial or meridional on $\operatorname{Bd}T_\alpha$. Let Δ^0 be a 2-simplex and let f be defining map for Δ, from Δ^0 onto Δ, locally a homeomorphism at each point of $f^{-1}[\Delta \cap \operatorname{Bd}T_\alpha]$. Let E^0 be the closure of the component of $\Delta^0 - f^{-1}[\Delta \cap \operatorname{Bd}T_\alpha]$ that contains $\operatorname{Bd}\Delta^0$ and let E be $f[E^0]$; $f[E^0]$ is a singular punctured disc. Since each curve of intersection of Δ with $\operatorname{Bd}T_\alpha$ is either trivial or meridional on $\operatorname{Bd}T_\alpha$, there is a singular punctured disc E^* in T_α^* such that E^* has as boundary the copy γ_j^* of γ_j in $T_{\alpha j}^*$, and E^* covers E once. Let γ_{j+1}^* denote the copy of γ_{j+1} in $T_{\alpha(j+1)}^*$.

Now we shall prove that γ_{j+1}^* intersects E^*. Since γ_{j+1}^* is compact, there is a polyhedral disc F in T_α^* such that (1) $\operatorname{Bd}F \subset \operatorname{Bd}T_\alpha^*$ and $\operatorname{Bd}F \not\sim 0$ on $\operatorname{Bd}T_\alpha^*$, and (2) F and γ_{j+1}^* are disjoint. Construct a singular disc as follows: If μ is a boundary curve of E^* such that $\mu \sim 0$ on $\operatorname{Bd}T_\alpha^*$, then fill in μ with a singular disc δ_μ on $\operatorname{Bd}T_\alpha^*$ and attach δ_μ to E^*. If μ is a boundary curve of E^* such that $\mu \not\sim 0$ on $\operatorname{Bd}T_\alpha^*$, then μ is homotopic, on $\operatorname{Bd}T_\alpha^*$, to some multiple of $\operatorname{Bd}F$. For such a curve μ, if $\mu \sim k\cdot(\operatorname{Bd}F)$ on $\operatorname{Bd}T_\alpha^*$, construct a singular disc δ_μ from (1) a singular annulus on $\operatorname{Bd}T_\alpha^*$ having as boundary curves μ and $k\cdot(\operatorname{Bd}F)$ and (2) a singular disc lying in F and having as boundary $k\cdot(\operatorname{Bd}F)$. Then fill in μ with δ_μ. This yields a singular disc lying in $E^* \cup (\operatorname{Bd}T_\alpha^*) \cup F$ and a slight adjustment yields a polyhedral singular disc E' such that (1) $\operatorname{Bd}E' = \gamma_j^*$, (2) $E' \subset \operatorname{Int}T_\alpha^*$, and (3) E' lies in E^* together with a small neighborhood of $F \cup \operatorname{Bd}T_\alpha^*$ missing γ_{j+1}^*. In particular, since $\gamma_j \subset T_{\alpha j}$ and $\gamma_j \not\sim 0$ in $T_{\alpha j}$, and $\gamma_{j+1} \subset T_{\alpha(j+1)}$ and $\gamma_{j+1} \not\sim 0$ in $T_{\alpha(j+1)}$ then γ_j^* and γ_{j+1}^* are linked. It follows that γ_{j+1}^* intersects E^*.

Now we shall show that γ_j^* intersects D_1. Since D_1 intersects $T_{\alpha j}^*$ meridionally, there is a component D_1' of $D_1 \cap T_{\alpha j}^*$ such that D_1' is a punctured disc with one, and only one, boundary curve non-trivial on $\operatorname{Bd}T_{\alpha j}^*$. The trivial boundary curves of D_1' can be filled in by singular disc on $\operatorname{Bd}T_{\alpha j}^*$, yielding a singular disc D_1'' whose boundary is meridional on $\operatorname{Bd}T_{\alpha j}^*$. Since $\gamma_j \not\sim 0$ in $T_{\alpha j}$, γ_j^* intersects D_1''. But $D_1'' \cap \operatorname{Int}T_{\alpha j}^* \subset D_1$. Hence γ_j^* intersects D_1. A similar argument shows that γ_{j+1}^* intersects D_2.

Let x be a point of $\gamma_j^* \cap D_1$, x'' be the corresponding point of D_2, and y be a point of $\gamma_{j+1}^* \cap D_2$. Let x' be a point of $E^* \cap \gamma_{j+1}^*$. There is a path λ_1 in E^* from x to x', and there is a path λ_2 in γ_{j+1}^* from x' to y. If D_0^1 and D_0^2 denote the copies of D_0 in D_1 and D_2, respectively, then since $D \cap (T_{\alpha j} \cup T_{\alpha(j+1)}) \subset D_0$, $x \in D_0^1$ and $y \in D_0^2$. Hence $x'' \in D_0^2$ and there is an arc λ_3 in D_0^2 from y to x''. See Figure 4. If $\gamma = \varphi[\lambda_1 \cup \lambda_2 \cup \lambda_3]$, it follows that $\gamma \subset U$ since $E \subset U$, $\gamma_{j+1} \subset U$, and $D_0 \subset U$. Clearly $\gamma \subset T_\alpha$ and $\gamma \not\sim 0$ in T_α.

LEMMA 7. *Suppose α is an index, Δ is a polyhedral singular disc such that* (1) *Δ and $\operatorname{Bd}T_\alpha$ are in relative general position and* (2) *Δ intersects $\operatorname{Bd}T_\alpha$ non-trivially but only in trivial and meridional curves of intersection, and V is an open set in E^3 such that $\Delta \subset V$. Then there is a polyhedral meridional disc D in T_α such that there is a punctured disc D_0 in $D \cap V$ containing $D \cap (\bigcup_{i=1}^{m_\alpha} T_{\alpha i})$.*

Proof. Since Δ intersects $\operatorname{Bd}T_\alpha$ non-trivially but in no curve with non-zero longitudinal component, there is a singular subdisc Δ' of Δ such that (1) $\operatorname{Bd}\Delta'$ is meridional on $\operatorname{Bd}T_\alpha$ and (2) every curve of intersection of Δ' with $\operatorname{Bd}T_\alpha$ distinct from $\operatorname{Bd}\Delta'$ is trivial on $\operatorname{Bd}T_\alpha$. There is, therefore, a singular punctured disc Γ in Δ' such that $\operatorname{Bd}\Delta'$ is a boundary curve of Γ and every other boundary curve of Γ is trivial on $\operatorname{Bd}T_\alpha$, and $\Gamma \subset T_\alpha$.

There is a polyhedral solid torus T'_a concentric with T_a, lying in $\operatorname{Int} T_a$, and such that $\bigcup_{i=1}^{m_a} T_{ai} \subset \operatorname{Int} T'_a$. Let W denote $(\operatorname{Int} T_a) - T'_a$.

For each boundary curve μ of Γ distinct from $\operatorname{Bd} \Delta'$, $\mu \sim 0$ on $\operatorname{Bd} T_a$ and hence μ bounds a singular disc δ_μ on $\operatorname{Bd} T_a$. Let Γ'' be a polyhedral

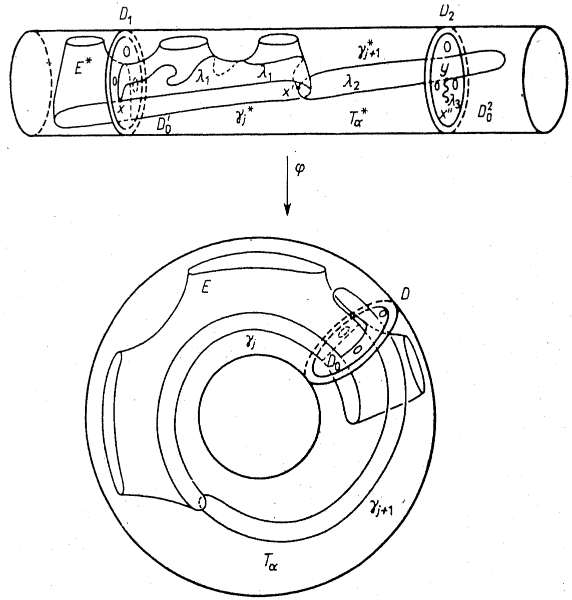

Fig. 4

singular disc obtained by adding each such δ_μ to Γ and then pushing δ_μ slightly into $\operatorname{Int} T_a$ so that the adjusted δ_μ lies in W. We may assume Γ'' has the following properties:

(1) $\Gamma'' \subset (\operatorname{Int} T_a) \cup (\operatorname{Bd} \Delta')$, (2) $\Gamma'' - \Delta' \subset W$, (3) $(\bigcup_{i=1}^{m_a} T_{ai}) \cap \Gamma'' \subset \Delta'$.

Recall that $\Delta' \subset V$.

By the loop theorem and Dehn's lemma [10], [11], [14], there is a polyhedral disc E such that (1) $\operatorname{Bd} E \subset \operatorname{Bd} T_a$, (2) $\operatorname{Bd} E \nsim 0$ on $\operatorname{Bd} T_a$, (3) $\operatorname{Int} E \subset \operatorname{Int} T_a$, and (4) E lies in the union of Γ'' and a small neighborhood of the singularities of Γ'', the latter chosen so that $E \cap T'_a \subset V$. Clearly E is meridional in T_a, and we assume E and $\operatorname{Bd} T'_a$ are in relative general position.

Now it will be shown that there is a component γ' of $E \cap \operatorname{Bd} T'_a$ such that γ' is meridional on $\operatorname{Bd} T'_a$. First, no component τ of $E \cap \operatorname{Bd} T'_a$ has non-zero longitudinal component on $\operatorname{Bd} T'_a$. For if there is such a component τ, then by Lemma 1, there is a polyhedral disc R such that $\operatorname{Bd} R = \operatorname{Bd} E$, $\operatorname{Int} R \subset \operatorname{Int} T_a$, and R is disjoint from T'_a. Since T'_a and T_a are concentric, this is impossible. Hence each non-trivial component of $E \cap \operatorname{Bd} T'_a$ is meridional.

Suppose that there is no meridional component of $E \cap \operatorname{Bd} T'_a$. Then each component of $E \cap \operatorname{Bd} T'_a$ is trivial on $\operatorname{Bd} T'_a$, and it is easy to construct a singular disc E' such that $\operatorname{Bd} E' = \operatorname{Bd} E$ and $E' \subset T_a - \operatorname{Int} T'_a$. But since T_a and T'_a are concentric, this is impossible. Therefore, there is a component γ' of $E \cap \operatorname{Bd} T'_a$ such that γ' is meridional on $\operatorname{Bd} T'_a$.

Let γ be a component of $E \cap \operatorname{Bd} T'_a$ which is meridional on $\operatorname{Bd} T'_a$ and is an innermost component of $E \cap \operatorname{Bd} T'_a$ which is meridional on $\operatorname{Bd} T'_a$. Let E_0 denote the disc on E bounded by γ. Each component of $(\operatorname{Bd} T'_a) \cap (\operatorname{Int} E_0)$ is trivial on $\operatorname{Bd} T'_a$, and it follows that the component of $E_0 \cap T'_a$ containing γ is a punctured polyhedral disc having γ as a boundary curve and such that every other boundary curve of D_0 is trivial on $\operatorname{Bd} T'_a$. Hence each boundary curve of D_0 distinct from γ bounds a disc on $\operatorname{Bd} T'_a$ missing γ.

Let γ_1 be a boundary curve of D_0 distinct from γ such that if Q_1 is the disc on $(\operatorname{Bd} T'_a) - \gamma$ bounded by γ_1, $\operatorname{Int} Q_1$ does not intersect D_0. Then add Q_1 to D_0, adjust the result to get a polyhedral punctured disc by pushing Q_1 slightly into $\operatorname{Int} T'_a$, and let D_1 denote the resulting punctured disc. The adjustment is to be made so that $D_1 \cap (\bigcup_{i=1}^{m_a} T_{ai}) \subset D_0$. If D_1 has a boundary curve distinct from γ, the process above is repeated. After finitely many repetitions, there results a polyhedral disc D' such that $\operatorname{Bd} D' = \gamma$, $\operatorname{Int} D' \subset \operatorname{Int} T'_a$, $D_0 \subset D'$, and $D' \cap (\bigcup_{i=1}^{m_a} T_{ai}) \subset D_0$.

There is a polyhedral annulus A such that $\operatorname{Bd} A = (\operatorname{Bd} E) \cup \gamma$ and $\operatorname{Int} A \subset W$. Let D denote $A \cup D'$. Then D satisfies the conclusion of Lemma 7.

6. Arrays. In order to handle a slightly complicated situation which occurs later, we introduce configurations we call *arrays*. In this section, we define them and establish an elementary property of arrays.

Suppose that α is an index, F_a is a polyhedral meridional disc in T_a and if $1 \leqslant j \leqslant m_a$, F_a and $\operatorname{Bd} T_{aj}$ are in relative general position.

(1) By a *basic array of type* I is meant a configuration of the form

$$(T_a, F_a)$$
$$(T_{ai}, X_{ai})$$

where $1 \leqslant i \leqslant m_a$ and X_{ai} is either (i) a polygonal loop γ_{ai} in T_{ai} such that $\gamma_{ai} \not\sim 0$ in T_{ai}, or (ii) a polyhedral meridional disc F_{ai} in T_{ai} such that (a) there is a punctured disc F^0_{ai} in F_{ai} such that $F_{ai} \cap (\bigcup_{j=1}^{m_{ai}} T_{aij}) \subset F^0_{ai}$ and (b) if $1 \leqslant j \leqslant m_{ai}$, F_{ai} and $\operatorname{Bd} T_{aij}$ are in relative general position. We call (T_a, F_a) and (T_{ai}, X_{ai}) the *top* and *bottom pairs*, respectively, of this array.

(2) By a *basic array of type* II is meant a configuration of the form

$$(T_a, F_a)$$

$$(T_{ai}, X_{ai}) \qquad (T_{a(i+1)}, X_{a(i+1)})$$

where $1 \leqslant i \leqslant m_a$ and X_{ai} is either a polygonal loop γ_{ai} in T_{ai} as above or a polyhedral meridional disc F_{ai} in T_{ai} as above, and $X_{a(i+1)}$ satisfies analogous conditions relative to $T_{a(i+1)}$. We call (T_a, F_a) the *top pair*, and (T_{ai}, X_{ai}) and $(T_{a(i+1)}, X_{a(i+1)})$ the *bottom pairs*, respectively, of this array.

By a *basic array* is meant either a basic array of type I or one of type II.

Suppose that

$$(T_a, F_a)$$

$$(T_{ai}, X_{ai})$$

is a basic array of type I. This array satisfies *case* I if and only if there exist adjacent copies F^1_a and F^2_a of F_a in T^*_a and a copy T^*_{ai} of T_{ai} in T^*_a such that T^*_{ai} intersects both F^1_a and F^2_a meridionally.

Suppose that

$$(T_a, F_a)$$

$$(T_{ai}, X_{ai}) \qquad (T_{a(i+1)}, X_{a(i+1)})$$

is a basic array of type II. This satisfies *case* II if and only if there exist adjacent copies F^1_a and F^2_a of F_a in T^*_a and adjacent copies T^*_{ai} and $T^*_{a(i+1)}$ of T_{ai} and $T_{a(i+1)}$, respectively, such that T^*_{a1} intersects F^1_a meridionally and $T^*_{a(i+1)}$ intersects F^2_a meridionally.

Suppose that n is a positive integer. By an *array* of $n+1$ rows we mean a diagram of $n+1$ non-empty rows such that for some index a,

(1) the top row is

$$(T_a, F_a)$$

where F_a is a polyhedral meridional disc in T_a as above,

(2) the first two rows form a basic array,

(3) if $1 < k \leqslant n+1$, each entry on the kth row is a bottom pair for some basic array whose top pair is an entry on row $k-1$,

(4) if $1 < k < n+1$ and $(T_{ai_1i_2\ldots i_k}, X_{ai_1i_2\ldots i_k})$ is an entry on the kth row, there is one and only one entry $(T_{ai_1i_2\ldots i_{k-1}}, X_{ai_1i_2\ldots i_{k-1}})$ on the $(k-1)$st row such that $(T_{ai_1i_2\ldots i_{k-1}}, X_{ai_1i_2\ldots i_{k-1}})$ and $(T_{ai_1i_2\ldots i_k}, X_{ai_1i_2\ldots i_k})$ are top and bottom pairs, respectively, of some one basic array, and $X_{ai_1i_2\ldots i_{k-1}}$ is necessarily a meridional disc in $T_{ai_1i_2\ldots i_{k-1}}$,

(5) if $1 \leqslant k \leqslant n$ and $(T_{ai_1i_2\ldots i_k}, F_{ai_1i_2\ldots i_k})$ is an entry on the kth row where $F_{ai_1i_2\ldots i_k}$ is a meridional disc in $T_{ai_1i_2\ldots i_k}$, then $(T_{ai_1i_2\ldots i_k}, F_{ai_1i_2\ldots i_k})$ is the top pair of some basic array whose bottom pair or pairs appear on row $(k+1)$, and

(6) each basic array of type I that appears in the diagram satisfies case I, and each basic array of type II that appears satisfies case II.

Suppose now that $\mathfrak{A}$ is an array of $n+1$ rows, with first row (T_a, F_a), and U is an open set in E^3. $\mathfrak{A}$ is *admissible with respect to* U if and only if the following conditions hold:

(1) If (T_β, γ_β) is an entry of $\mathfrak{A}$, then $\gamma_\beta \subset U$.

(2) If (T_β, F_β) is an entry of $\mathfrak{A}$, then the punctured disc F^0_a associated with F_β lies in U.

LEMMA 8. *Suppose a is an index, U is a simply connected open set in E^3, n is a positive integer, and $\mathfrak{A}$ is an array of $n+1$ rows, admissible with respect to U, with first row (T_a, F_a). Suppose that if $T_{aj_1j_2\ldots j_n}$ is a solid torus on row $n+1$ of $\mathfrak{A}$, there is a polygonal loop $\gamma_{aj_1j_2\ldots j_n}$ in $T_{aj_1j_2\ldots j_n} \cap U$ such that $\gamma_{j_1j_2\ldots j_n} \not\sim 0$ in $T_{aj_1j_2\ldots j_n}$. Then there is a polygonal loop γ_a in $T_a \cap U$ such that $\gamma_a \not\sim 0$ in T_a.*

Proof. Suppose that

$$(T_{aj_1j_2\ldots j_{n-1}}, F_{aj_1j_2\ldots j_{n-1}})$$

$$(T_{aj_1j_2\ldots j_{n-1}j_n}, X_{aj_1j_2\ldots j_{n-1}j_n}) \qquad (T_{aj_1j_2\ldots j_{n-1}(j_n+1)}, X_{aj_1j_2\ldots j_{n-1}(j_n+1)})$$

is a subarray of $\mathfrak{A}$ of type II. If $k = j_n$ or j_n+1, there exists, by hypothesis, a polygonal loop $\gamma_{j_1j_2\ldots j_{n-1}k}$ in $T_{aj_1j_2\ldots j_{n-1}k} \cap U$ such that $\gamma_{aj_1j_2\ldots j_{n-1}k} \not\sim 0$ in $T_{aj_1j_2\ldots j_{n-1}k}$. Since $\mathfrak{A}$ is admissible with respect to U, there is a punctured disc F_0 in $F_{aj_1j_2\ldots j_{n-1}} \cap U$ that contains $F_{aj_1j_2\ldots j_{n-1}} \cap (\bigcup_{i=1}^{m_{aj_1j_2\ldots j_{n-1}}} T_{aj_1j_2\ldots i})$. Since the basic array considered satisfies case II, then by Lemma 6, there is a polygonal loop $\gamma_{j_1j_2\ldots j_{n-1}}$ in $T_{aj_1j_2\ldots j_{n-1}} \cap U$ such that $\gamma_{j_1j_2\ldots j_{n-1}} \not\sim 0$ in $T_{aj_1j_2\ldots j_{n-1}}$.

Now suppose that

$$(T_{aj_1j_2\ldots j_{n-1}}, F_{aj_1j_2\ldots j_{n-1}})$$

$$(T_{aj_1j_2\ldots j_{n-1}j_n}, X_{aj_1j_2\ldots j_{n-1}j_n})$$

is a subarray of $\mathfrak{A}$ of type I. Since $\mathfrak{A}$ is admissible with respect to U, there is a punctured disc F_0' in $F_{aj_1j_2...j_{n-1}}$ that contains $F_{aj_1j_2...j_{n-1}} \cap (\bigcup_{i=1}^{m_{aj_1j_2...j_{n-1}}} T_{aj_1j_2...i})$. Since the basic array considered satisfies case I, then by Lemma 5, there is a polygonal loop $\gamma_{j_1j_2...j_{n-1}}$ in $T_{aj_1j_2...j_{n-1}} \cap U$ such that $\gamma_{j_1j_2...j_{n-1}} \nsim 0$ in $T_{aj_1j_2...j_{n-1}}$.

Hence, in either case considered, there is a polygonal loop $\gamma_{j_1j_2...j_{n-1}}$ in $T_{aj_1j_2...j_{n-1}} \cap U$ such that $\gamma_{j_1j_2...j_{n-1}} \nsim 0$ in $T_{aj_1j_2...j_{n-1}}$. Now let $\mathfrak{A}'$ denote the array obtained from $\mathfrak{A}$ by deleting the last row of $\mathfrak{A}$. $\mathfrak{A}'$ satisfies hypotheses analogous to those assumed for $\mathfrak{A}$ so the process above may be repeated.

It follows by induction that there is a polygonal loop γ_a in $T_a \cap U$ such that $\gamma_a \nsim 0$ in T_a.

7. The main lemmas. In this section we shall establish the main lemmas for the proof of Theorem 1.

LEMMA 9. *Suppose that a is an index and U is a simply connected open set in E^3 such that U is a union of elements of G. Suppose that Δ is a polyhedral singular disc in U such that* (1) $\mathrm{Bd}\,\Delta$ *and T_a are disjoint,* (2) Δ *and* $\mathrm{Bd}\,T_a$ *are in relative general position, and* (3) Δ *intersects* $\mathrm{Bd}\,T_a$ *non-trivially. Then there is a polygonal loop γ_a in $T_a \cap U$ such that $\gamma_a \nsim 0$ in T_a.*

Proof. Suppose first that there is a curve of intersection γ of Δ with $\mathrm{Bd}\,T_a$ such that γ has non-zero longitudinal component. Then $\gamma \subset T_a \cap U$ and $\gamma \nsim 0$ in T_a.

Hence, in the remainder of the proof, we shall suppose that each non-trivial curve of intersection of Δ with $\mathrm{Bd}\,T_a$ is meridional. Let V be an open set such that $\Delta \subset V$, $\overline{V} \subset U$, $\overline{V}$ is a union of elements of G, and $\overline{V}$ is compact. Such an open set may be constructed as follows: $P[U]$ is open in E^3/G since U is a union of elements of G. $P[\Delta]$ is compact and $P[\Delta] \subset P[U]$. Since $P[H_G]$ is a 0-dimensional set in E^3/G, there is an open set W such that $P[\Delta] \subset W$, $\overline{W} \subset U$, $\overline{W}$ is compact, and $\mathrm{Bd}\,W$ is disjoint from $P[H_G]$. Let V denote $P^{-1}[W]$; since $\overline{V} = P^{-1}[\overline{W}]$, it is easy to see that V has the properties stated above.

By Lemma 7, there is a polygonal meridional disc F_a in T_a such that there is a punctured disc F_a^0 in $F_a \cap V$ containing $F_a \cap (\bigcup_{i=1}^{m_a} T_{ai})$.

Now there are two cases to be considered. By Lemma 2, either (1) for some i_1, $1 \leqslant i_1 \leqslant m_a$, some copy $T_{ai_1}^*$ of T_{ai_1} in T^* intersects two adjacent copies F_a^1 and F_a^2 of F_a in T^*, or (2) for some i_1, $1 \leqslant i_1 \leqslant m_a$, some adjacent copies $T_{ai_1}^*$ and $T_{a(i_1+1)}^*$ of T_{ai_1} and $T_{a(i_1+1)}$, respectively, in T_a^*, and some adjacent copies F_a^1 and F_a^2 of F_a in T_a^*, $T_{ai_1}^*$ intersects F_a^1 meridionally and $T_{a(i_1+1)}^*$ intersects F_a^2 meridionally. If (1) holds, then by Lemma 5, there is a polygonal loop γ_a in $T_a \cap U$ such that $\gamma_a \nsim 0$ in T_a. Hence we shall suppose (2) holds.

We have begun the construction of an array. At this point, we have only the first row:

$$(T_a, F_a)\,.$$

Suppose $q = i_1$ or i_1+1, and consider F_a and T_{aq}. Either (3) some curve of intersection of F_a and $\mathrm{Bd}\,T_{aq}$ has non-zero longitudinal component, or (4) each curve of intersection of F_a and $\mathrm{Bd}\,T_{aq}$ is either trivial or meridional. Suppose first that (4) holds. Recall that there is a punctured disc F_a^0 in F_a such that $F_a^0 \subset V$ and $F_a \cap (\bigcup_{i=1}^{m_a} T_{ai}) \subset F_a^0$. It follows that $F_a \cap T_{aq} \subset V$. Hence by Lemma 7, there is a polyhedral meridional disc F_{aq} in T_{aq} such that (i) if $i = 1, 2, \ldots$ or m_{aq}, F_{aq} and $\mathrm{Bd}\,T_{aqi}$ are in relative general position, and (ii) there is a punctured disc F_{aq}^0 in $F_{aq} \cap V$ containing $F_{aq} \cap (\bigcup_{i=1}^{m_a} T_{aqi})$. It follows that regardless of whether (3) or (4) holds, either (5) there is a polygonal loop γ_{aq} in $T_{aq} \cap U$ such that $\gamma_{aq} \nsim 0$ in T_{aq} or (6) there is a disc F_{aq} as described above.

We may now construct the second row of the array. If there exist polygonal loops γ_{ai_1} and $\gamma_{a(i_1+1)}$, the second row is

$$(T_{ai_1}, \gamma_{ai_1}) \quad (T_{a(i_1+1)}, \gamma_{a(i_1+1)})\,.$$

If there exist a loop and a disc, it is the appropriate one of

$$(T_{ai_1}, \gamma_{ai_1}) \quad (T_{a(i_1+1)}, F_{a(i_1+1)})$$

and

$$(T_{ai_1}, F_{ai_1}) \quad (T_{a(i_1+1)}, \gamma_{a(i_1+1)})\,.$$

In the remaining case, there exist two discs, and the second row is

$$(T_{ai_1}, F_{ai_1}) \quad (T_{a(i_1+1)}, F_{a(i_1+1)})\,.$$

If the first of the three alternatives described in the preceding paragraph holds, the process terminates. Otherwise, for each pair of the type (T_{aq}, F_{aq}) on the second row, we repeat the preceding procedure. and construct a third row.

Suppose then that t is a positive integer and the $(t+1)$st row has been constructed and is non-void. Each pair on the $(t+1)$st row is either of the type $(T_{aq_1q_2...q_t}, \gamma_{aq_1q_2...q_t})$ where $\gamma_{aq_1q_2...q_t}$ is a polygonal loop in $T_{aq_1q_2...q_t} \cap U$, non-trivial in $T_{aq_1q_2...q_t}$, or of the type $(T_{aq_1q_2...q_t}, F_{aq_1q_2...q_t})$ where $F_{aq_1q_2...q_t}$ is a meridional disc in $T_{aq_1q_2...q_t}$ having certain properties. If for each pair on the $(t+1)$st row, the second term is a loop, the process terminates. Otherwise, for each pair whose second term is a disc, the process described previously is used to obtain entries for the $(t+2)$nd row.

Either this construction terminates or it continues indefinitely. Suppose that it continues indefinitely. For each positive integer t, let K_t be the union of all the solid tori appearing on the t-th row. Then for each t, K_t is a compact non-empty set, and $K_{t+1} \subset K_t$. Let K denote $\bigcap_{t=1}^{\infty} K_t$; K is compact, non-empty, and is a union of sets of G. Further, $K \subset U$. To establish this, suppose that g is an element of G contained in K. There exists a sequence $T_a, T_{aj_1}, T_{aj_1j_2}, \ldots$ of solid tori such that (1) for each r, $T_{aj_1j_2\ldots j_r}$ appears on row $r+1$ and (2) $g = \bigcap_{r=1}^{\infty} T_{aj_1j_2\ldots j_r}$. Now for each r, the second term of the pair of row $r+1$ whose first term is $T_{aj_1j_2\ldots j_r}$ is necessarily $F_{aj_1j_2\ldots j_r}$, because if that second term is $\gamma_{aj_1j_2\ldots j_r}$, then there are no tori on row $r+2$ that lie in $T_{aj_1j_2\ldots j_r}$. Hence each of $T_a, T_{aj_1}, T_{aj_1j_2}, \ldots$ intersects V and therefore g intersects $\overline{V}$. Since $\overline{V} \subset U$ and U is a union of elements of G, then $g \subset U$. Consequently, $K \subset U$.

Since K is compact, there is a positive integer r such that $K_r \subset U$. Let us consider a solid torus T_β that appears on row r. If T_β intersects K, then $T_\beta \subset U$ and hence there is a polygonal loop γ_β in $T_\beta \cap U$ such that $\gamma_\beta \not\sim 0$ in T_β.

Now suppose T_β does not intersect K. Then there exists a positive integer s such that $s \geqslant r$ and there is no solid torus in row s that lies in T_β. If not, then for each positive integer s, K_s intersects T_β and hence K intersects T_β. This is contrary to supposition, so such an s exists. Let k be the least positive integer s such that $s \geqslant r$ and no solid torus on row s lies in T_β. It is necessarily true that if T_λ is a solid torus in row $k-1$ that lies in T_β, the second term of the pair of row $k-1$ whose first term is T_λ is a polygonal loop γ_λ in $T_\lambda \cap U$ such that $\gamma_\lambda \not\sim 0$ in T_λ. By Lemma 8, there is a polygonal loop γ_β in $T_\beta \cap U$ such that $\gamma_\beta \not\sim 0$ in T_β.

Suppose, on the other hand, that the construction above terminates, and let r be the positive integer such that it terminates on row r. If T_β is a solid torus on row r, then necessarily there exists a polygonal loop γ_β in $T_\beta \cap U$ such that $\gamma_\beta \not\sim 0$ in T_β.

In either case, consider the array consisting of the first r rows constructed above. This array satisfies the hypothesis of Lemma 8. Hence there exists a polygonal loop γ_a in $T_a \cap U$ such that $\gamma_a \not\sim 0$ in T_a.

LEMMA 10. *Suppose that a is an index and if $i = 1, 2, \ldots,$ or m_a, Δ_i is a polyhedral singular disc in E^3 such that if $i = 1, 2, \ldots,$ or m_a, (1) $\mathrm{Bd}\,\Delta_i \subset T_{ai}$ and $(\mathrm{Bd}\,\Delta_i) \not\sim 0$ in T_{ai}, (2) Δ_i and $\mathrm{Bd}\,T_a$ are in relative general position and each curve of intersection of Δ_i with $\mathrm{Bd}\,T_a$ is trivial on $\mathrm{Bd}\,T_a$. Then there is a polygonal loop γ in T_a such that $\gamma \subset T_a \cap (\bigcup_{i=1}^{m_a} \Delta_i)$ and γ circles T_a n_a times.*

Proof. Let Δ^0 be a 2-simplex. If $i = 1, 2, \ldots,$ or m_a, let f_i be a defining map for Δ_i such that f_i is a local homeomorphism at points of $f_i^{-1}[\Delta_i \cap \mathrm{Bd}\,T_a]$. Let Δ_i^0 be the component of $\Delta^0 - f_i^{-1}[\Delta_i \cap \mathrm{Bd}\,T_a]$ that contains $\mathrm{Bd}\,\Delta^0$. Let Δ_i' denote $f_i[\Delta_i^0]$.

Now we shall show that if $i = 1, 2, \ldots,$ or m_a, Δ_i' and $\mathrm{Bd}\,\Delta_{i+1}$ intersect. Suppose that for some i, Δ_i' and $\mathrm{Bd}\,\Delta_{i+1}$ are disjoint. Let Δ_i'' be a singular disc obtained as follows: If μ is a boundary curve of Δ_i' distinct from $\mathrm{Bd}\,\Delta_i$, then $\mu \subset \mathrm{Bd}\,T_a$ and $\mu \sim 0$ on $\mathrm{Bd}\,T_a$. For each such μ, add to Δ_i' a singular disc on $\mathrm{Bd}\,T_a$ bounded by μ. The resulting singular disc will be denoted by Δ_i''. Since Δ_i' and $\mathrm{Bd}\,\Delta_{i+1}$ are disjoint, it follows that Δ_i'' and $\mathrm{Bd}\,\Delta_{i+1}$ are also disjoint. However, (1) $\mathrm{Bd}\,\Delta_i'' = \mathrm{Bd}\,\Delta_i$, (2) $\mathrm{Bd}\,\Delta_i \subset T_{ai}$ and $\mathrm{Bd}\,\Delta_i \not\sim 0$ in T_{ai}, and (3) $\mathrm{Bd}\,\Delta_{i+1} \subset T_{a(i+1)}$ and $\mathrm{Bd}\,T_{a(i+1)} \not\sim 0$ in $T_{a(i+1)}$.

Let T_a^* denote the universal covering space of T_a, and let T_{ai}^* and $T_{a(i+1)}^*$ be adjacent copies of T_{ai} and $T_{a(i+1)}$, respectively, in T_a^*. Then T_{ai}^* and $T_{a(i+1)}^*$ are linked in T_a^*. Δ_i'' lifts to a singular disc Ω_i'' with $\mathrm{Bd}\,\Omega_i''$ the copy of $\mathrm{Bd}\,\Delta_i$ in T_{ai}^*. Since $\mathrm{Bd}\,\Delta_{i+1} \subset T_{a(i+1)}$, $\mathrm{Bd}\,\Delta_{i+1}$ lifts to a loop γ_{i+1}^* in $T_{a(i+1)}^*$ such that $\gamma_{i+1}^* \not\sim 0$ in $T_{a(i+1)}^*$. It then follows that Ω_i'' and γ_{i+1}^* intersect. Since Δ_i'' and $\mathrm{Bd}\,\Delta_{i+1}$ are disjoint, this is a contradiction. Hence if $i = 1, 2, \ldots,$ or m_a, Δ_i' and $\mathrm{Bd}\,\Delta_{i+1}$ intersect.

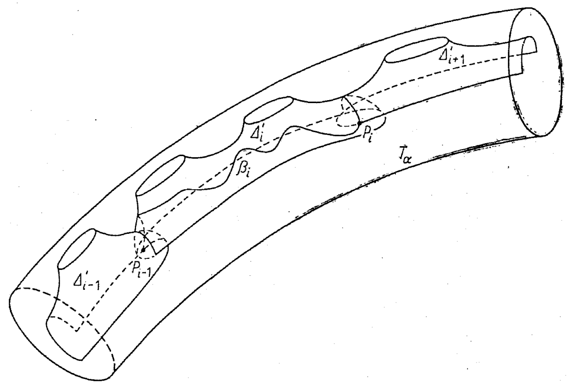

Fig. 5

For each i, let p_i be a point of $\Delta_i' \cap \mathrm{Bd}\,\Delta_{i+1}$. Now for each i, $f_i^{-1}[\Delta_i']$ is a punctured disc having $\mathrm{Bd}\,\Delta^0$ as one boundary component, $f_i^{-1}(p_{i-1})$ is a point of $\mathrm{Bd}\,\Delta^0$, and $f_i^{-1}(p_i) \in f_i^{-1}[\Delta_i']$. There is an arc β_i^0 in $f_i^{-1}[\Delta_i']$ joining $f_i^{-1}(p_{i-1})$ and $f_i^{-1}(p_i)$. Let β_i denote $f_i[\beta_i^0]$; β_i is a path on Δ_i' from p_{i-1} to p_i. See Figure 5.

Let γ denote $\beta_1 \cup \beta_2 \cup \ldots \cup \beta_{m_a}$; γ is to be regarded as a loop. We shall show that γ circles T_a n_a times. Since the chain $\{T_{a1}, T_{a2}, \ldots, T_{am_a}\}$

circles T_a n_a times and for each i, $\mathrm{Bd}\,\Delta_i \not\sim 0$ in T_{ai}, it follows that $\{\mathrm{Bd}\,\Delta_1, \mathrm{Bd}\,\Delta_2, \ldots, \mathrm{Bd}\,\Delta_{m_a}\}$ circles T_a n_a times.

Now consider the universal covering space T_a^* of T_a. Recall that for each i, Δ_i' lies on a singular disc Δ_i'' in T_a. Let $\Delta_1^*, \Delta_2^*, \ldots$, and $\Delta_{m_a}^*$ denote adjacent copies, in T_a^*, of $\Delta_1'', \Delta_2'', \ldots$, and Δ_{m_a}'', respectively. Let $p_{m_a}^*, p_1^*, p_2^*, \ldots$, and $p_{m_a-1}^*$ denote the copies of $p_{m_a}, p_1, p_2, \ldots$, and p_{m_a-1}, respectively, on $\mathrm{Bd}\,\Delta_1^*, \mathrm{Bd}\,\Delta_2^*, \mathrm{Bd}\,\Delta_3^*, \ldots$, and $\mathrm{Bd}\,\Delta_{m_a}^*$, respectively. Let $p_{m_a}^{**}$ denote the copy of p_{m_a} on $\Delta_{m_a}^{**}$.

If $1 < i < m_a$, β_i lies on Δ_i'' and hence β_i lifts to a path β_i^* from p_{i-1}^* to p_i^*. Similarly, β_1 and $\beta_{m_a}^*$ lift to paths β_1^* from $p_{m_a}^*$ to p_1^* and $\beta_{m_a}^*$ from $p_{m_a-1}^*$ to $p_{m_a}^{**}$, respectively. Then $\beta_1^* \cup \beta_2^* \cup \ldots \cup \beta_{m_a}^*$ is a path in T_a^* from $p_{m_a}^*$ to $p_{m_a}^{**}$. Since $\{\mathrm{Bd}\,\Delta_1, \mathrm{Bd}\,\Delta_2, \ldots, \mathrm{Bd}\,\Delta_{m_a}\}$ circles T_a n_a times, it follows that if λ is any arc in T_a^* from $p_{m_a}^*$ to $p_{m_a}^{**}$, $\varphi[\lambda]$ circles T_a n_a times. Since $\varphi[\beta_1^* \cup \beta_2^* \cup \ldots \cup \beta_{m_a}^*] = \gamma$, γ circles T_a n_a times.

LEMMA 11. *Suppose a is an index, $i = 1, 2, \ldots$, or m_a, and γ_i is a polygonal loop in T_{ai} such that $\gamma_i \not\sim 0$ in T_{ai}. Suppose γ_i bounds a polygonal singular disc Δ_i in E^3 such that* (1) *Δ_i is in general position relative to each of $\mathrm{Bd}\,T_a$ and $\mathrm{Bd}\,T_{a(i+1)}$ and* (2) *each curve of intersection of Δ_i with $\mathrm{Bd}\,T_a$ is trivial on $\mathrm{Bd}\,T_a$. Then Δ_i intersects $\mathrm{Bd}\,T_{a(i+1)}$ non-trivially.*

Proof. Suppose that each curve of intersection of Δ_i with $\mathrm{Bd}\,T_{a(i+1)}$ is trivial on $\mathrm{Bd}\,T_{a(i+1)}$. If μ is a curve of intersection of Δ_i with $\mathrm{Bd}\,T_a$ let δ_μ be a singular disc on $\mathrm{Bd}\,T_a$ with boundary μ. For each such μ, replace the singular subdisc of Δ_i bounded by μ by δ_μ. If λ is a curve of intersection of Δ_i with $\mathrm{Bd}\,T_{a(i+1)}$, let δ_λ be a singular disc on $\mathrm{Bd}\,T_{a(i+1)}$ bounded by λ. For each such λ, replace the singular subdisc of Δ_i bounded by λ by δ_λ. There results a singular disc Δ_i' with boundary γ_i, lying in T_a and disjoint from $\mathrm{Int}\,T_{a(i+1)}$. Adjust Δ_i' slightly into T_a and slightly away from $T_{a(i+1)}$. If Ω_i denotes the resulting disc, then Ω_i has boundary γ_i, $\Omega_i \subset \mathrm{Int}\,T_a$, and Ω_i and $T_{a(i+1)}$ are disjoint.

Let T_{ai}^* and $T_{a(i+1)}^*$ be adjacent copies of T_{ai} and $T_{a(i+1)}$, respectively, in T_a^*. By definition, T_{ai}^* and $T_{a(i+1)}^*$ are linked. Let Ω_i^* denote the copy in T_a^* of Ω_i such that the boundary of Ω_i^* lies in T_{ai}^*. Since $\gamma_i \not\sim 0$ in T_{ai} and Ω_i has γ_i as its boundary, it follows that Ω_i^* intersects $T_{a(i+1)}^*$. Since Ω_i and $T_{a(i+1)}$ are disjoint, this is a contradiction. Hence Lemma 11 is established.

LEMMA 12. *Suppose that a is an index, U is a simply connected open set in E^3, and U is a union of elements of G. Suppose there exist an integer i, $1 \leqslant i \leqslant m_a$, and a polygonal loop γ_{ai} in $T_{ai} \cap U$ such that $\gamma_{ai} \not\sim 0$ in T_{ai}. Then exists a polygonal loop γ_a in $T_a \cap U$ such that $\gamma_a \not\sim 0$ in T_a.*

Proof. Let Δ_i be a polyhedral singular disc bounded by γ_{ai}, lying in U, and such that if $1 \leqslant j \leqslant m_a$, Δ_i is in general position relative to $\mathrm{Bd}\,T_a$ and $\mathrm{Bd}\,T_{aj}$. If there exists a curve of intersection γ of Δ_i with $\mathrm{Bd}\,T_a$ such that γ has non-zero longitudinal component, then γ is a polygonal loop in $T_a \cap U$ such that $\gamma \not\sim 0$ in T_a. If there exists no curve of intersection of Δ_i with $\mathrm{Bd}\,T_a$ having non-zero longitudinal component, but Δ_i intersects $\mathrm{Bd}\,T_a$ non-trivially, then by Lemma 9, there is a polygonal loop γ_a in $T_a \cap U$ such that $\gamma_a \not\sim 0$ in T_a.

Therefore we shall suppose that each curve of intersection of Δ_i with $\mathrm{Bd}\,T_a$ is trivial on $\mathrm{Bd}\,T_a$. Now consider Δ_i and $T_{a(i+1)}$. Since $\gamma_{ai} \not\sim 0$ in T_{ai}, then by Lemma 11, Δ_i intersects $\mathrm{Bd}\,T_{a(i+1)}$ non-trivially. If some curve of intersection of Δ_i with $\mathrm{Bd}\,T_{a(i+1)}$ has non-zero longitudinal component on $\mathrm{Bd}\,T_{a(i+1)}$, then there is a polygonal loop $\gamma_{a(i+1)}$ in $T_{a(i+1)} \cap U$ such that $\gamma_{a(i+1)} \not\sim 0$ in $T_{a(i+1)}$. Otherwise, by Lemma 9, there is a polygonal loop $\gamma_{a(i+1)}$ in $T_{(ai+1)} \cap U$ such that $\gamma_{a(i+1)} \not\sim 0$ in $T_{a(i+1)}$.

The argument above may be repeated using $\gamma_{a(i+1)}$ and some singular disc Δ_{i+1}, lying in U and bounded by $\gamma_{a(i+1)}$. After finitely many repetitions, we find that we need to consider only the following situation: For each integer j, $1 \leqslant j \leqslant m_a$, there exists a polyhedral singular disc Δ_j in U such that if $\gamma_{aj} = \mathrm{Bd}\,\Delta_j$, then $\gamma_{aj} \subset T_{aj} \cap U$ and $\gamma_{aj} \not\sim 0$ in T_{aj}, and each curve of intersection of Δ_j with $\mathrm{Bd}\,T_a$ is trivial on $\mathrm{Bd}\,T_a$. Then by Lemma 10, there is a polygonal loop γ_a in $T_a \cap U$ such that $\gamma_a \not\sim 0$ in T_a. Hence Lemma 12 is established.

8. The main result.

THEOREM 1. *Suppose that G is a pointlike simple toroidal decomposition of E^3 as described in Section 3 and such that, in the notation of Section 3, for each index a, $m_a < 2n_a$. If g is a point of E^3/G belonging to $\mathrm{Cl}\,P[H_G]$, there is no simply connected open set W in E^3/G such that $g \in W$ and $W \subset P[\mathrm{Int}\,T_0]$.*

Proof. Suppose there exists a point g of E^3/G belonging to $\mathrm{Cl}\,P[H_G]$ and such that there exists a simply connected open set W in E^3/G such that $g \in W$ and $W \subset P[\mathrm{Int}\,T_0]$. Let U denote $P^{-1}[W]$; since G is a pointlike decomposition of E^3, then by ([12], Theorem 2.1), U is simply connected. Clearly $U \subset \mathrm{Int}\,T_0$.

Since $g \in W$, it follows that in E^3, $g \subset U$. Further, since $g \in \mathrm{Cl}\,P[H_G]$, then $g \subset \mathrm{Cl}\,H_G$. Now $\mathrm{Cl}\,H_G$ is

$$\bigcap_{i=1}^{\infty} \left(\bigcup \{T_{j_1 j_2 \ldots j_i}\colon\ j_1 j_2 \ldots j_i \text{ is an index}\}\right).$$

Thus there exist a positive integer n and a torus $T_{i_1 i_2 \ldots i_n}$ such that $g \subset T_{i_1 i_2 \ldots i_n}$ and $T_{i_1 i_2 \ldots i_n} \subset U$.

There exists, then, a polygonal loop γ_n in $T_{i_1 i_2 \ldots i_n} \cap U$ such that $\gamma_n \not\sim 0$ in $T_{i_1 i_2 \ldots i_n}$. By Lemma 12, there is a polygonal loop γ_{n-1} in $T_{i_1 i_2 \ldots i_{n-1}} \cap U$ such that $\gamma_{n-1} \not\sim 0$ in $T_{i_1 i_2 \ldots i_{n-1}}$. It follows by induction and Lemma 12 that there is a polygonal loop γ_0 in $T_0 \cap U$ such that $\gamma_0 \not\sim 0$ in T_0.

Since U is simply connected, $\gamma_0 \sim 0$ in U. However, $U \subset \operatorname{Int} T_0$ and $\gamma_0 \not\sim 0$ in T_0. This is a contradiction and Theorem 1 is proved.

COROLLARY 1. *If G is a decomposition of E^3 satisfying the hypothesis of the theorem, then E^3/G is not strongly locally simply connected.*

CORROLLARY 2. *If m and n are positive integers, G is a pointlike simple toroidal decomposition of E^3 such that E^3/G is an (m, n)-space, and $m < 2n$, then E^3/G is not strongly locally simply connected.*

COROLLARY 3. *Suppose that G is the pointlike decomposition of E^3 described in Section 3 of* [7]. *Then E^3/G is not strongly locally simply connected.*

COROLLARY 4. *Suppose that G is the pointlike decomposition described in Section 2 of* [4]. *Then E^3/G is not strongly locally simply connected.*

9. Concluding remarks and questions. The condition, in the hypothesis of the theorem above, that the decomposition G of E^3 be pointlike is used only to insure that if W is a simply connected open set in E^3/G, then $P^{-1}[W]$ is simply connected. It is known that such a proposition holds for a larger class of decompositions than pointlike ones. In particular, if each element of G is a compact absolute retract and W is a simply connected open set in E^3/G, then $P^{-1}[W]$ is simply connected; see [3]. Therefore, by the proof of Theorem 1, we may establish the following result.

THEOREM 2. *Suppose G is a simple toroidal decomposition of E^3 into compact absolute retracts such that, in the notation of Section 3, for each index α, $m_\alpha < 2n_\alpha$. Then E^3/G is not strongly locally simply connected.*

The following two question are suggested by the results of this paper.

QUESTION 1. *Suppose that G is a pointlike simple toroidal decomposition of E^3 such that E^3/G is strongly locally simply connected. Is E^3/G homeomorphic to E^3?*

QUESTION 2. *Suppose G is a toroidal decomposition of E^3 such that E^3/G is strongly locally simply connected. Is E^3/G homeomorphic to E^3?*

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