A Kakutani type coincidence theorem

by

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1. Introduction. Brouwer's classical fixed point theorem for single-valued maps of the n-ball B^n into itself has been extended to the case of coincidences of two maps in [9]. There it is shown that if $f\colon B^n \to B^n$ maps the boundary of B^n onto itself with non-zero degree, then every pair $f, g\colon B^n \to B^n$ has a coincidence. Here we prove a theorem which extends in a similar manner Kakutani's fixed point theorem [7] to coincidences of "acyclic" upper semi-continuous functions. The main result is stated in Theorem 5.1 below.

This result and its proof depend on the concept of the degree of an acyclic upper semi-continuous function. It was shown by J. W. Jaworowski [6] how an acyclic upper semi-continuous function φ on an n-sphere S^n induces a homomorphism $\varphi_*\colon H_n(S^n) \to H_n(S^n)$. This homomorphism is used to define for such a function a degree which is a generalization of the Hopf degree of a single-valued map (§ 2). Various properties of the degree of φ are described in § 3; some are used in the later proofs, the others serve mainly to supply some examples of functions which satisfy the assumptions of the coincidence Theorems 4.2 and 5.1.

We work with Čech cohomology rather than Vietoris homology as was done in [6], but all results quoted from [6] carry over immediately.

2. Definition of the degree. The term upper semi-continuous will be defined in 2.1. This definition is the same as used e.g. in [1], p. 109 and equivalent to "weakly continuous" in [12]. Lemma 2.2 shows that it is also equivalent to the definition of continuity in [6].

Definition 2.1. A multifunction $\varphi\colon X \to Y$ from a topological space X to a topological space Y is a correspondence which assigns to each point of X at least one point of Y. — $\varphi\colon X \to Y$ is usc (*upper semi-continuous*) at the point $x_0 \in X$ if for every open set $V \subset Y$ containing $\varphi(x_0)$ there exists an open set $U \subset X$ containing x_0 such that $\varphi(U) \subset V$. — $\varphi\colon X \to Y$ is usc if it is usc at each point $x \in X$ and if also $\varphi(x)$ is compact for each $x \in X$.

We reserve the term map for continuous single-valued functions. — All spaces and all pairs of spaces will be assumed to be compact Hausdorff.

The following well-known property of usc functions will be frequently used.

LEMMA 2.2. *φ: $X \to Y$ is usc if and only if it is closed (i.e. if and only if its graph $\{(x,y)|\ x \in X, y \in Y, y \in \varphi(x)\}$ in $X \times Y$ is closed).* (See [1], p. 112.)

Let $\tilde{H}$ denote reduced Čech cohomology with integer coefficients. Then we say that the function φ: $X \to Y$ is acyclic if $\tilde{H}^q(\varphi(x)) = 0$ for all $x \in X$ and all integer q. Note that this definition is equivalent to that in [6].

Both the definition of the degree of an acyclic usc function (see 2.4 below) and the proof of the results of this paper rely heavily on the Vietoris-Begle mapping theorem, which we use in the following form.

PROPOSITION 2.3 (Vietoris–Begle mapping theorem). *Let f: $(X, A) \to (Y, B)$ be a closed surjective map such that $\tilde{H}^q(f^{-1}(y)) = 0$ for all $y \in Y$ and every integer q. Then f induces isomorphisms $\tilde{H}^q(Y) \simeq \tilde{H}^q(X)$ and $\tilde{H}^q(Y, B) \simeq H^q(X, A)$ for every integer q.* (See [11], p. 344.)

We now define the degree of an acyclic usc function φ: $S_1^n \to S_2^n$, where S_i^n $(i = 1, 2)$ are two spaces which have the cohomology of an n-sphere.

Let $G \subset S_1^n \times S_2^n$ be the graph of φ, and let p_i: $G \to S_i^n$ $(i = 1, 2)$ be the restriction of the projections $S_1^n \times S_2^n \to S_i^n$ to G. It follows from Lemma 2.2 that the p_i are closed. They induce homomorphisms p_i^*: $\tilde{H}^n(S_i^n) \to \tilde{H}^n(G)$, and from the Vietoris–Begle mapping theorem 2.3 it follows that

$$p_1^*\colon\ \tilde{H}^n(S_1^n) \simeq \tilde{H}^n(G)$$

is an isomorphism. Then

$$\varphi^* = p_1^{*-1} p_2^*\colon\ \tilde{H}^n(S_2^n) \to \tilde{H}^n(S_1^n)$$

is the homomorphism induced by φ (see [6], p. 263). Orient the S_i^n by choosing generators $z_i \in \tilde{H}^n(S_i^n)$.

DEFINITION 2.4. The *degree* $\deg\varphi$ of φ: $S_1^n \to S_2^n$ is the unique integer which satisfies

$$\varphi^*(z_2) = (\deg\varphi) z_1 .$$

If φ is single-valued then p_1 is a homeomorphism and $\varphi = p_2 \circ p_1^{-1}$, so that definition 2.4 is equivalent to the usual definition of degree in the single-valued case. As maps are acyclic usc functions, we see that $\deg\varphi$ can assume all possible integer values.

3. Some properties of the degree. Most of the properties of $\deg\varphi$ described in this paragraph are straightforward consequences of results in [6].

Let us first investigate the invariance of $\deg\varphi$ under homotopy. It was already pointed out in [6], p. 261 that a restriction should be placed on the homotopies considered, as e.g. the identity map of S^1 is "multi-homotopic" to zero ([13], corollary 1.1). A more drastic result is shown in the following proposition, in which I denotes the unit interval $0 \leqslant t \leqslant 1$.

PROPOSITION 3.1. *If φ: $X \to Y$ is an arbitrary usc function, then there exists an usc function Φ: $X \times I \to Y$ such that $\Phi(x, 0) = \varphi(x)$ and $\Phi(x, 1)$ is a constant map.*

Proof. Let c: $X \to Y$ be a constant map given by $c(x) = y_0$ for all $x \in X$. Construct Φ: $X \times I \to Y$ by

$$\Phi(x, t) = \begin{cases} \varphi(x) & \text{for} \quad 0 \leqslant t < 1/3 , \\ Y & \text{for} \quad 1/3 \leqslant t \leqslant 2/3 , \\ y_0 & \text{for} \quad 2/3 < t \leqslant 1 . \end{cases}$$

Lemma 2.2 shows that $\Phi(x, t)$ is usc as its graph in $X \times I \times Y$ is the union of the three closed sets $\{(x, t, y)|\ x \in X, 0 \leqslant t \leqslant 1/3, y \in \varphi(x)\}$, $X \times [1/3, 2/3] \times Y$ and $X \times [2/3, 1] \times \{y_0\}$, and hence closed.

Therefore it seems sensible to restrict homotopies between acyclic usc functions to "acyclic" homotopies, i.e. to define that two acyclic usc functions φ_0, φ_1: $X \to Y$ are acyclically homotopic if there exists an acyclic usc function Φ: $X \times I \to Y$ such that $\Phi(x, 0) = \varphi_0(x)$ and $\Phi(x, 1) = \varphi_1(x)$, as was done in [6], p. 265. The homotopy invariance of $\deg\varphi$ is then contained in the following theorem, which is a consequence of [6], Theorem 3.

THEOREM 3.2. *If φ_0, φ_1: $S_1^n \to S_2^n$ are acyclically homotopic, then*

$$\deg\varphi_0 = \deg\varphi_1 .$$

Note that we do not prove the converse of Theorem 3.2 although it seems likely to be true. It is equivalent to the fact that every acyclic usc function between n-spheres is acyclically homotopic to a map. But the existence of such a homotopy has so far been only established in special cases; the most general one known to me is that of "cellular" functions [2], [8].

If f: $S_1^n \to S_2^n$ and g: $S_2^n \to S_3^n$ are maps, then

$$\deg(g \circ f) = \deg f \cdot \deg g .$$

It is in general not possible to extend this formula to acyclic usc functions, as the composite $\psi \circ \varphi$ of two acyclic functions φ and ψ need not by acyclic. But in special cases it can be generalized.

PROPOSITION 3.4. *If* $f\colon \Sigma_1^n \to S_1^n$ *is a map,* $\varphi\colon S_1^n \to S_2^n$ *an acyclic usc function and* $h\colon S_2^n \to \Sigma_2^n$ *a homeomorphism between cohomology* n*-spheres, then*

(i) $$\deg(\varphi \circ f) = \deg f \cdot \deg \varphi ,$$

(ii) $$\deg(h \circ \varphi) = \deg \varphi \cdot \deg h .$$

Proof. (i) is an immediate consequence of [6], Theorem 2, and (ii) follows at once from the definition 2.4 of $\deg\varphi$.

Theorem 1 in [6] implies the following result.

PROPOSITION 3.5. *If the map* $f\colon S_1^n \to S_2^n$ *is a selection of the acyclic usc function* $\varphi\colon S_1^n \to S_2^n$ *(i.e. if* $f(x) = \varphi(x)$ *for every* $x \in S_1^n$*), then* $\deg\varphi = \deg f$.

We finally determine $\deg\,\varphi$ for some special functions.

PROPOSITION 3.6. *If the acyclic usc function* $\iota\colon S^n \to S^n$ *on a cohomology sphere is an identity (i.e. if* $x \in \iota(x)$ *for all* $x \in S^n$*) then* $\deg\iota = 1$.

Proof. From proposition 3.5.

PROPOSITION 3.7. *If the acyclic usc function* $\varphi\colon S^n \to S^n$ *on a cohomology sphere is an involution (i.e. if* $y \in \varphi(x)$ *implies* $x \in \varphi(y)$*) then* $\deg\varphi = \pm 1$.

Proof. As the graph G of the involution φ is symmetric about the diagonal of $S^n \times S^n$ we see that in this case not only p_1, but also $p_2\colon G \to S^n$ satisfies the assumptions of the Vietoris–Begle mapping Theorem 2.3., so that both p_1^* and p_2^* are isomorphisms. Hence $\varphi^*(z_2) = \pm z_1$ in the Definition 2.4 of $\deg\varphi$.

Note that in the remaining two cases S^n has to be a proper sphere and not only a cohomology sphere.

PROPOSITION 3.8. *If the acyclic usc function* $\alpha\colon S^n \to S^n$ *on a sphere is antipodal (i.e. if* $-x \in \alpha(x)$ *for all* $x \in S^n$*) then* $\deg\alpha = (-1)^{n+1}$.

Proof. Let $a\colon S^n \to S^n$ be the antipodal map given by $a(x) = -x$. As it is a selection of α the assertion follows from Proposition 3.5 and the fact that $\deg a = (-1)^{n+1}$.

PROPOSITION 3.9. *If the acyclic usc function* $\varphi\colon S^n \to S^n$ *on a sphere has no fixed point (i.e. if for no* $x \in S^n$ *is* $x \in \varphi(x)$*) then* $\deg\varphi = (-1)^{n+1}$.

Proof. It is shown in [6], lemma p. 266 that if for every $x \in S^n$ the set $\varphi(x)$ does not contain the antipode of a map $f\colon S^n \to S^n$ then φ is acyclically homotopic to f. Hence the fixed point free function φ is acyclically homotopic to the antipodal map $a\colon S^n \to S^n$, so that $\deg\varphi = \deg a = (-1)^{n+1}$.

4. A coincidence theorem for the n-ball. We now come to the main subject of the paper, the coincidence theorems. First we prove a special case concerning coincidences of functions of n-balls.

DEFINITION 4.1. A *coincidence* of a pair of functions $\varphi, \psi\colon X \to Y$ is a point $x \in X$ with $\varphi(x) \cap \psi(x) \neq \emptyset$.

Let $B^n = \{x \in R^n \mid \|x\| \leqslant 1\}$ denote the n-ball in Euclidean n-space R^n.

PROPOSITION 4.2. *Every pair* $\varphi, \psi\colon B^n \to B^n$ *of acyclic usc functions in which* φ *transforms the boundary of* B^n *onto itself with non-zero degree has a coincidence.*

Proof. Let

$$G = \{(x, y, z) \in B^n \times B^n \times B^n \mid x \in B^n, y \in \varphi(x), z \in \psi(x)\}$$

be the graph of the product function $\varphi \times \psi\colon B^n \to B^n \times B^n$. Denote the centre of B^n by c, the boundary of B^n by $\dot{B}^n$, and define

$$F = \{(x, y, z) \in \dot{B}^n \times \dot{B}^n \times B^n \mid x \in \dot{B}^n, y \in \varphi(x), z \in \psi(x)\} ,$$

$$E = \{(x, y, c) \in \dot{B}^n \times \dot{B}^n \times c \mid x \in \dot{B}^n, y \in \varphi(x)\} ,$$

$$D = \{(x, y, z) \in B^n \times B^n \times B^n \mid y = z\} .$$

We assume that the pair φ, ψ has no coincidence on $\dot{B}^n$—otherwise there is nothing to prove — and consider the following diagram

(4.3)
$$\begin{array}{ccccc}
 & & \tilde{H}^n(G, F) & \xleftarrow{\;j^*\;} & \tilde{H}^n(B^n\times B^n\times B^n,\ B^n\times B^n\times B^n\setminus D) \\
 & & \uparrow{\scriptstyle \partial'} & & \uparrow{\scriptstyle \partial} \\
 & & \tilde{H}^{n-1}(F) & \xleftarrow{\;j'^*\;} & \tilde{H}^{n-1}(B^n\times B^n\times B^n\setminus D) \\
 & & \uparrow{\scriptstyle q^*} & & \downarrow{\scriptstyle i^*} \\
\tilde{H}^{n-1}(\dot{B}^n\times c\times c) & \xrightarrow{\;p_1^*\;} & \tilde{H}^{n-1}(E) & & \tilde{H}^{n-1}(B^n\times \dot{B}^n\times c) \\
 & & \nwarrow{\scriptstyle p_2^*} & & \nearrow{\scriptstyle p^*} \\
 & & & \tilde{H}^{n-1}(c\times \dot{B}^n\times c) . &
\end{array}$$

The homomorphisms are defined as follows: i^*, j^*, and j'^* are induced by inclusions, ∂, ∂' are coboundary operators, p_1^* is induced by $p_1(x, y, c) = (x, c, c)$, p_2^* and p^* are induced by $p_2(x, y, c) = (c, y, c)$, q^* is induced by $q(x, y, z) = (x, y, c)$.

If the pair φ, ψ has no coincidence on B^n, then $G \subset B^n \times B^n \times B^n \setminus D$. We prove that this cannot happen by showing that j^* cannot be the zero homomorphism.

(i) *The diagram is commutative*: Clearly $j^* \circ \partial = \partial' \circ j'^*$. To see the commutativity of the rest, consider the deformation retraction of $B^n \times B^n \times B^n \setminus D$ to $B^n \times \dot{B}^n \times c$ defined by

(4.4) $$f_t\big(x, \lambda z + (1-\lambda)y, z\big) = \big(x, (1-t)\lambda z + (1-\lambda+\lambda t)y, (1-t)z + tc\big) ,$$

where $x \in B^n$, $y \in \dot{B}^n$, $z \in B^n$, $y \neq z$, $0 \leqslant \lambda < 1$, $0 \leqslant t \leqslant 1$. It shows that i^* is an isomorphism, and that its inverse

$$r^*\colon \tilde{H}^{n-1}(B^n \times \dot{B}^n \times c) \to \tilde{H}^{n-1}(B^n \times B^n \times B^n \backslash D)$$

is induced by the retraction $r = f_1$. Hence if $(x, y, z) \in F$, then (with $\lambda = 0$ in (4.4))

$$p \circ r \circ j'(x, y, z) = (c, y, c) = p_2 \circ q(x, y, z)\,,$$

so that

$$j'^* \circ i^{*-1} \circ p^* = q^* \circ p_2^*\,.$$

(ii) *The homomorphisms i^*, p_1^*, p^*, q^*, ∂, and ∂' are isomorphisms*: That i^* is an isomorphism was proved in (i). The Vietoris–Begle mapping theorem 2.3 shows that p_1^*, p^*, and q^* are isomorphisms. ∂ is an isomorphism as $B^n \times B^n \times B^n$ is contractible. Finally, the map $p_1 \circ q\colon (G, F) \to (B^n \times c \times c, \dot{B}^n \times c \times c)$ satisfies the assumptions of the Vietoris–Begle mapping theorem 2.3 because of the Künneth formula and hence induces isomorphisms q_1^* and q_2^* in the commutative diagram

$$\begin{array}{ccc} \tilde{H}^n(G, F) & \xleftarrow{q_1^*} & \tilde{H}^n(B^n \times c \times c, \dot{B}^n \times c \times c) \\ \uparrow{\scriptstyle \partial'} & & \uparrow{\scriptstyle \partial''} \\ \tilde{H}^{n-1}(F) & \xleftarrow{q_2^*} & \tilde{H}^{n-1}(\dot{B}^n \times c \times c)\,. \end{array}$$

The coboundary operator ∂'' of the exact sequence of $(B^n \times c \times c, \dot{B}^n \times \times c \times c)$ is an isomorphism, and therefore ∂' is an isomorphism.

(iii) Select a generator z of the infinite cyclic group

$$\tilde{H}^n(B^n \times B^n \times B^n,\ B^n \times B^n \times B^n \backslash D)\,.$$

Then $p^{*-1} \circ i^* \circ \partial^{-1}(z) = z_2$ is a generator of $\tilde{H}^{n-1}(c \times \dot{B}^n \times c)$, and by the definition 2.4 of $\deg\varphi$ we have

$$p_1^{*-1} \circ p_2^*(z_2) = (\deg\varphi) \cdot z_1 \neq 0\,, \tag{4.5}$$

where z_1 is a generator of $\tilde{H}^{n-1}(\dot{B}^n \times c \times c)$. But (4.3) shows that

$$j^*(z) = \partial' \circ q^* \circ p_2^* \circ p^{*-1} \circ i^* \circ \partial^{-1}(z) = \partial' \circ q^* \circ p_2^*(z_2)\,. \tag{4.6}$$

If the pair φ, ψ had no coincidence on B^n, then j^* would be the zero homomorphism, and (4.6) would imply $p_2^*(z_2) = 0$ in contradiction to (4.5). Hence Proposition 4.2 follows.

Holsztyński [5] called a map $f\colon X \to Y$ universal for all maps of X into Y if every map $g\colon X \to Y$ has a coincidence with f. The more precise term coincidence producing for the same property was suggested in [10].

Using the latter terminology, we can formulate the following corollary to Propositions 4.2 and 3.6 to 3.9.

COROLLARY 4.7. *If the acyclic usc function $\varphi\colon B^n \to B^n$ transforms the boundary of B^n onto itself and is on this boundary either an identity, an involution, antipodal, or fixed point free, then it is coincidence producing for all acyclic usc functions of B^n onto itself.*

5. The main theorem. We now derive the general case concerning coincidences of acyclic usc functions on convex subsets of R^n. The proof uses the well-known result ([3], p. 31) that every compact convex subset C of R^n is homeomorphic to an r-ball ($r \leqslant n$). Hence the boundary of C is a cohomology $(r-1)$-sphere, so that the degree of an acyclic usc function of the boundary of C into itself is defined.

THEOREM 5.1. *Every pair $\varphi, \psi\colon C \to C$ of acyclic usc functions of a compact convex subset of R^n into itself in which φ transforms the boundary of C onto itself with non-zero degree has a coincidence.*

Proof. Let $h\colon C \to B^r$ be a homeomorphism and consider the pair $h \circ \varphi \circ h^{-1}$, $h \circ \psi \circ h^{-1}\colon B^r \to B^r$. By Proposition 3.4 it satisfies the assumptions of Proposition 4.1, so that there exists a point $p \in B^r$ with $h \circ \varphi \circ h^{-1}(p) \cap h \circ \psi \circ h^{-1}(p) \neq \emptyset$. Then $h^{-1}(p) = x \in C$ has the property $\varphi(x) \cap \psi(x) \neq \emptyset$, and therefore Theorem 5.1 is proved.

If φ is the identity map, we obtain the result shown in [4]:

COROLLARY 5.2. *Let C be a compact convex subset of R^n and $\psi\colon C \to C$ an acyclic usc function. Then ψ has a fixed point.*

As every convex subset of R^n is acyclic, Kakutani's theorem [7] is obtained as the following special case.

COROLLARY 5.3 (Kakutani). *Let C be a compact convex subset of R^n and $\psi\colon C \to C$ an usc function such that the set $\psi(x)$ is convex for every $x \in C$. Then ψ has a fixed point.*

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On minimal regular digraphs with given girth

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Introduction. A problem in graph theory which has received much attention in recent years is the determination of the smallest number $f(r, n)$ of vertices that a graph G may posses such that G has degree r and girth n. (See [1], for example.) With few exceptions, the numbers $f(r, n)$ are unknown for $r \geqslant 3$ and $n \geqslant 5$. The purpose of this article is to study the analogous problem for digraphs (directed graphs).

The Function $g(r, n)$. For a vertex v of a digraph D, we denote by $\mathrm{id}\, v$ and $\mathrm{od}\, v$ the indegree and outdegree, respectively, of v. If $\mathrm{id}\, v = \mathrm{od}\, v = r$, then we speak of the *degree* of v and write $\deg v = r$. If every vertex of D has degree r, then D is said to be *regular of degree* r or simply r*-regular*.

The *girth* of a digraph D containing (directed) cycles is the length of the smallest cycle in D. For $n \geqslant 2$ and $r \geqslant 1$, we define $g(r, n)$ as the minimum number of vertices in an r-regular digraph D having girth n. It is obvious that $g(1, n) = n$ since the n-cycle has the desired properties and is clearly minimal. The cycle is a member of a more general class of regular digraphs which we now describe.

For $r \geqslant 1$ and $n \geqslant 2$ we denote by $D(r, n)$ the digraph whose $r(n-1)+1$ vertices are labeled v_i, $i = 1, 2, \ldots, r(n-1)+1$, and such that $v_i v_j$ is an arc if and only if $j = i+1, i+2, \ldots, i+r$, where the numbers are expressed modulo $r(n-1)+1$. The digraphs $D(2, 5)$ and $D(3, 3)$ are shown in Figure 1.

Clearly, $D(r, n)$ is r-regular and, furthermore, it is easily seen that $D(r, n)$ contains cycles of every length k, $n \leqslant k \leqslant r(n-1)+1$ but of no length k, $k < n$, so that $D(r, n)$ has girth n. This construction implies the following.

THEOREM 1. *For each* $r \geqslant 1$ *and* $n \geqslant 2$, *the number* $g(r, n)$ *exists and, moreover,*

$$g(r, n) \leqslant r(n-1)+1 . \tag{1}$$