Fixed points, index, and degree for some set valued functions

by

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A topological space X is said to have the fixed point property if every continuous self mapping of the space leaves some point fixed, i.e. if $f\colon X\to X$ is continuous then $a=f(a)$ for at least one $a\in X$.

A continuous function $f\colon S^n\to S^n$ has degree k if for a "sufficiently close" simplicial approximation of f, say a, and some simplex S in the range space of a the set of simplexes in the domain space that map onto S is $\{A_1, A_2, \ldots, A_m\}$ and $k=\sum_{i=1}^{m} o(A_i)$, where $o(A_i)$ is the orientation of f restricted to A_i.

A continuous function $f\colon S^n\to R^{n+1}$ has index k with respect to the origin in R^{n+1}, 0 if $0\notin f(S^n)$ and the degree of $g(z)=f(z)/\|z\|$ is k. f has index k with respect to a point $p\in R^{n+1}$ if $g(z)=f(z)-p$ has index k with respect to 0.

It is the purpose of this paper to expand the above interrelated ideas with the use of semi-closure operators. The reader is referred to [4] for proofs of the theorems for the continuous cases which will be assumed here.

By a semi-closure operator on a topological space X we mean a set valued function T that assigns to every subset A of X a closed subset $T(A)$ of X for which (1) $A\subset T(A)$ for $A\subset X$, and (2) if $A\subset T(B)$ then $T(A)\subset T(B)$.

Examples:

(1) $C(A)=$ the convex hull of A for X convex metric.

(2) $T(A)=\bar{A}=$ the closure of A.

(3) X is a continuum and $T(A)$ is the intersection of all subcontinua of X that contain A.

(4) X is a continuum, $p\in X$, and $T(A)=X$ if $p\in\bar{A}$, $T(A)$ is the closure of the complement of the component of $X-\bar{A}$ containing p if $p\notin\bar{A}$.

Fixed point theory seems to have its origin in the intermediate value theorem of elementary calculus. We notice that the intermediate value theorem is dealt with easily with the purely topological notions of continuity and connectivity. However, the sufficiency of continuity and connectivity to deal with fixed points seems to end here. In fact all known proofs that the n-cell has the fixed point property for $n>1$, resort to some form of discrete and indeed finite mathematics. The inductive step common to all proofs of the Brouwer fixed-point theorem, (as can be found in [9] for example) is "*If a set with $n+1$ elements in mapped into a set with n elements then either 0 or 2 proper subsets are mapped onto.*".

A statement of the Brouwer theorem that would be more inclined to remove some of the mystery of the proof would be "Let X be the first $k+1$ non-negative integers and let $f\colon X^n \to X^n$ for some positive integer n. Then there is a unit cube D such that any cube that contains $f(D \cap X^n)$ intersects D."

The apparent lack of cohesiveness between the idea of fixed points and the techniques used in the proofs becomes more apparent with the continuing production of counter examples to conjectures that attempt to use spaces that have the fixed point property to build new spaces with the fixed point property. Many of these counter examples are discussed in [2]. The purpose of this section of the paper is to suggest a way to broaden the approach to fixed point theory so as to incorporate most of the existing theory into a new theory that places the theory closer to the proofs and at the same time avoids some of the standard counter examples.

DEFINITION. Let T be a semi-closure operator on a space Z and let $f\colon X \to Z$. Then $f_T\colon X \to 2^Z$ is defined by $f_T(x) = \bigcap\{T[f(U)]\colon x \in \operatorname{int} U\}$. If A is a set we denote $\bigcup\{f_T(x)\colon x \in A\}$ by $f_T(A)$. A space X has the fixed point property with respect to a semi-closure operator T if for $f\colon X\to X$ there is at least one $x \in X$ for which $x \in f_T(x)$. A function $r\colon X \to A$ is called a *semi-retraction* if $A \subset X$, $r(x) = x$ for $x \in A$, and r is continuous at each $a \in A$. A is called a *semi-retract* of X.

THEOREM (Eilenberg and Montgomery). *Let f be an upper semi-continuous set valued function defined on a compact convex subset X of R^n. If $f(x)$ is an acyclic subcontinuum of X for each $x \in X$ then f has a fixed point.*

THEOREM. *Suppose X has the fixed point property with respect to a semi-closure operator T and $A = T(A)$ is a semi-retract of X. Then A has the fixed point property with respect to T.*

Proof. Let $f\colon A \to A$. Then $fr\colon X \to A \subset X$ has a fixed point x_0. Since r is continuous at x_0 we have $f_T(x_0) \supset [f(r)]_T(x_0)$.

LEMMA. *Let T be a semi-closure operator on Y and let $f\colon X \to Y$. f_T is upper semi-continuous if each $x \in X$ has a neighborhood U for which $T(f(U))$ is compact.*

Proof. Suppose $f_T(x) \subset V$ for some open set $V \subset Y$. Let U_0 be a neighborhood of x for which $T[f(U_0)]$ is compact. Then

$$\emptyset = (f_T(x)) \cap (Y - V) = \bigcap\{(T[f(U \cap U_0)]) \cap (Y-V)\colon x \in \operatorname{int} U\}.$$

Since $(T[f(U \cap U_0)]) \cap (Y-V)$ is compact for $x \in \operatorname{int} U$ there is a finite set of neighborhoods $\mathfrak{U}$ such that

$$\bigcap\{(T[f(U \cap U_0)]) \cap (Y-V)\colon U \in \mathfrak{U}\} = \emptyset.$$

If $U_1 = \bigcap\{U \cap U_0\colon U \in \mathfrak{U}\}$ then $(T[f(U_1)]) \cap (Y-V) = \emptyset$. We now have $T[f(U_1)] \subset V$. Therefore, $f_T(U_1) \subset V$.

THEOREM. *Let T be a semi-closure operator on a compact space $X \subset R^n$ such that $T(A)$ is acyclic for $A \subset X$. Then X has the fixed point property with respect to T.*

Proof. Let $r\colon R^n \to X$ so that $\|r(x) - x\|$ is minimal for $x \in R^n$. r is clearly a semi-retraction.

Applications:

(1) Let X be a treelike continuum and let $T(A)$ be the intersection of all subcontinua of X that contain A. Then X has the fixed point property with respect to T.

(2) Let X be the 2-simplex in R^2 and let $T(A)$ be the intersection of all compact subsets of X that contain A and have connected complements in R^2.

The interest here is in the class of functions that preserve connected sets. That is, if $f(A)$ is connected when A is connected then there is an $x \in X$ such that every compact set that contains the image of a neighborhood of x and does not separate R^2 contains x. O. H. Hamilton in [6] proved that if X is the n-cell and $f\colon X \to X$ is such that the graph of connected sets is connected then f has a fixed point. However, the image of connected sets being connected does not guarantee a fixed point even in the case $n = 1$.

(3) X is a compact, convex subset of R^n. $T(A)$ is the closed convex hull of A. This is easily equivalent to the Kakutani fixed point theorem.

DEFINITION. Let T be a semi-closure operator on a space X and let $\mathfrak{U}$ be a collection of subsets of X. Two functions $f, g\colon Y \to X$ are said to be *homotopic with respect to T and $\mathfrak{U}$* if there is a function $F\colon Y \times [0,1] \to X$ such that $F[(y,0)] = f(y)$ and $F[(y,1)] = g(y)$ for $y \in Y$ and $F_T[(y,s)]$ is contained in some $U_{(y,s)} \in \mathfrak{U}$ for $(y,s) \in Y \times [0,1]$. If $\mathfrak{U} = \{U\}$ we shall say f and g are homotopic with respect to U. We write $f \sim_T g$ where $\mathfrak{U}$ or U is understood. We write $f \sim g$ to mean f and g are homotopic in the usual sense.

LEMMA. *$f \sim_T g$ is an equivalence relation on $\{h\colon h\colon Y \to X$ and $h_T(y) \in U_y \in \mathfrak{U}$ for $y \in Y\}$.*

DEFINITION. For n a fixed positive integer and $p \in R^{n+1}$ define $H_p^n = \{f: f\colon S^n \to R^{n+1}$ and $p \notin f_C(S^n)\}$ where $C(A)$ is the closed convex hull of A. Here $\mathfrak{U} = \{R^{n+1} - \{p\}\}$.

LEMMA. *If $f \in H_p^n$ then there is a $g \in H_p^n$ such that g is continuous and $f \sim_C g$.*

Proof. Let $\mathfrak{V}$ be an open cover of S^n for which $p \notin C[f(V)]$ for $V \in \mathfrak{V}$. Let $\mathfrak{T}$ be a simplicial decomposition of S^n for which each n-cell n $\mathfrak{T}$ is contained in some $V \in \mathfrak{V}$. Let g be the map obtained by restricting f to the vertices of $\mathfrak{T}$ and extending this restriction linearly on each n-cell of $\mathfrak{T}$. $g \sim_C f$ by $F[(x, s)] = (g(x))s + (1-s)f(x)$.

LEMMA. *If $f, g \in H_p^n$ are continuous then $f \sim_C g$ iff $f \sim g$ in $R^{n+1} - \{p\}$.*

Proof. Trivially if $f \sim g$ in $R^{n+1} - \{p\}$ then $f \sim_C g$. If $f \sim_C g$ with homotopy $F\colon S^n \times [0,1] \to R^{n+1}$ then let $\mathfrak{T}$ be a simplicial decomposition of $S^n \times [0,1]$ for which $p \notin C[F(S)]$ for $S \in \mathfrak{T}$. Then restrict F to $S^n \times \{0,1\} \cup V$ where V is the set of vertices of $\mathfrak{T}$. Then re-extend F to $S^n \times [0,1]$ so that each $S \in \mathfrak{T}$ is mapped into $C[F(S)]$.

DEFINITION. If $f \sim_C g \in H_p^n$ and g is continuous we define the index of f to be the index of g with respect to p.

LEMMA. *$f \sim_C g$ iff f and g have the same index.*

DEFINITION. Let $K^n = \{f: f\colon S^n \to S^n$ and $0 \notin f_C(S^n)\}$. Let C' be the semi-closure operator on S^n defined by $C'(A) = \{z/\|z\|: z \in C(A)\}$ if $0 \notin C(A)$ and $C'(A) = S^n$ if $0 \in C(A)$. For $f \in K^n$ the degree of f is defined to be the index of f with respect to C' and $R^{n+1} - \{0\}$.

LEMMA. *If $\mathfrak{U}$ is the collection of proper subsets of S^n then $f \sim_{C'} g$ with respect to $\mathfrak{U}$ is an equivalence relation on K^n.*

LEMMA. *$f \in K^n \to f \sim_{C'} g$ for some continuous function g.*

LEMMA. *$f \sim_{C'} g$ iff f and g have the same degree.*

THEOREM. *For $f \in K^n$ the following are equivalent:*

(1) *The degree of f is zero.*

(2) *$f \sim_{C'} g$ for some constant function g.*

(3) *There is an extension of f to then $n+1$ unit ball, E^{n+1}, F such that $0 \notin F_C(E^{n+1})$, i.e. $F_{C'}(x)$ is defined for $x \in E^{n+1}$.*

CONJECTURE. *Let $f\colon S^n \to S^n$ have degree $k \neq 0$. Let $D_f = \{x \in S^n$: cardinal $f_{C'}^{-1}(x) < |k|\}$ $(f_{C'}^{-1}(x) = \{z: x \in f_{C'}(z)\})$. If $A \subset D_f$ and $\emptyset = f_{C'}^{-1}(x) \cap f_{C'}^{-1}(y)$ for $x, y \in A$ and $x = y$, then*

(1) *the interior of A is empty;*

(2) *the dimension of $A \leqslant n-2$.*

(2) is of course stronger than (1).

In the case where f is continuous, light, interior and locally sense preserving (1) is known (see [8]).

In the case where f is continuous and $n = 2$ A is finite (see [1]). In fact if D_x is defined to be the maximum of k-cardinal $f_{C'}^{-1}(x)$ and 0 for $x \in S^2$ then $\sum_{x \in A} D_x \leqslant 2(|k|-1)$. The proof used in the continuous case contains the proof for the case $f \in K^2$.

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