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On minimal regular digraphs with given girth

by

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Introduction. A problem in graph theory which has received much attention in recent years is the determination of the smallest number $f(r, n)$ of vertices that a graph G may posses such that G has degree r and girth n. (See [1], for example.) With few exceptions, the numbers $f(r, n)$ are unknown for $r \geqslant 3$ and $n \geqslant 5$. The purpose of this article is to study the analogous problem for digraphs (directed graphs).

The Function $g(r, n)$. For a vertex v of a digraph D, we denote by $\mathrm{id}\,v$ and $\mathrm{od}\,v$ the indegree and outdegree, respectively, of v. If $\mathrm{id}\,v = \mathrm{od}\,v = r$, then we speak of the *degree* of v and write $\deg v = r$. If every vertex of D has degree r, then D is said to be *regular of degree* r or simply r-*regular*.

The *girth* of a digraph D containing (directed) cycles is the length of the smallest cycle in D. For $n \geqslant 2$ and $r \geqslant 1$, we define $g(r, n)$ as the minimum number of vertices in an r-regular digraph D having girth n. It is obvious that $g(1, n) = n$ since the n-cycle has the desired properties and is clearly minimal. The cycle is a member of a more general class of regular digraphs which we now describe.

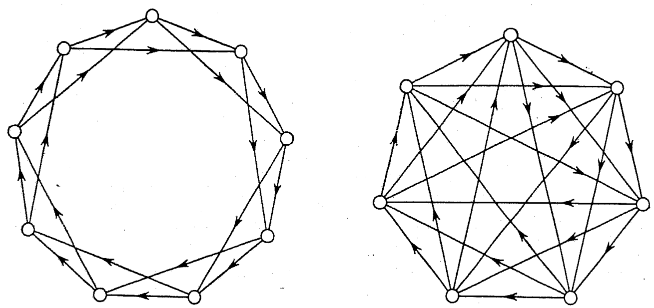
For $r \geqslant 1$ and $n \geqslant 2$ we denote by $D(r, n)$ the digraph whose $r(n-1)+1$ vertices are labeled v_i, $i = 1, 2, \ldots, r(n-1)+1$, and such that v_iv_j is an arc if and only if $j = i+1, i+2, \ldots, i+r$, where the numbers are expressed modulo $r(n-1)+1$. The digraphs $D(2, 5)$ and $D(3, 3)$ are shown in Figure 1.

Clearly, $D(r, n)$ is r-regular and, furthermore, it is easily seen that $D(r, n)$ contains cycles of every length k, $n \leqslant k \leqslant r(n-1)+1$ but of no length k, $k < n$, so that $D(r, n)$ has girth n. This construction implies the following.

THEOREM 1. *For each $r \geqslant 1$ and $n \geqslant 2$, the number $g(r, n)$ exists and, moreover,*

$$g(r, n) \leqslant r(n-1)+1 . \tag{1}$$

Although there are no known values of r and n for which the s inequality in (1) holds, there are several cases in which equality ca proved. We now consider these, beginning with $n=2$ and $n=3$. *complete symmetric digraph* K_p has p vertices and for each two vertice and v, both uv and vu are arcs of K_p. A *tournament* is a digraph D su that for each two vertices u and v of D, exactly one of the arcs uv and belongs to D. A digraph D is *transitive* if whenever uv and vw are arc of D then uw is also an arc of D.



$D(2,5)$ $D(3,3)$

Fig. 1

THEOREM 2. (i) *An r-regular digraph with girth 2 has $g(r,2)$ vertices if and only if D is the complete symmetric digraph K_{r+1}.*

(ii) *An r-regular digraph with girth 3 has $g(r,3)$ vertices if and only if D is a non-transitive, regular tournament with $2r+1$ vertices.*

Proof. (i) Any r-regular digraph has at least $r+1$ vertices. The only such regular digraph with $r+1$ vertices is the complete symmetric digraph K_{r+1}, which has girth 2.

(ii) Let D be an r-regular digraph having girth 3. Thus D contains no symmetric pair of arcs so that any vertex of D is necessarily adjacent to and from $2r$ distinct vertices. The only digraphs with these properties having $2r+1$ vertices are regular tournaments which contain 3-cycles. These are precisely, however, the non-transitive regular tournaments with $2r+1$ vertices.

We call a digraph D an $[r,n]$ digraph provided it is r-regular, has girth n, and has $g(r,n)$ vertices. From what we have seen, it now follows that there is only one $[1,n]$ digraph, namely the n-cycle, and that the $[r,2]$ digraph is also unique, namely the complete symmetric digraph K_{r+1}. We shall see that, in general, the $[r,3]$ digraphs are not unique.

For a fixed $n \geqslant 4$ and an arbitrary r, the number $g(r,n)$ is not known. We consider the number $g(r,4)$ in somewhat more detail. Of course, by Theorem 1,

$$g(r,4) \leqslant 3r+1 .$$

We now give a lower bound for $g(r,4)$. We use here the well known fact that if D is a digraph having no cycles and which fails to consist only of isolated vertices, then D contains a transmitter (a vertex with positive outdegree and zero indegree) and a receiver (a vertex with positive indegree and zero outdegree).

THEOREM 3. *For $r>1$,*

$$g(r,4) \geqslant (5r+4)/2 .$$

Proof. Let D be a $[r,4]$ digraph and let v be any vertex of D. Since D has no 2-cycles, the set V_1 of vertices adjacent to v and the set V_2 of vertices adjacent from v are disjoint. Hence $g(r,4) \geqslant 2r+1$. Because D is r-regular, the number of arcs emenating from the vertices in V_2 totals r^2. Since D has no 3-cycles, no vertex of V_2 can be adjacent to a vertex in V_1; thus none of the aforementioned r^2 arcs can lead to any vertex in the set $V_1 \cup \{v\}$. The subdigraph $\langle V_2 \rangle$ of D induced by the set V_2 (i.e. the subdigraph with vertex set V_2 and arc set consisting of those arcs of D joining two vertices in V_2) contains less than $r(r-1)/2$ arcs or is a tournament. In the last case $\langle V_2 \rangle$ has no cycles because $\langle V_2 \rangle$ has no 3-cycles, therefore from the previous remark $\langle V_2 \rangle$ contains a receiver. Hence in every case at least one vertex u of $\langle V_2 \rangle$ has outdegree less than $\frac{1}{2}(r-1)$. Therefore, u is adjacent to at least $(r+2)/2$ vertices, no one of which belongs to $V_1 \cup V_2 \cup \{v\}$. This, however, implies that D has at least $2r+1+(r+2)/2 = (5r+4)/2$ vertices so that $g(r,4) \geqslant (5r+4)/2$.

Combining this result with Theorem 1, we have the following.

COROLLARY 3a. *For $r=1$, 2, and 3,*

$$g(r,4) = 3r+1 .$$

There is one additional pair (r,n) for which $g(r,n)$ is known, namely $(r,n)=(4,4)$. We consider this next.

THEOREM 4. $g(4,4)=13$.

Proof. By Theorem 1, $g(4,4) \leqslant 13$. Suppose $g(4,4)=k<13$. Thus there exists a $[4,4]$ digraph D having k vertices. Let v_1 be a vertex of D adjacent from the vertices in $V_1 = \{v_2, v_3, v_4, v_5\}$ and adjacent to the vertices in $V_2 = \{v_6, v_7, v_8, v_9\}$. We now distinguish two cases, depending on whether both induced subdigraphs $\langle V_1 \rangle$ and $\langle V_2 \rangle$ contain cycles.

Case 1. *Suppose one of $\langle V_1 \rangle$ or $\langle V_2 \rangle$ fails to contain a cycle, say $\langle V_1 \rangle$.* In this case, $\langle V_1 \rangle$ has a transmitter in $\langle V_1 \rangle$, say v_2. However, v_2 cannot be adjacent from any of the vertices v_i, $1 \leqslant i \leqslant 9$. Hence there exist at least four additional vertices of D which are distinct from the vertices v_i, $1 \leqslant i \leqslant 9$; thus $k \geqslant 13$, producing a contradiction.

Case 2. *Each of $\langle V_1\rangle$ and $\langle V_2\rangle$ contains a cycle.* Since D has girth 4, both $\langle V_1\rangle$ and $\langle V_2\rangle$ are cycles of length 4. Observe now that each vertex of V_2 has outdegree 1 and, therefore, must be adjacent to three other vertices, none of which is v_i, $1 \leqslant i \leqslant 9$; hence $k = 12$. Suppose v_{10}, v_{11}, and v_{12} are vertices which are adjacent from all vertices of V_2. At this point each vertex of V_1 has indegree 1; thus each of these vertices must be adjacent from three additional vertices. Because D has girth 4, we must have v_{10}, v_{11}, and v_{12} adjacent to each vertex in V_1. Thus far every vertex in V_1 has insufficient outdegree while every vertex in V_2 has insufficient indegree, but all other vertices have degree 4. Thus a vertex in V_1 must be adjacent to a vertex in V_2, but this produces a 3-cycle and a contradiction.

This completes the proof.

We now turn our attention to specific values of $r > 1$. For $r = 2$, the number $g(r, n)$ has already been determined for $n = 2$, 3, and 4, namely $g(2, n) = 2n-1$.

We now show that this formula holds for $n = 5$.

THEOREM 5. $g(2, 5) = 9$.

Proof. Let D be a $[2, 5]$ digraph and v_1 a vertex of D. Denote by v_2 and v_3 the vertices of D which are adjacent to v_1; denote by v_4 and v_5 those vertices of D adjacent from v_1. Necessarily, the five vertices v_i, $1 \leqslant i \leqslant 5$, are distinct. Since D has girth 5, at least one of v_4 and v_5 is adjacent to two other vertices; say v_4 is adjacent to v_6 and v_7. Moreover, at least one of v_2 and v_3 is adjacent from two vertices different from either v_2 or v_3; say v_2 is such a vertex. It is now easily checked that v_2 is not adjacent from any of the vertices v_i, $1 \leqslant i \leqslant 7$; thus there exist two vertices v_8 and v_9 distinct from the v_i, $1 \leqslant i \leqslant 7$. This implies that $g(2, 5) \geqslant 9$, so by Theorem 1, $g(2, 5) = 9$.

Uniqueness. We have determined the number $g(r, n)$ for several values of r and n, and in each case we have shown that $g(r, n) = r(n-1)+1$. We conclude here by making some comments regarding the uniqueness of $[r, n]$ digraphs.

It has already been noted that for each $n \geqslant 2$, there is precisely one $[1, n]$ digraph and, furthermore, there is exactly one $[r, 2]$ digraph for each $r \geqslant 1$.

For other $[r, n]$ digraphs, the situation is not entirely clear. For example, it can be proved that for $(r, n) = (2, 3)$ and $(2, 4)$, there is only one $[r, n]$ digraph. Such is not the case, however, for $(r, n) = (3, 3)$,

for the digraph $D(3, 3)$ (shown in Figure 1) and the digraph of Figure 2 are non-isomorphic $[3, 3]$ digraphs.

We conclude with the following.

CONJECTURE. *For all $r \geqslant 1$, $n \geqslant 2$,*

$$g(r, n) = r(n-1)+1 .$$

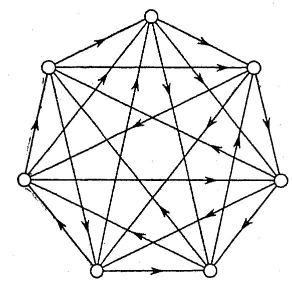


Fig. 2

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