Correction to the paper
"On the hyperspace of subcontinua of a finite graph, I"

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by

R. Duda (Wrocław)

Professor Jack Segal has observed that Lemmas 6.1 and 6.2 are incorrect as they stand.

The aim of this note is to show that after some modifications these lemmas can still be used in the proofs of all the theorems in which they were originally applied, and that all those theorems remain true.

Here are the details.

Let X be a finite graph. If $A \subset B$ is a pair of X, then each segment of $\overline{B-A}$ both end-points of which belong to A will be called *closing for the pair* $A \subset B$.

If X is acyclic, then obviously X contains no pair with a closing segment.

Lemma 6.1 should read:

6.1. *Let $A \subset B$ and $D \subset E$ be two distinct pairs of X such that $A \subset D \subset E \subset B$. If each segment closing for the pair $D \subset E$ is also closing for the pair $A \subset B$, then $\mathfrak{M}_{D\subset E}$ is a ball lying on the surface of the ball $\mathfrak{M}_{A\subset B}$.*

Proof of the old version of 6.1 remains valid (the hypothesis that each segment closing for the pair $D \subset E$ is also closing for the pair $A \subset B$ is needed to ensure that if $k+1 \leqslant i \leqslant l$, then the sets T_i defined for the pair $A \subset B$ remain the same for the pair $D \subset E$).

Note that if X is acyclic or if A contains all internal vertices of X, then the assumption on segments is satisfied and so Lemma 6.1 in these two cases is valid.

Lemma 6.2 should read:

6.2. *Let $A \subset B$ and $A' \subset B'$ be two distinct pairs of X. If the two balls $\mathfrak{M}_{A\subset B}$ and $\mathfrak{M}_{A'\subset B'}$ meet, then their common part either is equal to a ball $\mathfrak{M}_{D\subset E}$, where $D \subset E$ is a pair of X (this is the case if, for instance, X is acyclic or if both pairs $A \subset B$ and $A' \subset B'$ contain all internal vertices of X) or consists of finitely many disjoint balls, each lying on the surface of both $\mathfrak{M}_{A\subset B}$ and $\mathfrak{M}_{A'\subset B'}$.*

Proof. Since $\mathfrak{M}_{A\subset B}\cap\mathfrak{M}_{A'\subset B'}\neq 0$ by assumption, there exists a non-empty continuum C such that

$$A\subset C\subset B \quad \text{and} \quad A'\subset C\subset B',$$

whence

(1) $$A\cup A'\subset C\subset B\cap B'.$$

Therefore

(2) $$\mathfrak{M}_{A\subset B}\cap\mathfrak{M}_{A'\subset B'}\subset\{C\in C(X)\colon A\cup A'\subset C\subset B\cap B'\}.$$

We shall consider three cases.

I. $A=0=A'$. In this case B and B' are distinct segments and since, by (1), they meet, they have a common vertex v (by the assumption of (γ), one only) and $B\cap B'=(v)$. Clearly, (v) is the only subcontinuum of both B and B' and so

$$\mathfrak{M}_{A\subset B}\cap\mathfrak{M}_{A'\subset B'}=(v).$$

On the other hand, however, $(v)\subset(v)$ is a pair of X and

$$(v)=\mathfrak{M}_{(v)\subset(v)}.$$

The two equalities imply our lemma in the case under consideration.

II. $A\cup A'$ is non-empty and connected.

In this case we shall show that $\mathfrak{M}_{A\subset B}\cap\mathfrak{M}_{A'\subset B'}$ either is the ball $\mathfrak{M}_{A\cup A'\subset B\cap B'}$ or consists of finitely many disjoint balls, each lying on the surface of $\mathfrak{M}_{A\subset B}$ (in view of the symmetry, we may leave out the case of $\mathfrak{M}_{A'\subset B'}$). For this purpose, let

(3) $$L_1, L_2, \ldots, L_m$$

be the sequence of all segments of $\overline{B-A}$. Denoting, if necessary, by L_0 the empty set, we may assume that the segments (3) are ordered in such a way that

(4) $$A\cup A'=A\cup\bigcup_{i=0}^{k}L_i \quad \text{and} \quad B\cap B'=A\cup\bigcup_{i=0}^{l}L_i,$$

where $0\leqslant k\leqslant l\leqslant m$.

According to Lemma 5.2 there exists a homeomorphism h from the ball $\mathfrak{M}_{A\subset B}$ onto the Cartesian product $\bigtimes_{i=1}^{m}T_i$, where each T_i is either the segment $[0,1]$ (if L_i meets A at one end-point only) or the triangle of vertices $(0,0)$, $(0,1)$ and $(1,0)$ with the hypotenuse contracted to a single point (if $L_i\cap A$ consists of two end-points of L_i). Let $(0)_i$ denote the point 0 in the first case and the point $(0,0)$ in the second, and let $(1)_i$ denote the point 1 in the first case and the point of contraction in the second.

We proceed to show that under the homeomorphism h the common part $\mathfrak{M}_{A\subset B}\cap\mathfrak{M}_{A'\subset B'}$ corresponds to the set $\bigtimes_{i=1}^{m}T'_i$, where each T'_i either is equal to the whole of T_i or is a subball on the boundary of T_i or consists of two end-points of T_i (the latter case only if T_i is the segment $[0,1]$).

For $0<i\leqslant k$ put $T'_i=(1)_i$ and for $l<i\leqslant m$ put $T'_i=(0)_i$. It remains to consider $i=k+1, k+2, \ldots, l$. By virtue of (2) and (4) each $C\in\mathfrak{M}_{A\subset B}\cap\mathfrak{M}_{A'\subset B'}$ can be written in the form

$$C=A\cup A'\cup\bigcup_{i=k+1}^{l}C\cap L_i.$$

Consider segment L_i and denote its end-points by a and b, $L_i=\overline{ab}$. Since L_i is contained in both B and B', it meets both A and A'.

In view of the symmetry, the following four cases must now be considered (note, however, that for X acyclic one has only subcase 3°, and for A and A' both containing all internal vertices of X one has only subcases 1° and 3°):

1° $a\in A\cap A'$, $b\in A\cap A'$,

2° $a\in A\cap A'$, $b\in A-A'$,

3° $a\in A\cap A'$, $b\notin A\cup A'$,

4° $a\in A-A'$, $b\in A'-A$.

Subcase 1°. In this case C meets L_i along two arcs (virtually they may be reduced to end-points of L_i or cover the whole of L_i),

$$C\cap L_i=L_i(t')\cup L_i(t''_i),$$

where $0\leqslant t'_i+t''_i\leqslant 1$.

Here $T'_i=T_i$ is a triangle with the hypotenuse contracted to a point.

Subcase 2°. By virtue of (2) both end-points of L_i belong to C. We shall show that the component of $L_i\cap C$ containing b is equal either to b itself or to the whole segment L_i.

Suppose, to the contrary, that $C\cap L_i$ has a component N containing b and distinct from both L_i and b. Let M be the component of $\overline{C-A'}$ containing N. Since C is a continuum and $A'\subset C$ by assumption, we have $A'\cap M\neq 0$, and since $N\cap A'=0$ by assumption, any arc in M joining b to a point of $A'\cap M$ must contain a segment (A' is a subgraph of X and so $A'\cap M$ consists of vertices of X). In view of (α) this implies that $\varrho^1(M, A'\cap M)>1$, contrary to $C\in\mathfrak{M}_{A'\subset B'}$.

Thus we have proved what we claimed, and this means that the common part $C\cap L_i$ is a union of an arc $L_i(t)$ (which contains a and has length $t\leqslant 1$) and of the point b of C which also belongs to $A\cup A'$.

Here T_i is again a triangle with the hypotenuse contracted to a point. Let T'_i be the subset of T_i consisting of all points $(t,0)$ with $0\leqslant t\leqslant 1$.

Subcase 3°. We have $C \cap L_i = L_i(t)$, where $L_i(t)$ is a subarc of L_i containing a and having length $t \leqslant 1$.

Here T_i is the real segment $[0, 1]$. Let $T'_i = T_i$.

Subcase 4°. Reasoning twice as in subcase 2° we come to the conclusion that $C \cap L_i$ is equal either to the end-points a and b of L_i or to the whole of L_i.

Here T_i is the real segment $[0, 1]$. Let T'_i consist of the end-points $(0)_i$ and $(1)_i$ of this segment.

It is not difficult to see that under the homeomorphism h from $\mathfrak{M}_{A\subset B}$ onto $\bigtimes_{i=1}^{m} T_i$ (cf. Lemma 5.2) the common part $\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'}$ goes onto $\bigtimes_{i=1}^{m} T'_i$.

If all segments L_i are of the type considered in subcase 3° (which happens if X is acyclic) or in subcases 1° and 3° (which happens if both A and A' contain all internal vertices of X; in particular, if both pairs $A \subset B$ and $A' \subset B'$ are maximal fine — cf. § 7), then $T'_i = T_i$ for each $i = k+1, k+2, \ldots, l$, and since, by virtue of (4),

$$A \cup A' \cup \bigcup_{i=k+1}^{l} L_i = B \cap B' ,$$

we have the equality

$$\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'} = \mathfrak{M}_{A\cup A'\subset B\cap B'} .$$

III. $A \cup A'$ is non-empty and not connected, i.e. $A \neq 0 \neq A'$ and $A \cap A' = 0$. If $\mathfrak{M}_{A\subset B} \cap \mathfrak{M}_{A'\subset B'} \neq 0$, then, as follows from the inclusions

$$A \cup A' \subset B \cap B' \subset Q(A, 1) \cap Q(A', 1) ,$$

we must also have $A \subset Q(A', 1)$ and $A' \subset Q(A, 1)$, i.e. $\varrho^1(A, A') \leqslant 1$. Since A and A' are disjoint, both A and A' must be vertices lying at a distance 1 from each other. By assumptions (α) and (γ), X must then contain a segment D joining them. But $D \subset D$ is clearly a pair and so, by virtue of (3), the proof will be completed if we show that $B \cap B' = D$. And this is obvious, because if S a segment distinct from D one vertex of which is, say, A, then, by assumptions (α) and (γ), for any point $x \in S-(A)$ we have $\varrho(x, A') > 1$. In other words, $x \notin B'$.

Thus the proof of 6.2 is completed. (Parts I and III were repeated, part II is new. The remark following 6.2 in the paper remains valid.)

There are three theorems the proofs of which were based upon lemmas 6.1 and 6.2.

Theorem 6.3 remains true, because nothing has been altered if X is acyclic.

Theorem 6.4 remains true, because $\bigcup \mathfrak{M}_{A\subset B}$ is a decomposition of $C(X)$ into a finite number of topological balls any two of which meet in such a way that their union forms a polyhedron or their common part is empty.

Theorem 7.5 remains true, because the pairs considered there are maximal fine and for such pairs new 6.1 and 6.2 hold. In the proof of 7.5 one should only replace the sentence "Hence ... see 6.2, case II" appearing in lines 7–11 on p. 279 by the following one "Hence $A \cup A' \subset X$ is a pair and by 6.2 (see 6.2, case II, and the Remark following 6.2)

$$\mathfrak{M}_{A\subset X} \cap \mathfrak{M}_{A'\subset X} = \mathfrak{M}_{A\cup A'\subset X} ."$$

I should like to express my gratitude to Professor Jack Segal for pointing out the incorrectness of the old Lemmas 6.1 and 6.2.

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