Free products of lattices

by

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1. Introduction. The main result of this paper is a solution to the "word problem" of the free product L of the lattices L_λ, $\lambda \in \Lambda$. This means that an algorithm is given which decides $A \leqslant B$ in L for the lattice polynomials A and B, of course modulo the structure of the lattices L_λ.

This theorem can be considered a structure theorem for free products. Its usefulness will be illustrated in several applications.

The basic idea is the introduction of "covers". This goes back to R. P. Dilworth [3], see esp. Theorem 2.2. The importance of this idea was emphasized in [1], where extensions and simplifications of the results of [3] were proved.

This paper seems to stretch this method to its natural limit in this setting.

An essentially new idea, namely the replacement of "elements as covers" by "ideals as covers", surfaced in R. A. Dean [2] and was systematically exploited in H. Lakser ([5], [6]). The papers of H. Lakser ([5], [6]) contain many extensions of our joint results.

We will use $\wedge$, $\vee$ for lattice meet and join, and $\bigwedge$, $\bigvee$ for infinite meet and join. *The lattices L_λ, $\lambda \in \Lambda$, will always be assumed to be pairwise disjoint.* Set theoretic operations will be denoted by $\cup, \cap, -, \bigcup, \bigcap$. The operational symbols for $\wedge$ and $\vee$ will be $\boldsymbol{\wedge}, \boldsymbol{\vee}$; these will be used in the formation of polynomials.

2. Construction of the free product. Given a set X we recall the definition of the concepts *lattice polynomial over* X and *length*.

DEFINITION 1. (i) If $x \in X$ then x is a lattice polynomial of length 1; we write $l(x) = 1$.

(ii) If A_0, A_1 are lattice polynomials of length l_0, l_1 respectively, then $A_0 \boldsymbol{\vee} A_1$ and $A_0 \boldsymbol{\wedge} A_1$ are lattice polynomials of length $l_0 + l_1$; $l(A_0 \boldsymbol{\vee} A_1) = l(A_0 \boldsymbol{\wedge} A_1) = l(A_0) + l(A_1)$.

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(iii) The only lattice polynomials are those obtained from a finite sequence of applications of (i) and (ii).

Remark. The name "lattice polynomial" should not mislead the reader; the axioms of lattices were not taken into consideration in the definition. Lattice polynomials are formal sequences of symbols, commonly called "polynomial symbols" or "terms" used in the construction of the first order language with two binary operation symbols. They could also be identified with the elements of the "absolutely free algebra" generated by X.

We note that $l(A)$ is a natural number and that $l(A) = 1$ if and only if $A \in X$. In the sequel we often prove theorems by induction on the length of polynomials.

Now let $Q = \bigcup(L_\lambda | \ \lambda \in \Lambda)$, and let us denote the set of lattice polynomials over Q by $P(Q)$. For each $\lambda \in \Lambda$ we define the concept of upper and lower λ-cover.

Definition 2. For each $A \in P(Q)$ and each $\lambda \in \Lambda$, existence and value of the *upper λ-cover*, $A^{(\lambda)}$, and the *lower λ-cover*, $A_{(\lambda)}$, are defined as follows:

(i) If $A \in L_\lambda$ then $A_{(\lambda)}$ and $A^{(\lambda)}$ exist, and they are both equal to A; $A_{(\mu)}$, $A^{(\mu)}$ do not exist for $\mu \neq \lambda$.

(ii) If $A = B \vee C$ then $A^{(\lambda)}$ exists if and only if $B^{(\lambda)}$ and $C^{(\lambda)}$ both exist and in this event $A^{(\lambda)} = B^{(\lambda)} \vee C^{(\lambda)}$ (the join is in L_λ, of course). Furthermore, $A_{(\lambda)}$ exists if and only if at least one of $B_{(\lambda)}$, $C_{(\lambda)}$ exists; $A_{(\lambda)} = B_{(\lambda)}$ (respectively $C_{(\lambda)}$) if only $B_{(\lambda)}$ (respectively $C_{(\lambda)}$) exists, and $A_{(\lambda)} = B_{(\lambda)} \vee C_{(\lambda)}$ if both $B_{(\lambda)}$, $C_{(\lambda)}$ exist.

(iii) If $A = B \wedge C$ then $A_{(\lambda)}$ exists if and only if $B_{(\lambda)}$ and $C_{(\lambda)}$ both exist and in this event $A_{(\lambda)} = B_{(\lambda)} \wedge C_{(\lambda)}$. $A^{(\lambda)}$ exists if and only if at least one of $B^{(\lambda)}$, $C^{(\lambda)}$ exists; $A^{(\lambda)} = B^{(\lambda)}$ (respectively $C^{(\lambda)}$) if only $B^{(\lambda)}$ (respectively $C^{(\lambda)}$) exists, and $A^{(\lambda)} = B^{(\lambda)} \wedge C^{(\lambda)}$ if both $B^{(\lambda)}$, $C^{(\lambda)}$ exist.

Remark. $A^{(\lambda)}$ will turn out to be the smallest element of L_λ containing the element represented in the free product by A, and dually for $A_{(\lambda)}$. Note that $A^{(\lambda)}$, $A_{(\lambda)}$ are always in L_λ and that if both exist then $A_{(\lambda)} \leqslant A^{(\lambda)}$ in L_λ.

We state two lemmas that are useful in applications:

Lemma 1. *If $A \in P(Q)$, $\lambda, \mu \in \Lambda$ and $A_{(\lambda)}$, $A^{(\mu)}$ both exist then $\lambda = \mu$.*

Proof. We proceed by induction on $l(A)$. If $l(A) = 1$ then $A \in L_\nu$ for some $\nu \in \Lambda$ and so $\lambda = \nu = \mu$. If $A = B \vee C$ then both $B^{(\mu)}$, $C^{(\mu)}$ exist and one of $B_{(\lambda)}$, $C_{(\lambda)}$ exists. Since $l(B)$, $l(C) < l(A)$ we find that $\lambda = \mu$. The dual argument applies if $A = B \wedge C$.

If Λ consists of only two elements we have the following complement to Lemma 1:

Lemma 2. *If Λ consists of two elements, $\Lambda = \{\lambda, \mu\}$, then for each $A \in P(Q)$, if $A_{(\lambda)}$ is undefined then $A^{(\mu)}$ exists, and dually.*

Proof. This also follows by induction on $l(A)$. If $l(A) = 1$ then, since $Q = L_\lambda \cup L_\mu$ and $A_{(\lambda)}$ is undefined, $A \in L_\mu$ and so $A^{(\mu)}$ exists. If $A = B \vee C$ then neither $B_{(\lambda)}$ nor $C_{(\lambda)}$ exist, and so, by induction, both $B^{(\mu)}$, $C^{(\mu)}$ exist; hence $A^{(\mu)}$ exists. If $A = B \wedge C$ then one of $B_{(\lambda)}$, $C_{(\lambda)}$, say $B_{(\lambda)}$, does not exist and so $B^{(\mu)}$ exists; thus $A^{(\mu)}$ exists.

Remark. The hypothesis that Λ consists of only two elements is essential. As an example, let $\Lambda = \{0, 1, 2\}$ and let $a_i \in L_i$ for each $i \in \Lambda$. Then $(a_0 \vee a_1) \wedge (a_0 \vee a_2) \wedge (a_1 \vee a_2)$ has no upper or lower covers.

Definition 3. For any $A, B \in P(Q)$ we define by induction on $l(A) + l(B)$ the relation $A \subseteq B$ to hold if and only if at least one of the conditions (1) to (6) below holds:

(1) $A = B$;

(2) there is a $\lambda \in \Lambda$ such that $A^{(\lambda)}$, $B_{(\lambda)}$ exist and $A^{(\lambda)} \leqslant B_{(\lambda)}$;

(3) $A = A_0 \vee A_1$, where $A_0 \subseteq B$ and $A_1 \subseteq B$;

(4) $A = A_0 \wedge A_1$, where $A_0 \subseteq B$ or $A_1 \subseteq B$;

(5) $B = B_0 \vee B_1$, where $A \subseteq B_0$ or $A \subseteq B_1$;

(6) $B = B_0 \wedge B_1$, where $A \subseteq B_0$ and $A \subseteq B_1$.

Set $A \cong B$ if $A \subseteq B$ and $B \subseteq A$.

Lemma 3. *Let $A \subseteq B$ and $\lambda \in \Lambda$. If $A_{(\lambda)}$ exists then $B_{(\lambda)}$ exists and $A_{(\lambda)} \leqslant B_{(\lambda)}$; if $B^{(\lambda)}$ exists then $A^{(\lambda)}$ exists and $A^{(\lambda)} \leqslant B^{(\lambda)}$.*

Proof. Let $A_{(\lambda)}$ exist. If $A \subseteq B$ follows by (2), then, by Lemma 1, $A^{(\lambda)} \leqslant B_{(\lambda)}$ and so $A_{(\lambda)} \leqslant A^{(\lambda)} \leqslant B_{(\lambda)}$. Otherwise we proceed by induction on $l(A) + l(B)$ using (3)–(6). The second half is proved in a dual manner.

Theorem 1. (i) *The relation $\subseteq$ is a quasi-order (that is, $\subseteq$ is reflexive and transitive) and thus $\cong$ is an equivalence relation.*

(ii) *Given $A \in P(Q)$ let $\langle A \rangle$ denote the equivalence class of A under $\cong$, and let $L = \{\langle A \rangle | \ A \in P(Q)\}$. Define the binary relation $\leqslant$ on L by $\langle A \rangle \leqslant \langle B \rangle$ if and only if $A \subseteq B$. Then $\leqslant$ is a partial order on L with respect to which L is a lattice. Moreover, $\langle A \rangle \vee \langle B \rangle = \langle A \vee B \rangle$ and $\langle A \rangle \wedge \langle B \rangle = \langle A \wedge B \rangle$.*

(iii) *For each $\lambda \in \Lambda$ the mapping φ_λ: $L_\lambda \to L$, given by $\varphi_\lambda(x) = \langle x \rangle$, is a 1-1 lattice homomorphism, and $\langle (\varphi_\lambda | \ \lambda \in \Lambda); L \rangle$ is the free product of the family $(L_\lambda | \ \lambda \in \Lambda)$.*

(iv) *For each $\lambda \in \Lambda$ and $A \in P(Q)$, $A_{(\lambda)}$ exists if and only if $\{x \in L_\lambda | \ \langle x \rangle \leqslant \langle A \rangle\} \neq \varnothing$ and in this event $A_{(\lambda)} = \bigvee (x \in L_\lambda | \ \langle x \rangle \leqslant \langle A \rangle)$, and dually for $A^{(\lambda)}$.*

Proof. The proof is essentially the same as that of Theorem 2.2 in [3]. We present only an outline of the proof.

To show that $\subseteq$ is a quasi-order we need only establish transitivity. If $A \subseteq B$ and $B \subseteq C$ and if at least one of these relations is due to (2) of Definition 3 we apply Lemma 3 and so $A \subseteq C$ follows by (2). Otherwise we proceed as in [3], by induction on $l(A)+l(B)+l(C)$.

Since $\subseteq$ is a quasi-order it is well-known that $\cong$ is an equivalence relation and that $\leqslant$ is a partial order on L. That $\langle A\rangle \vee \langle B\rangle = \langle A \vee B\rangle$ follows from (3) and (5), and dually for $\langle A\rangle \wedge \langle B\rangle$. Thus L is a lattice.

Clearly, given $x, y \in L_\lambda$, $x \leqslant y$ in L_λ if and only if $x \subseteq y$ and so φ_λ is 1-1. That $x \vee y \cong x \vee y$ can be proved using the fact that $(x \vee y)_{(\lambda)} = x \vee y$ and thus $(x\vee y)^{(\lambda)} \leqslant (x \vee y)_{(\lambda)}$. This fact and the dual establish that φ_λ is a lattice homomorphism.

To show that L is the free product of the L_λ it is enough to observe that L is generated by the L_λ in the freest possible manner; two polynomials were identified only if it followed from the lattice axioms that they be identified.

If one wants to proceed formally, then let K be a lattice and let $f_\lambda\colon L_\lambda \to K$, $\lambda \in \Lambda$, be a family of lattice homomorphisms. The f_λ define a map $f_0\colon Q \to K$ and so we define $F\colon P(Q) \to K$ inductively by:

(a) if $x \in Q$ then $F(x) = f_0(x)$;

(b) $F(A \vee B) = F(A) \vee F(B)$;

(c) $F(A \wedge B) = F(A) \wedge F(B)$.

Now we have to show that F factors through $\cong$ and that it defines a lattice homomorphism of L into K extending the f_λ. The formal computations are a special case of the computations in the proof of Theorem 4, and are therefore omitted here.

Since $A_{(\lambda)} \in L_\lambda$ it follows from (2) that $A_{(\lambda)} \subseteq A$ if $A_{(\lambda)}$ exists. Also, if there is an $x \in L_\lambda$ such that $x \subseteq A$ then, by Lemma 3, $A_{(\lambda)}$ exists and $x \leqslant A_{(\lambda)}$. This argument and the dual establish (iv).

To relate this result to the solution of the word problem in free lattices (P. M. Whitman [8]) and to its generalization, the solution of the word problem for the free product of chains due to Yu. I. Sorkin [7], we observe:

COROLLARY. *Let $\Lambda' \subseteq \Lambda$ be such that L_λ is a chain for each $\lambda \in \Lambda'$. Then* (2) *of Definition* 3 *can be replaced by the two conditions*:

(2a) $A, B \in L_\lambda$, $\lambda \in \Lambda'$, *and* $A \leqslant B$;

(2b) *there is a* $\lambda \in \Lambda - \Lambda'$ *such that* $A^{(\lambda)}, B_{(\lambda)}$ *exist and* $A^{(\lambda)} \leqslant B_{(\lambda)}$.

Proof. We need only show that if $A, B \in P(Q)$, $\lambda \in \Lambda'$, $A^{(\lambda)} \leqslant B_{(\lambda)}$, and either $A \notin L_\lambda$ or $B \notin L_\lambda$, then $A \subseteq B$ can be derived from an application of a rule of Definition 3 other than rule (2).

Assume that $A \notin L_{(\lambda)}$ (the dual argument applies if $B \notin L_\lambda$). Since $A \notin L_\lambda$ and $A^{(\lambda)}$ exists it follows that $l(A) > 1$. If $A = A_0 \vee A_1$, then by (5) and the transitivity of $\subseteq$, we get $A_0, A_1 \subseteq B$ and so $A \subseteq B$ by (3). If $A = A_0 \wedge A_1$, then at least one of $A_0^{(\lambda)}$ and $A_1^{(\lambda)}$, say $A_0^{(\lambda)}$, exists; if only $A_0^{(\lambda)}$ exists then $A_0^{(\lambda)} = A^{(\lambda)} \leqslant B_{(\lambda)}$ and so $A_0 \subseteq B$, and $A \subseteq B$ follows from (4). If both $A_0^{(\lambda)}, A_1^{(\lambda)}$ exist then, since L_λ is a chain, we may assume that $A_0^{(\lambda)} \leqslant A_1^{(\lambda)}$; thus $A^{(\lambda)} = A_0^{(\lambda)}$ and $A \subseteq B$ follows by (4) as above.

Sorkin's result now follows by taking $\Lambda' = \Lambda$ and P. M. Whitman's by taking $\Lambda' = \Lambda$, and $L_\lambda = 1$ for all $\lambda \in \Lambda$.

For simplicity's sake, we will henceforth identify $A \in L_\lambda$ with $\langle A\rangle$, and thus the lattices L_λ, $\lambda \in \Lambda$ will be considered as sublattices of the free product.

3. Applications. As an application of our methods we prove a theorem of B. Jónsson [4]. Jónsson's result, which holds for any class of algebras satisfying the amalgamation property, is proved in [4] in an entirely different manner.

THEOREM 2. *For each $\lambda \in \Lambda$ let L_λ^* be a sublattice of L_λ and let L^* be the sublattice of the free product of the L_λ generated by the L_λ^*. Then L^* is the free product of the L_λ^*.*

Proof. In view of Theorem 1, (iv), we only have to observe that if $A \in P(Q^*)$, where $Q^* = \bigcup (L_\lambda^* |\ \lambda \in \Lambda)$, then $A^{(\lambda)}$ and $A_{(\lambda)}$ (if they exist) are in L_λ. Thus for $A, B \in P(Q^*)$, $A \subseteq B$ does not change its meaning when passing from Q to Q^*. Thus $\{\langle A\rangle |\ A \in P(Q^*)\} = L^*$ is by Theorem 1 the free product of the L_λ^*, $\lambda \in \Lambda$.

One of the most important properties of the free generators of a lattice is that they are join-, and meet-irreducible (P. M. Whitman [8]). (This implies among other things the uniqueness of the free generating set.) This was generalized by Yu. I. Sorkin [7] to free products of chains. The next result shows that no further generalization is possible.

THEOREM 3. *Let Λ consist of more than one element and let $\Lambda' \subseteq \Lambda$. Let L be the free product of $(L_\lambda |\ \lambda \in \Lambda)$. Then $L - \bigcup (L_\lambda |\ \lambda \in \Lambda')$ is a sublattice of L if and only if L_λ is a chain for each $\lambda \in \Lambda'$.*

Proof. We observe, by Theorem 1, (iv), that if $A \in P(Q)$ and $\lambda \in \Lambda$ then $\langle A\rangle \in L_\lambda$ if and only if $A^{(\lambda)} \leqslant A_{(\lambda)}$.

Let $\lambda \in \Lambda'$ and let L_λ not be a chain. Thus there are x, y, and $z = x \wedge y$ in L_λ such that x, y, z are all distinct. Let $\mu \neq \lambda$ and let $w \in L_\mu$. Let $A = (x \wedge w) \vee z$, $B = (y \wedge w) \vee z$; then $A_{(\lambda)} = B_{(\lambda)} = z$, $A^{(\lambda)} = x$, and $B^{(\lambda)} = y$. Since $A^{(\nu)}, B^{(\nu)}$ are undefined for all $\nu \neq \lambda$ we conclude that $\langle A\rangle$, $\langle B\rangle \in L - \bigcup (L_\nu |\ \nu \in \Lambda')$. However, $(A \wedge B)^{(\lambda)} = x \wedge y = z = (A \wedge B)_{(\lambda)}$. Thus $\langle A\rangle \wedge \langle B\rangle \in L_\lambda$; consequently $L - \bigcup (L_\nu |\ \nu \in \Lambda')$ is not a sublattice of L.

Conversely, let L_λ be a chain for each $\lambda \in \Lambda'$. Let $\lambda \in \Lambda'$, $\langle A\rangle, \langle B\rangle \in L$ and $\langle A\rangle \wedge \langle B\rangle \in L_\lambda$. Then $(A \wedge B)^{(\lambda)}, (A \wedge B)_{(\lambda)}$ exist and $(A \wedge B)^{(\lambda)} = (A \wedge B)_{(\lambda)}$. Thus $(A \wedge B)^{(\lambda)} \leqslant A_{(\lambda)}, B_{(\lambda)}$. If only $A^{(\lambda)}$ exists then $A^{(\lambda)} \leqslant A_{(\lambda)}$ and so $\langle A\rangle \in L_\lambda$. If, on the other hand, both $A^{(\lambda)}, B^{(\lambda)}$ exist, then since L_λ is a chain, we may assume that $A^{(\lambda)} \leqslant B^{(\lambda)}$ and again $(A \wedge B)^{(\lambda)} = A^{(\lambda)}$ implying that $\langle A\rangle \in L_\lambda$. Consequently $\langle A\rangle, \langle B\rangle \in L - \bigcup(L_\lambda|\ \lambda \in \Lambda')$ implies that $\langle A\rangle \wedge \langle B\rangle \in L - \bigcup(L_\lambda|\ \lambda \in \Lambda')$. In view of the principle of duality the theorem is proved.

4. Sorkin's theorem. In [7] Yu. I. Sorkin established a rather surprising result: if L, K and L_λ, $\lambda \in \Lambda$, are lattices, L is the free product of the L_λ, and if, for each $\lambda \in \Lambda$, $f_\lambda\colon L_\lambda \to K$ is an isotone map (but not necessarily a lattice homomorphism) then there is an isotone map f from L into K extending all of the f_λ. Sorkin's proof was rather long and involved; we present a very simple proof of this result.

Let $Q = \bigcup(L_\lambda|\ \lambda \in \Lambda)$ and let $f_0\colon Q \to K$ be defined by $f_0(x) = f_\lambda(x)$ if $x \in L_\lambda$. By induction on the length of the polynomials in $P(Q)$ we define $F\colon P(Q) \to K$:

(i) if $A \in Q$ then $F(A) = f_0(A)$;

(ii) if $A = B \vee C$ and $A_{(\lambda)}$ is defined for no $\lambda \in \Lambda$ then $F(A) = F(B) \vee \vee F(C)$; otherwise $F(A) = \bigvee(f_\lambda(A_{(\lambda)})|\ \lambda \in \Lambda$ and $A_{(\lambda)}$ exists$) \vee F(B) \vee F(C)$;

(iii) if $A = B \wedge C$ and $A^{(\lambda)}$ is defined for no $\lambda \in \Lambda$ then $F(A) = F(B) \wedge \wedge F(C)$; otherwise $F(A) = \bigwedge(f_\lambda(A^{(\lambda)}|\ \lambda \in \Lambda$ and $A^{(\lambda)}$ exists$) \wedge F(B) \wedge F(C)$.

We observe that by Definition 2 $A^{(\lambda)}$ or $A_{(\lambda)}$ exist for only finitely many $\lambda \in \Lambda$, and thus the definition of F makes sense.

We define $f\colon L \to K$ by requiring that $f(\langle A\rangle) = F(A)$. To establish that f is well-defined and isotone we need only show that $A \subseteq B$ implies $F(A) \leqslant F(B)$.

We first show that if $A \in P(Q)$ and $A^{(\lambda)}$ is defined then $F(A) \leqslant f_\lambda(A^{(\lambda)})$ (and dually). We proceed by induction on $l(A)$. If $l(A) = 1$ then $A \in L_\lambda$ and so $F(A) = f_\lambda(A^{(\lambda)})$. If $A = B \wedge C$ then $F(A) = \bigwedge(f_\mu(A^{(\mu)})|\ A^{(\mu)}$ exists$) \wedge F(B) \wedge F(C) \leqslant f_\lambda(A^{(\lambda)})$. If $A = B \vee C$ then, by Lemma 1, if $A_{(\mu)}$ exists then $\mu = \lambda$. If $A_{(\lambda)}$ exists then, since $A_{(\lambda)} \leqslant A^{(\lambda)}$ and f_λ is isotone,

$$f_\lambda(A_{(\lambda)}) \leqslant f_\lambda(A^{(\lambda)}) .$$

Now $B^{(\lambda)}$, $C^{(\lambda)}$ exist and, by induction,

$$F(B) \leqslant f_\lambda(B^{(\lambda)}) \leqslant f_\lambda(A^{(\lambda)}) , \quad F(C) \leqslant f_\lambda(C^{(\lambda)}) \leqslant f_\lambda(A^{(\lambda)}) .$$

Thus

$$F(A) = f_\lambda(A_{(\lambda)}) \vee F(B) \vee F(C) \leqslant f_\lambda(A^{(\lambda)}) .$$

If $A_{(\lambda)}$ is undefined then, as above,

$$F(A) = F(B) \vee F(C) \leqslant f_\lambda(A^{(\lambda)}) .$$

We now show, by induction on $l(A) + l(B)$, that if $A, B \in P(Q)$ and $A \subseteq B$ then $F(A) \leqslant F(B)$.

If $l(A) + l(B) = 2$ then $A, B \in Q$ and so $A \subseteq B$ follows by (2) of Definition 3; thus there is a $\lambda \in \Lambda$ such that $A^{(\lambda)} \leqslant B_{(\lambda)}$ and so

$$F(A) \leqslant f_\lambda(A^{(\lambda)}) \leqslant f_\lambda(B_{(\lambda)}) \leqslant F(B) .$$

Now let $l(A) + l(B) > 2$. If $A \subseteq B$ follows by rule (1) of Definition 3 the result is clear. If by (2), then the proof is identical with that above.

If $A \subseteq B$ follows by (3), that is, $A = C \vee D$, where $C, D \subseteq B$, then $F(C) \leqslant F(B)$, $F(D) \leqslant F(B)$.

If $\lambda \in \Lambda$ and $A_{(\lambda)}$ exists then, by Lemma 3, $B_{(\lambda)}$ exists and $A_{(\lambda)} \leqslant B_{(\lambda)}$; thus $f_\lambda(A_{(\lambda)}) \leqslant f_\lambda(B_{(\lambda)}) \leqslant F(B)$.

Consequently

$$F(A) = \bigvee(f_\lambda(A_{(\lambda)})|\ \lambda \in \Lambda \text{ and } A_{(\lambda)} \text{ exists}) \vee F(C) \vee F(D) \leqslant F(B) .$$

If $A \subseteq B$ follows by (4) then $A = C \wedge D$ and, say, $C \subseteq B$. Since $A \subseteq C$ we find, by induction, that

$$F(A) \leqslant F(C) \leqslant F(B) .$$

Since conditions (5) and (6) are the duals of (3) and (4) we have shown that $A \subseteq B$ implies $F(A) \leqslant F(B)$.

The mapping $f\colon L \to K$ extends each f_λ since $F(A) = f_\lambda(A)$ if $A \in L_\lambda$.

We note finally that if f_λ is a $\vee$-morphism for each $\lambda \in \Lambda$ (that is, $f_\lambda(x \vee y) = f_\lambda(x) \vee f_\lambda(y)$ for each $x, y \in L_\lambda$) then f is a $\vee$-morphism. Indeed, if $A \vee B$ has no lower covers then $F(A \vee B) = F(A) \vee F(B)$ by definition. On the other hand, if $(A \vee B)_{(\lambda)}$ exists from some $\lambda \in \Lambda$ then at least one of $A_{(\lambda)}, B_{(\lambda)}$ exists. If only $A_{(\lambda)}$ exists then

$$f_\lambda\big((A \vee B)_{(\lambda)}\big) = f_\lambda(A_{(\lambda)}) \leqslant F(A) .$$

If both $A_{(\lambda)}, B_{(\lambda)}$ exist then

$$f_\lambda\big((A \vee B)_{(\lambda)}\big) = f_\lambda(A_{(\lambda)} \vee B_{(\lambda)}) = f_\lambda(A_{(\lambda)}) \vee f_\lambda(B_{(\lambda)}) \leqslant F(A) \vee F(B) ,$$

using the hypothesis that f_λ is a $\vee$-morphism. Thus, in either case,

$$\bigvee\big(f_\lambda((A \vee B)_{(\lambda)})|\ \lambda \in \Lambda \text{ and } (A \vee B)_{(\lambda)} \text{ exists}\big) \leqslant F(A) \vee F(B) .$$

Consequently $F(A \vee B) = F(A) \vee F(B)$ and so f is a $\vee$-morphism.

By the principle of duality, if f_λ is a $\wedge$-morphism for each $\lambda \in \Lambda$ then f is a $\wedge$-morphism. Thus we obtain the following result:

THEOREM 4 (Yu. I. Sorkin [7]). *If $f_\lambda\colon L_\lambda \to K$ is an isotone map for each $\lambda \in \Lambda$, and L is the free product of the L_λ, $\lambda \in \Lambda$, then there is an isotone map f from L to K extending all of the f_λ. If f_λ is a $\vee$-morphism for all $\lambda \in \Lambda$ then f is a $\vee$-morphism, and dually.*

Finally, we note that the first statement of Theorem 4 (concerning isotone maps) is trivial from Theorem 1 if K has a zero. If K has a zero define

$$f(\langle A\rangle) = \bigvee \big(f_\lambda(A_{(\lambda)})\,\big|\ \lambda \in \Lambda\big)\,.$$

f obviously satisfies all the requirements. If no $A_{(\lambda)}$ exists, then, of course, $f(\langle A\rangle) = 0$.

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Non-existence of certain Borel structures

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This note conceptually simplifies the proofs and extends the theorems of [1] and puts them in a more general setting.

Let $(X, \boldsymbol{B})$ be any separable (countably generated and containing singletons) Borel space, where to avoid trivialities X is assumed to be uncountable. Sets in $\boldsymbol{B}$ are to be called Borel subsets of X. Throughout, $\boldsymbol{B}$ is fixed.

THEOREM 1. *For any σ-algebra $\boldsymbol{\Sigma}$ on X containing $\boldsymbol{B}$, the following are equivalent*:

(i) *Any one-one real $\boldsymbol{\Sigma}$-measurable function on X coincides with a $\boldsymbol{B}$-measurable function on an uncountable Borel subset of X.*

(ii) *Any separable σ-algebra $\boldsymbol{S}$ on X with $\boldsymbol{B} \subset \boldsymbol{S} \subset \boldsymbol{\Sigma}$ coincides with $\boldsymbol{B}$ on an uncountable Borel subset of X, that is, on some uncountable Borel subset of X the restrictions of $\boldsymbol{B}$ and $\boldsymbol{S}$ coincide.*

Proof: Given (i), we can prove (ii) by looking at the Marczewski function associated with any countable generator for $\boldsymbol{S}$. Conversely, given (ii), we can prove (i) by looking at the separable σ-algebra induced by the given function and $\boldsymbol{B}$.

DEFINITION 1. A σ-algebra $\boldsymbol{\Sigma}$ on X containing $\boldsymbol{B}$ and satisfying any one of the above two equivalent conditions is said to be a $\boldsymbol{B}$-*Souslin σ-algebra for X* (with due respect to the work done by Souslin).

DEFINITION 2. A σ-algebra $\boldsymbol{Z}$ on X is said to be $\boldsymbol{B}$-*mixing* if $\boldsymbol{Z}$ contains $\boldsymbol{B}$ and any uncountable Borel subset of X contains an element of $\boldsymbol{Z} - \boldsymbol{B}$.

From the above definitions and Theorem 1, we have the following theorem, which can be easily proved by contradiction.

THEOREM 2. *Let $\boldsymbol{Z}$ be any $\boldsymbol{B}$-mixing σ-algebra on X. Let $\boldsymbol{\Sigma}$ be any $\boldsymbol{B}$-Souslin σ-algebra containing $\boldsymbol{Z}$. Then there is no separable σ-algebra on X containing $\boldsymbol{Z}$ and contained in $\boldsymbol{\Sigma}$. Consequently, no separable σ-algebra containing $\boldsymbol{Z}$ can be a $\boldsymbol{B}$-Souslin σ-algebra.*

Remark 1. Throughout this paragraph let X be I the unit interval, $\boldsymbol{B}$ its usual Borel σ-algebra, $\boldsymbol{Z} = \boldsymbol{A}$ the σ-algebra generated by its usual