A Kakutani type coincidence theorem

by

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1. Introduction. Brouwer's classical fixed point theorem for single-valued maps of the n-ball $B^n$ into itself has been extended to the case of coincidences of two maps in [9]. There it is shown that if $f: B^n \rightarrow B^n$ maps the boundary of $B^n$ onto itself with non-zero degree, then every pair $f, g: B^n \rightarrow B^n$ has a coincidence. Here we prove a theorem which extends in a similar manner Kakutani's fixed point theorem [7] to coincidences of "acyclic" upper semi-continuous functions. The main result is stated in Theorem 5.1 below.

This result and its proof depend on the concept of the degree of an acyclic upper semi-continuous function. It was shown by J. W. Jaworowski [6] how an acyclic upper semi-continuous function $\varphi$ on an $s$-sphere $S^n$ induces a homomorphism $\varphi_\ast: H_\ast(S^n) \rightarrow H_\ast(S^n)$. This homomorphism is used to define for such a function $\varphi$ a degree which is a generalization of the Hopf degree of a single-valued map (§ 2). Various properties of the degree of $\varphi$ are described in § 3; some are used in the later proofs, the others serve mainly to supply some examples of functions which satisfy the assumptions of the coincidence Theorems 4.2 and 5.1.

We work with Čech cohomology rather than Vietoris homology as was done in [6], but all results quoted from [9] carry over immediately.

2. Definition of the degree. The term upper semi-continuous will be defined in 2.1. This definition is the same as used e.g. in [1], p. 109 and equivalent to "weakly continuous" in [12]. Lemma 2.2 shows that it is also equivalent to the definition of continuity in [6].

Definition 2.1. A multifunction $\varphi: X \rightrightarrows Y$ from a topological space $X$ to a topological space $Y$ is a correspondence which assigns to each point of $X$ at least one point of $Y$. — $\varphi: X \rightrightarrows Y$ is use (upper semi-continuous) at the point $x_0 \in X$ if for every open set $V \subseteq Y$ containing $\varphi(x_0)$ there exists an open set $U \subseteq X$ containing $x_0$ such that $\varphi(U) \subseteq V$. — $\varphi: X \rightrightarrows Y$ is use if it is use at each point $x \in X$ and if also $\varphi(x)$ is compact for each $x \in X$.

We reserve the term map for continuous single-valued functions. — All spaces and all pairs of spaces will be assumed to be compact Hausdorff.
The following well-known property of use functions will be frequently used.

**Lemma 2.2.** \(\psi: X \to Y\) is use if and only if it is closed (i.e., if and only if its graph \((x, y) \in X \times Y, y \in \psi(x)\) in \(X \times Y\) is closed). (See [1], p. 112.)

Let \(\overline{B}\) denote reduced Čech cohomology with integer coefficients. Then we say that the function \(\psi: X \to Y\) is acyclic if \(\overline{B}^q(\psi(x)) = 0\) for all \(x \in X\) and all integer \(q\). Note that this definition is equivalent to that in [6].

Both the definition of the degree of an acyclic use function (see 2.4 below) and the proof of the results of this paper rely heavily on the Vietoris–Begle mapping theorem, which we use in the following form.

**Proposition 2.3 (Vietoris–Begle mapping theorem).** Let \(f: (X, A) \to (Y, B)\) be a closed surjective map such that \(\overline{H}^q(f^{-1}(y)) = 0\) for all \(y \in Y\) and every integer \(q\). Then \(f\) induces isomorphisms \(\overline{H}^q(Y) = \overline{H}^q(X)\) and \(\overline{H}^q(Y, B) = \overline{H}^q(X, A)\) for every integer \(q\). (See [11], p. 344.)

We now define the degree of an acyclic use function \(\psi: S^i \to S^j\), where \(S^i\) (i = 1, 2) are two spaces which have the cohomology of an \(n\)-sphere.

Let \(G \subset S^i \times S^j\) be the graph of \(\psi\), and let \(p_i: G \to S^i\) (i = 1, 2) be the restriction of the projections \(S^i \times S^j \to S^i\) to \(G\). It follows from Lemma 2.2 that the \(p_i\) are closed. They induce homomorphisms \(p_i^*: \overline{H}^q(S^i) \to \overline{H}^q(G)\), and from the Vietoris–Begle mapping theorem 2.3 it follows that \(p_i^*: \overline{H}^q(S^i) \cong \overline{H}^q(G)\) is an isomorphism. Then

\[
\varphi^* = p_1^{-1}p_2^*: \overline{H}^q(S^i) \to \overline{H}^q(S^j)
\]

is the homomorphism induced by \(\varphi\) (see [6], p. 263). Orient the \(S^i\) by choosing generators \(e_1 \in \overline{H}^q(S^i)\).

**Definition 2.4.** The degree \(\deg\) of \(\psi: S^i \to S^j\) is the unique integer which satisfies

\[
\varphi^*(e_1) = (\deg \varphi) e_1.
\]

If \(\varphi\) is single-valued then \(p_1\) is a homeomorphism and \(\varphi = p_1 \circ p_1^{-1}\), so that definition 2.4 is equivalent to the usual definition of degree in the single-valued case. As maps are acyclic use functions, we see that \(\deg \varphi\) can assume all possible integer values.

3. Some properties of the degree. Most of the properties of \(\deg \varphi\) described in this paragraph are straightforward consequences of results in [6].

Let us first investigate the invariance of \(\deg \varphi\) under homotopy. It was already pointed out in [6], p. 261 that a restriction should be placed on the homotopies considered, as e.g. the identity map of \(S^i\) is "multi-homotopic" to zero (13], corollary 1.1). A more drastic result is shown in the following proposition, in which \(I\) denotes the unit interval \(0 \leq t \leq 1\).

**Proposition 3.1.** If \(\psi: X \to Y\) is an arbitrary use function, then there exists an use function \(\Phi: X \times I \to Y\) such that \(\Phi(x, 0) = \psi(x)\) and \(\Phi(x, 1)\) is a constant map.

**Proof.** Let \(c: X \to Y\) be a constant map given by \(c(x) = y_0\) for all \(x \in X\). Construct \(\Phi: X \times I \to Y\) by

\[
\Phi(x, t) = \left\{ \begin{array}{ll}
\psi(x) & \text{for } 0 \leq t < 1/3, \\
y_0 & \text{for } 1/3 \leq t < 2/3, \\
y_0 & \text{for } 2/3 \leq t \leq 1.
\end{array}\right.
\]

Lemma 2.2 shows that \(\Phi(x, t)\) is use as its graph in \(X \times I \times Y\) is the union of the three closed sets \(\{(x, t, y) \mid x \in X, 0 \leq t < 1/3, y \in \psi(x)\}\), \(X \times [1/3, 2/3] \times Y\) and \(X \times [2/3, 1] \times \{y_0\}\), and hence closed. Therefore it seems sensible to restrict homotopies between acyclic use functions to "acyclic" homotopies, i.e. to define that two acyclic use functions \(\psi_0, \psi_1: X \to Y\) are acyclically homotopic if there exists an acyclic use function \(\Phi: X \times I \to Y\) such that \(\Phi(x, 0) = \psi_0(x)\) and \(\Phi(x, 1) = \psi_1(x)\), as was done in [6], p. 265. The homotopy invariance of \(\deg \varphi\) is then contained in the following theorem, which is a consequence of [6], Theorem 3.

**Theorem 3.2.** If \(\psi_0, \psi_1: S^i \to S^j\) are acyclically homotopic, then

\[
\deg \psi_0 = \deg \psi_1.
\]

Note that we do not prove the converse of Theorem 3.2 although it seems likely to be true. It is equivalent to the fact that every acyclic use function between \(n\)-spheres is acyclically homotopic to a map. But the existence of such a homotopy has so far only been established in special cases; the most general one known to me is that of "cellular" functions [2], [8].

If \(f: S^i \to S^k\) and \(g: S^j \to S^m\) are maps, then

\[
\deg(g \circ f) = \deg g \cdot \deg f.
\]

It is in general not possible to extend this formula to acyclic use functions, as the composite \(\psi \circ \varphi\) of two acyclic functions \(\varphi\) and \(\psi\) need not be acyclic. But in special cases it can be generalized.
Proposition 3.4. If $f: \Sigma^n \to \Sigma^n$ is a map, $\varphi: \Sigma_n^2 \to \Sigma_n^2$ an acyclic use function and $h: \Sigma_n^2 \to \Sigma_n^2$ a homeomorphism between cohomology n-spheres, then

(i) $\deg(\varphi \cdot f) = \deg f \cdot \deg \varphi$,

(ii) $\deg(h \cdot \varphi) = \deg \varphi \cdot \deg h$.

Proof. (i) is an immediate consequence of [6], Theorem 2, and (ii) follows at once from the definition 2.4 of deg.$\varphi$.

Theorem 1 in [6] implies the following result.

Proposition 3.5. If the map $f: \Sigma^2 \to \Sigma^2$ is a selection of the acyclic use function $\varphi: \Sigma^2_1 \to \Sigma^2_1$ (i.e., if $f(x) = \varphi(x)$ for every $x \in \Sigma^2_1$) then deg.$\varphi = \deg f$.

We finally determine deg.$\varphi$ for some special functions.

Proposition 3.6. If the acyclic use function $\varphi: \Sigma^2 \to \Sigma^2$ is an identity (i.e. if $x = \varphi(x)$ for all $x \in \Sigma^2$) then deg.$\varphi = 1$.

Proof. From proposition 3.5.

Proposition 3.7. If the acyclic use function $\varphi: \Sigma^2 \to \Sigma^2$ on a cohomology sphere is an involution (i.e., if $y = \varphi(y)$) then deg.$\varphi = \pm 1$.

Proof. As the graph $G$ of the involution $\varphi$ is symmetric about the diagonal of $\Sigma^2 \times \Sigma^2$ we see that in this case not only $p_1$, but also $p_2, G \to \Sigma^2$ satisfies the assumptions of the Victoris-Begle mapping Theorem 2.3., so that both $p_1^* \varphi$ and $p_2^* \varphi$ are isomorphisms. Hence $\varphi(a \sigma) = \pm a \sigma$ in the Definition 2.4 of deg.$\varphi$.

Note that in the remaining two cases $\varphi$ has to be a proper sphere and not only a cohomology sphere.

Proposition 3.8. If the acyclic use function $\varphi: \Sigma^2 \to \Sigma^2$ on a sphere is antipodal (i.e., if $-x = \varphi(x)$ for all $x \in \Sigma^2$) then deg.$\varphi = (-1)^{n+1}$.

Proof. Let $a: \Sigma^2 \to \Sigma^2$ be the antipodal map given by $a(x) = -x$.

As it is a selection of $a$ the assertion follows from Proposition 3.5 and the fact that deg.$a = (-1)^{n+1}$.

Proposition 3.9. If the acyclic use function $\varphi: \Sigma^2 \to \Sigma^2$ on a sphere has no fixed point (i.e., if for no $x \in \Sigma^2$ is $x = \varphi(x)$) then deg.$\varphi = (-1)^{n+1}$.

Proof. It is shown in [6], lemma p. 266 that if for every $x \in \Sigma^2$ the set $\varphi(x)$ does not contain the antipode of a map $f: \Sigma^2 \to \Sigma^2$ then $\varphi$ is acyclically homotopic to $f$. Hence the fixed point free function $\varphi$ is acyclically homotopic to the antipodal map $a: \Sigma^2 \to \Sigma^2$, so that deg.$\varphi = \deg a = (-1)^{n+1}$.

4. A coincidence theorem for the n-ball. We now come to the main subject of the paper, the coincidence theorems. First we prove a special case concerning coincidences of functions of n-balls.

Definition 4.1. A coincidence of a pair of functions $\varphi, \psi: X \to Y$ is a point $x \in X$ with $\varphi(x) \equiv \psi(x) \neq \emptyset$.

Let $B^n = \{x \in \mathbb{R}^n, \|x\| \leq 1\}$ denote the n-ball in Euclidean n-space $\mathbb{R}^n$.

Proposition 4.2. Every pair $\varphi, \psi: B^n \to B^n$ of acyclic use functions in which $\varphi$ transforms the boundary of $B^n$ onto itself with non-zero degree has a coincidence.

Proof. Let

$$G = \{(x, y, z) \in B^n \times B^n \times B^n | x \in B^n, y \neq \varphi(x), z = \psi(x)\}$$

be the graph of the product function $\varphi \times \psi: B^n \times B^n \to B^n$. Denote the centre of $B^n$ by $\mathbf{0}$, the boundary of $B^n$ by $\partial B^n$, and define

$$E = \{(x, y, z) \in B^n \times B^n \times B^n | x \in \partial B^n, y = \varphi(x), z = \psi(x)\},$$

$$E = \{(x, y, z) \in B^n \times B^n \times B^n | x \in B^n, y = \varphi(x)\},$$

$$D = \{(x, y, z) \in B^n \times B^n \times B^n | y = \mathbf{0}\}.$$

We assume that the pair $\varphi, \psi$ has no coincidence on $\partial B^n$—otherwise there is nothing to prove—and consider the following diagram

$$\begin{array}{ccc}
\tilde{H}^n(G, B^n) & \xrightarrow{\varphi} & \tilde{H}^n(B^n \times B^n \times B^n, B^n \times B^n \setminus D) \\
\downarrow \varphi & & \downarrow\psi \\
\tilde{H}^n(E, B^n \times B^n \times B^n) & \xrightarrow{\varphi} & \tilde{H}^n(B^n \times B^n \times B^n, B^n \setminus \partial B^n) \\
\downarrow \varphi & & \downarrow\psi \\
\tilde{H}^n(D, B^n \times B^n \times B^n) & \xrightarrow{\varphi} & \tilde{H}^n(B^n \times B^n \times B^n, B^n) \\
\end{array}$$

(4.3)

The homomorphisms are defined as follows: $\varphi^*, \psi^*$ and $\varphi^* \psi^*$ are induced by inclusions, $p_2^*$ and $p_3^*$ are induced by $p_2(x, y, z) = (x, y, z), p_2^*$ and $p_3^*$ are induced by $p_3(x, y, z) = (c, y, z), p_2^*$ is induced by $g(x, y, z) = (x, y, z)$.

If the pair $\varphi, \psi$ has no coincidence on $B^n$, then $G \subseteq B^n \times B^n \times B^n \setminus D$.

We prove that this cannot happen by showing that $\varphi$ cannot be the zero homomorphism.

(i) The diagonal is commutative: Clearly $\varphi^* \psi^* = \varphi^* \psi^*$. To see the commutativity of the rest, consider the deformation retraction of $B^n \times B^n \times B^n, D$ to $B^n \times B^n \times B^n$ defined by

$$f[x, y, z] = (1-\lambda)[x, (1-\lambda)y, (1-\lambda)z] + \lambda[x, y, z],$$

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where $x \in B^n, y \in B^n, z \in B^n$, $y \neq z$, $0 \leq \lambda < 1$, $0 < t < 1$. It shows that $i^*$ is an isomorphism, and that its inverse
\[ r^*: \overline{H}^{n-1}(B^n \times B^n \times B^n) \rightarrow \overline{H}^{n-1}(B^n \times B^n) \]
is induced by the retraction $r = f_1$. Hence if $(x, y, z) \in F$, then $\{x = 0 \}$ in \((4.4)\)
\[ p \circ r \circ f = (x, y, z) = (x, y, z), \]
so that
\[ j^* \circ i^* = p^* = q^* \circ p^* \]
(ii) The homomorphisms $i^*, q^*, p^*$, $q^*$, $\delta$, and $\delta'$ are isomorphisms: That $i^*$ is an isomorphism was proved in (i). The Vietoris–Begle mapping theorem 2.3 shows that $p^*$, $q^*$, and $q^*$ are isomorphisms $\delta$ is an isomorphism as $B^n \times B^n \times B^n$ is contractible. Finally, the map $p_1 \times q_1: (B^n \times B^n) \rightarrow (B^n \times B^n)$ satisfies the assumptions of the Vietoris–Begle mapping theorem 2.3 because of the Künneth formula and hence induces isomorphisms $q^* \circ i^*$ and $q^* \circ i^*$ in the commutative diagram
\[ \overline{H}^n(G, F) \rightarrow \overline{H}^{n-1}(B^n \times B^n \times B^n) \]
\[ \overline{H}^{n-1}(G, F) \rightarrow \overline{H}^{n-1}(B^n \times B^n \times B^n) \]
The coboundary operator $\delta^*$ of the exact sequence of $(B^n \times B^n \times B^n) \cong \overline{H}^{n-1}(B^n \times B^n \times B^n)$ is an isomorphism, and therefore $\delta'$ is an isomorphism.
(iii) Select a generator $z$ of the infinite cyclic group
\[ \overline{H}^{n-1}(B^n \times B^n \times B^n) \rightarrow \overline{H}^{n-1}(B^n \times B^n \times B^n) \]
Then $p^* \circ i^* \circ \delta^{-1}(z) = z$ is a generator of $\overline{H}^{n-1}(B^n \times B^n \times B^n)$, and by the definition 2.4 of $\deg \psi$ we have
\[ (4.5) \quad (p^* \circ i^* \circ \delta^{-1}(z)) = (\deg \psi) \cdot z \neq 0, \]
where $z$ is a generator of $\overline{H}^{n-1}(B^n \times B^n \times B^n)$. But $(4.3)$ shows that
\[ (4.6) \quad \delta^* \circ i^* = p^* \circ i^* \circ \delta^{-1}(z) = \delta^* \circ q^* \circ p^* \circ i^* \circ \delta^{-1}(z), \]
If the pair $\varphi$, $\psi$ has no coincidence on $B^n$, then $f^*$ would be the zero homomorphism, and $(4.6)$ would imply $p_2 \circ i = 0$ in contradiction to $(4.5)$. Hence Proposition 4.2 follows.
Holstynski [5] called a map $f: X \rightarrow Y$ universal for all maps of $X$ into $Y$ if every map $g: X \rightarrow Y$ has a coincidence with $f$. The more precise term coincidence producing for the same property was suggested in [10].

Using the latter terminology, we can formulate the following corollary to Propositions 4.2 and 3.6 to 3.9.

**Corollary 4.7.** If the acyclic use function $\varphi: B^n \rightarrow B^n$ transforms the boundary of $B^n$ onto itself and is on this boundary either an identity, an inversion, an antipodal, or a fixed point free, then it is coincidence producing for all acyclic use functions of $B^n$ onto itself.

5. **The main theorem.** We now derive the general case concerning coincidences of acyclic use functions on convex subsets of $B^n$. The proof uses the well-known result [3], p. 31, that every compact convex subset $C$ of $B^n$ is homeomorphic to an $r$-ball $(r < n)$. Hence the boundary of $C$ is a homology $(n-1)$-sphere, so that the degree of an acyclic use function of the boundary of $C$ into itself is defined.

**Theorem 5.1.** Every pair $\varphi, \psi: C \rightarrow C$ of acyclic use function of a compact convex subset of $B^n$ into itself in which $\psi$ transforms the boundary of $C$ onto itself with non-zero degree has a coincidence.

**Proof.** Let $h: C \rightarrow B^n$ be a homeomorphism and consider the pair $h \circ \varphi \circ h^{-1}$, $h \circ \psi \circ h^{-1}: B^n \rightarrow B^n$. By Proposition 3.4 it satisfies the assumptions of Proposition 4.3, so that there exists a point $p \in B^n$ with $h \circ \varphi \circ h^{-1}(p) \cap h \circ \psi \circ h^{-1}(p) = \emptyset$. Then $h^{-1}(p) = x \in C$ has the property $\varphi(x) \cap \psi(x) \neq \emptyset$, and therefore Theorem 5.1 is proved.

If $\varphi$ is the identity map, we obtain the result shown in [4]:

**Corollary 5.2.** Let $C$ be a compact convex subset of $B^n$ and $\varphi: C \rightarrow C$ an acyclic use function. Then $\psi$ has a fixed point.

As every convex subset of $B^n$ is acyclic, Kakutani's theorem [7] is obtained as the following special case.

**Corollary 5.3 (Kakutani).** Let $C$ be a compact convex subset of $B^n$ and $\psi: C \rightarrow C$ an acyclic use function such that the set $\psi(x)$ is convex for every $x \in C$. Then $\psi$ has a fixed point.

**References**


16b
On minimal regular digraphs with given girth

by

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Introduction. A problem in graph theory which has received much attention in recent years is the determination of the smallest number $f(r, n)$ of vertices that a graph $G$ may possess such that $G$ has degree $r$ and girth $n$. (See [1], for example.) With few exceptions, the numbers $f(r, n)$ are unknown for $r \geq 3$ and $n \geq 5$. The purpose of this article is to study the analogous problem for digraphs (directed graphs).

The Function $g(r, n)$. For a vertex $v$ of a digraph $D$, we denote by $\text{id}v$ and $\text{od}v$ the indegree and outdegree, respectively, of $v$. If $\text{id}v = \text{od}v = r$, then we speak of the degree of $v$ and write $\text{deg}v = r$. If every vertex of $D$ has degree $r$, then $D$ is said to be regular of degree $r$ or simply $r$-regular.

The girth of a digraph $D$ containing (directed) cycles is the length of the smallest cycle in $D$. For $n \geq 2$ and $r \geq 1$, we define $g(r, n)$ as the minimum number of vertices in an $r$-regular digraph $D$ having girth $n$. It is obvious that $g(1, n) = n$ since the $n$-cycle has the desired properties and is clearly minimal. The cycle is a member of a more general class of regular digraphs which we now describe.

For $r \geq 1$ and $n \geq 2$ we denote by $D(r, n)$ the digraph whose $r(n-1)+1$ vertices are labeled $v_i$, $i = 1, 2, ..., r(n-1)+1$, and such that $v_iv_j$ is an arc if and only if $j = i+1, i+2, ..., i+r$, where the numbers are expressed modulo $n$. The digraphs $D(2, 5)$ and $D(3, 3)$ are shown in Figure 1.

Clearly, $D(r, n)$ is $r$-regular and, furthermore, it is easily seen that $D(r, n)$ contains cycles of every length $k$, $n \leq k \leq r(n-1)+1$, but of no length $k$, $k < n$, so that $D(r, n)$ has girth $n$. This construction implies the following.

Theorem 1. For each $r \geq 1$ and $n \geq 2$, the number $g(r, n)$ exists and, moreover,

$$g(r, n) \leq r(n-1)+1.$$